UPPER BOUND ON TOTAL CROSS SECTIONS. II*

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Using fewer assumptions, we improve on a recently derived asymptotic upper bound for total cross sections.

In a recent paper, we showed, under certain assumptions, that if the N-th derivative of the absorptive part of an elastic scattering amplitude exists in the interval $t_1 \leqslant t \leqslant 0$, then the total cross section has the following asymptotic upper bound [1]

$$\sigma_T(S) < S^{\frac{1}{N+1}}. \tag{1}$$

The purpose of this paper is to demonstrate that the bound of Eq. (1) can be improved by factors of LnS using one less assumption than that needed to obtain the inequality of Eq. (1).

For clarity and completeness, we list the required assumptions given in Ref. [1]. We denote the elastic scattering amplitude and its absorptive part, respectively, by $F(S, t)$ and $A(S, t)$. The following result is a rigorous consequence of the unitarity condition [2, 3]: If $A(S, t)$ has an m-th derivative at $t = 0$, then, the derivative satisfies the following asymptotic lower bound

$$\lim_{S \to \infty} A_m(S) > \left(\frac{1}{16\pi}\right)^m \cdot \frac{S}{(m+1)!} \cdot [\sigma_T(S)]^{m+1}, \tag{2}$$

where

$$A_m(S) = \left.\frac{d^m A(S, t)}{dt^m}\right|_{t=0}. \tag{2'}$$

In addition to unitarity, we assume that $F(S, t)$ and $A(S, t)$ fulfil the following two requirements:

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(I) $F(S, t)$ satisfies a twice subtracted dispersion relation. For simplicity, we shall consider $F(S, t)$ to be symmetric in S and U, and contain no pole terms; therefore [4]

$$F(S, t) = C(t)+\frac{S^2}{\pi}\int_{S_0}^{\infty}\frac{A_s(x, t)dx}{x^2(x-S)}+\frac{U^2}{\pi}\int_{S_0}^{\infty}\frac{A_u(x, t)dx}{x^2(x-U)}, \tag{3}$$

where

$$A_s(x, t) = A_u(x, t) = A(x, t). \tag{4}$$

(II) We assume that the N-th derivative of $A(S, t)$ exists and is continuous in the interval $t_1 \leqslant t \leqslant 0$, where t_1 is a negative real constant.

Using (II) and Taylor's theorem, we can express $A(S, t)$ in the following form [5]

$$A(x, t) = \sum_{m=0}^{N} A_m(x)t^m/m!+\eta(x, t), \tag{5}$$

where $A_m(x)$ is defined by Eq. (2′) and

$$\lim_{t\to 0}\left|\frac{\eta(x, t)}{t^N}\right| = 0. \tag{6}$$

The substitution of Eq. (5) into Eq. (3) gives

$$F(S, t) = C(t)+\frac{S^2}{\pi}\sum_{m=0}^{N}\left(\frac{t^m}{m!}\right)\int_{S_0}^{\infty}\frac{A_m(x)dx}{x^2(x-S)}+\frac{U^2}{\pi}\sum_{m=0}^{N}\left(\frac{t^m}{m!}\right)\int_{S_0}^{\infty}\frac{A_m(x)dx}{x^2(x-U)}$$

$$+\frac{S^2}{\pi}\int_{S_0}^{\infty}\frac{\eta(x, t)dx}{x^2(x-S)}+\frac{U^2}{\pi}\int_{S_0}^{\infty}\frac{\eta(x, t)dx}{x^2(x-U)}. \tag{7}$$

For fixed, finite S and t, all the integrals on the right-side of Eq. (7) exist and are bounded; in particular

$$\int_{S_0}^{\infty}\frac{A_m(x)dx}{x^2(x-S)} < \infty, \qquad 0 \leqslant m \leqslant N. \tag{8}$$

The inequality given by Eq. (8) implies the existence of a sequence of values $x = x_i$, where $x_i \to \infty$, such that for $x \to \infty$ along such a sequence, we have [6]

$$A_m(x) < x^2/\mathrm{Ln}\,(x)\,[\mathrm{LnLn}\,(x)]\,\ldots\,. \tag{9}$$

Comparison of Eqs. (2) and (9) gives, after some simplification, the result

$$\lim_{x\to\infty}\sigma_T(x) < C_1(m)\,\{x/[\mathrm{Ln}\,(x)\,\mathrm{LnLn}\,(x)\,\ldots]\}^{\frac{1}{m+1}}, \tag{10}$$

where $C_1(m)$ is a known function of m. If we change to the variable S and set $m = N$, we obtain

$$\lim_{S\to\infty} \sigma_T(S) < \text{constant}\, \{S/[\text{