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μ^+ DECAY FROM MUONIUM

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The decay rate of muonium is calculated, taking into account the presence of the bound electron. The corresponding correction involved in the high precision μSR and magnetic resonance experiments is discussed. The angular distribution is also calculated, and the initial polarization of the muon is separately studied. The decay rate of a muon decaying from muonium is found to be 0.004% greater than when it decays from the free state and this enhancement is not negligible.

1. Introduction

Muonium provides a system of great interest from the point of view of muon electro-dynamics, muonic weak interaction, atomic and molecular collisions, and solid state physics. In particular, the hyper-fine structure interval $\Delta\nu$ provides a sensitive measure of the magnetic interaction between the muon and the electron, and the Zeeman energy intervals can be used to determine the muon spin and magnetic moment in the μSR and magnetic resonance experiments.

The stability of the muonium atom is however determined by the life time of the positive muon that constitutes its nucleus, and hence the magnetic resonance experiments, and the μSR techniques involve the weak interaction of the muon in muonium as well. In fact, the natural line-width for transitions between two muonium states is [1] $\Delta f_{1/2} = R/\pi$, where R is the decay rate of the muon. The resonance line width is an important factor in the expected precision of an experiment.

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Usually R is taken to be the decay rate of the free muon. The high precision techniques used currently in experiments connected with muons demand more accurate estimates of the decay rate, for the muon decaying from muonium. The muonium decay rate will differ from the free muon case due to several factors. The binding reduces the total energy available for the decay and this effectively reduces the decay rate. Although the bound electron does not participate in the weak interaction process, it remains present after the decay and can hence take up recoil momentum. It thus provides an additional final state particle, and increases the accessible phase space, thus enhancing the decay rate. The change in kinematics also alters the spectrum of the emitted positron, and its angular distribution, from that of the free muon.

In this communication, we report the calculation of the muonium decay rate and the positron angular distribution for the unpolarised muon. We also study separately the effect on the cross-section when the muon is initially polarized. Finally, we discuss the influence of our results on the high precision magnetic resonance experiments.

2. *Mathematical formalism*

We represent the process schematically, and indicate the momenta in Fig. 1. We write the matrix element for the decay process as

$$M_{fi} = \langle \Psi_f | H_{int} | \Psi_i \rangle \tag{1}$$

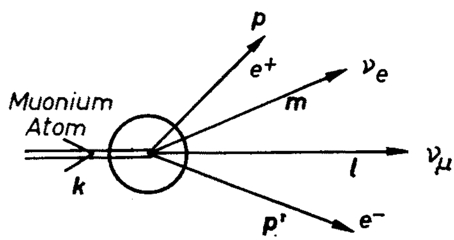

Fig. 1. Muonic decay from muonium

While H_{int} includes the electromagnetic interactions, the dominant contribution comes from the four-Fermi weak interaction H_F, where

$$H_F = (G/\sqrt{2}) \int [\bar{\psi}_\mu(k) O_\alpha \psi_{\nu_\mu}(l) \bar{\psi}_{\nu_e}(m) O_\alpha \psi_e(p)]\,(2\pi)^4 \times \delta^4(k-p-l-m) d^3\boldsymbol{p}\, d^3\boldsymbol{k}\, d^3\boldsymbol{l}\, d^3\boldsymbol{m} \tag{2}$$

with

$$O_\alpha = \gamma_\alpha(1+\gamma_5) \tag{3}$$

(we have used the standard (V–A) formalism).

To represent the bound muonium state, we use the field theoretic formalism developed by Roy [2], and write

$$|\bar{\Psi}_i\rangle = \int g_\mu(\boldsymbol{k}_1, \boldsymbol{k}) a_e^+(\boldsymbol{k}_1) B_\mu^+(\boldsymbol{k}) d^3\boldsymbol{k}_1 d^3\boldsymbol{k} |0\rangle, \tag{4}$$

where $a_e^+(\boldsymbol{k}_1)$, $B_\mu^+(\boldsymbol{k})$ correspond to the creation operators of the electron and muon of muonium, with momenta $\boldsymbol{k}_1$ and $\boldsymbol{k}$ respectively. $g_\mu(\boldsymbol{k}_1, \boldsymbol{k})$ is the bound state solution of the Schrödinger equation for the muon-electron system in momentum space [3].

$$
\begin{aligned}
g_\mu(\boldsymbol{k}_1, \boldsymbol{k}) &= \phi(\boldsymbol{k}_1)\delta^3(\boldsymbol{k}_1+\boldsymbol{k}-\boldsymbol{k}_2) \\
&= \int f(x') \exp(i\boldsymbol{k}_1 \cdot \boldsymbol{x})\delta^3(\boldsymbol{k}_1+\boldsymbol{k}-\boldsymbol{k}_2)d^3x' \qquad (5)
\end{aligned}
$$

($\boldsymbol{k}_2$ corresponds to the momentum of the centre of mass of the muonium atom), $f(x')$ corresponds to the wave function of the bound electron of muonium. For the final state, we assume all the particles to be free, and write

$$
|\Psi_f\rangle = a_e^+(\boldsymbol{k}_1)b^+(\boldsymbol{p})c_{\nu_\mu}^+(\boldsymbol{l})d_{\nu_e}^+(\boldsymbol{m})\,|0\rangle, \qquad (6)
$$

where $a^+(\boldsymbol{k}_1)$, $b^+(\boldsymbol{p})$, $c_{\nu_\mu}^+(l)$, $d_{\nu_e}^+(m)$ refer to the creation operators of the final electron, the created positron, and the two neutrinos of momenta $\boldsymbol{k}_1, \boldsymbol{p}, \boldsymbol{l}, \boldsymbol{m}$ respectively. Since H_F does not affect the bound electron, we can suppose its momentum to be representable by $\boldsymbol{k}_1$. Referring to (4), we integrate $|\Psi_i\rangle$ over k_1, to get

$$
|\Psi_i\rangle = \int \exp\{i(\boldsymbol{k}_2-\boldsymbol{k})\cdot \boldsymbol{x}'\}f(x')B_\mu^+(\boldsymbol{k})a_e^+(\boldsymbol{k}_2-\boldsymbol{k})d^3\boldsymbol{k}|0\rangle d^3\boldsymbol{x}'. \qquad (7)
$$

For the decay process, we consider the four weakly interacting leptons to be represented by plane waves. Taking the vacuum expectation values of the field operators, and integrating over k, x', we have finally for the matrix element:

$$
\begin{aligned}
M_{fi} = (G/\sqrt{2}) \int \overline{U}_\mu(\boldsymbol{k})O_\alpha U_p(\boldsymbol{p})\overline{U}_{\nu_\mu}(\boldsymbol{m})O_\alpha U_{\nu_e}(\boldsymbol{l})f(x) \\
\times \exp\{i(\boldsymbol{k}_2-\boldsymbol{p}-\boldsymbol{l}-\boldsymbol{m})\cdot \boldsymbol{x}\}d^3x. \qquad (8)
\end{aligned}
$$

3. The decay rate

We define the decay rate in the usual way as

$$
\begin{aligned}
R = (G^2/\sqrt{2}) \int \{d^3\boldsymbol{p}/(p_4(2\pi)^3)\}\{d^3\boldsymbol{p}'/(p_4(2\pi)^3)\}\{d^3\boldsymbol{l}/(l_4(2\pi)^3)\} \\
\times \{d^3\boldsymbol{m}/(m_4(2\pi)^3)\} \sum_{\text{spins}} |M_{fi}|^2, \qquad (9)
\end{aligned}
$$

where $d^3\boldsymbol{p}'/p_4'(2\pi)^3$ represents the phase space factor for the recoil electron, and $d^3\boldsymbol{p}/(2\pi)^3p_4$, $d^3\boldsymbol{l}/l_4(2\pi)^3$, $d^3\boldsymbol{m}/m_4(2\pi)^3$ correspond to the phase space factors of the positron and the two neutrinos respectively. We assume both the electron and the positron to have high momenta, so that their rest masses can be neglected, ad we also neglect the neutrino masses.

We evaluate the electron-muon, and $(\nu_\mu-\nu_e)$ parts of the decay rate separately, as is customary in the case of the free muon, or a muon bound in an atom [4]. We follow the same conventions for the γ matrices as Okun [5]. We assume for the present that the muon is unpolarised. Thus

$$
\sum_{\text{spins}} |M_{fi}|^2 = T_{e\mu}T_\nu II^*, \qquad (9')
$$

where

$$T_{e\mu} = \tfrac{1}{2} \sum_{\substack{\text{initial}\\ \text{spins}}} \sum_{\substack{\text{final}\\ \text{spins}}} \bar{U}_{\mu}(\boldsymbol{k}) O_{\alpha} U_{p}(\boldsymbol{p}) \bar{U}_{p}(\boldsymbol{p}) O_{\alpha} U_{\mu}(\boldsymbol{k}), \tag{10}$$

$$T_{\nu} = \sum_{\text{spins}} \sum_{\text{spins}} \bar{U}_{\nu_{\mu}}(\boldsymbol{m}) O_{\alpha} U_{\nu_e}(\boldsymbol{l}) \bar{U}_{\nu_e}(\boldsymbol{l}) O_{\alpha} U_{\nu_{\mu}}(\boldsymbol{m}), \tag{11}$$

$$I = \int f(x) \exp \{i(\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{l}-\boldsymbol{m}) \cdot \boldsymbol{x}\} d^3 x, \tag{12}$$

$$I^* = \int f(x') \exp \{i(\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{l}-\boldsymbol{m}) \cdot \boldsymbol{x}'\} d^3 \boldsymbol{x}', \tag{13}$$

with ω, E, $\omega_{\nu_{\mu}}$, ω_{ν_e} representing the energies of the electron, positron, muon-neutrino and electron neutrino respectively. We define

$$\boldsymbol{q}_b = \text{sum of the neutrino momenta} = \boldsymbol{l}+\boldsymbol{m} = -\boldsymbol{p}-\boldsymbol{p}'+\boldsymbol{k}_2 \tag{14}$$

(using the momentum conservation law).

After evaluating the traces, and performing the neutrino phase space integrations, we have finally

$$R = \{8G^2\pi/12(2\pi)^8\} \int d^3\boldsymbol{p} d^3\boldsymbol{p}' p_{\alpha} k_{\beta} (q_b^2 \delta_{\alpha\beta} + 2q_{b\alpha} q_{b\beta}) II^*/E\omega. \tag{15}$$

We assume the centre of mass of the initial atom to be at rest. Since the nucleus is 200 times heavier than the electron, we also assume the muon (nucleus) to be at rest, as in the case of the hydrogen problem.

$$q_b = 2E\omega(1-\cos\theta') \tag{16}$$

with $\cos\theta' = \boldsymbol{p} \cdot \boldsymbol{p}'$. So

$$p_{\alpha} k_{\beta} (q_b^2 \delta_{\alpha\beta} + 2q_{b\alpha} q_{b\beta}) = 2\omega E(1-u)(2E+\omega)W, \tag{17}$$

where $u = \cos\theta'$.

4. *Results*

Evaluating I, I^*, assuming the electron to be bound in the muonium in a Bohr orbit, we have

$$II^* = \mu_0^3 \cdot 64\pi^2 \mu_0^2/\pi(\mu_0^2+\omega^2)^4, \tag{18}$$

where

$$\mu_0 = Ze^2/\hbar c. \tag{19}$$

Dividing R by the free muon decay rate

$$R_{\text{f}} = G^2 m_{\mu}^5/192\pi^3 \tag{20}$$

we obtain

$$R' = R/R_{\text{f}} = \{2 \cdot (16^2) \cdot \mu_0^5/\pi m_{\mu}^5\} \int_0^W \omega^2 d\omega/\omega \int_0^{Em} E^2 dE/E \times \int_{-1}^{1} du \cdot 2\omega E(1-u)W(2E+\omega)/(\mu_0^2+\omega^2)^4. \tag{21}$$

The integration limits are obtained from the kinematics of the problem. Performing the phase-space integrations, we eventually obtain

$$R' = R/R_f = W^5/m_\mu^5 + 32W^2\mu_0/9\pi - W\mu_0^2. \tag{22}$$

The remaining terms are negligibly small, being smaller by order 10^{-8} than the last one. Numerically, this yields

$$R' = 1 + 0.00004131 - 0.1332\times10^{-8} = 1.00004132 \tag{23}$$

((W/m_μ) differs from unity in the order of 10^{-8}).

The decay rate of a muon from muonium is thus enhanced from the free value by about 0.004% which is not negligible.

Considering equation (21), the positron energy spectrum can be obtained from it by integrating over u and ω.

To obtain the angular distribution of the emitted positron, with respect to the direction of the emitted electron, we integrate (21) over E and ω.

The laborious evaluation of the ω integrals yield dR as a sum of more than 50 terms, some of which contain the factor $1/(W^2 + u\cdot\mu_0^2)$ where $\mu_0^2 \ll W$. It yields an approximately $(1-u)$ distribution.

Studying the angular spectrum for various fixed values of W and E also, it is in general found to be proportional to $(1-u)$, where $u = \cos\theta'$, θ' being the angle between the momentum directions of the emitted electron and positron.

5. *The polarised muon in muonium*

If we suppose the initial muon to be polarised, and denote its polarisation vector by ξ, the projection operator corresponding to the initial muon must be taken as

$$U\bar{U} = \tfrac{1}{2}(k\cdot\gamma + m_\mu)(1 - \gamma_5 s\cdot\gamma)$$

with $s\cdot\gamma = \gamma_\mu s_\mu$. s_μ is the covariant generalisation of ξ

$$s_4 = \boldsymbol{k}\cdot\xi/m_\mu, \qquad s = \xi + \boldsymbol{k}\cdot(\boldsymbol{k}\cdot\xi)/m_\mu(m_\mu+\omega).$$

Proceeding as for the unpolarised muon, taking traces etc., we have the additional contribution to R as

$$R_p = \int d^3\boldsymbol{p}/E \int d^3\boldsymbol{p}'/\omega \cdot \{64\pi\mu_0^5 2Wm_\mu G^2(W-E-\omega)\}$$
$$\times[\xi\cdot\boldsymbol{p} - \xi\cdot\boldsymbol{p}']/3(\mu_0^2+\omega^2)^4 \tag{24}$$

and the angular asymmetry of the decay positron emission with respect to the muon spin direction can be determined from $(R+R_p)$.

6. *Discussion*

Refering to equation (22), the $(+\mu_0)$ term occurs due to the presence of the catalytic electron, which makes available extra accessible phase space.

The $(-\mu_0^2)$ terms arises due to the reduction of available energy due to binding, and W is reduced from m_μ to $\sqrt{m_\mu^2-\mu_0^2}$, for the same reason. This effect is small, as the binding is small $(\mu_0^2 \approx 0.13\times 10^{-8})$ due to the small electronic mass, as compared to the muonic mass.

The correction to the decay-rate calculated by us is not negligible. Thus in the magnetic resonance experiments, the natural line width $\Delta f_{1/2} = R/\pi$ is increased from $1/\pi$ to $(1\ 00004)/\pi$, and the maximum displacement S, where $S = |b|^2/(4|b|^2+R^2)$ (where $|b|^2$ is the microwave power), is also affected. In the high precision resonance experiments that are performed, these differences should be detectable.

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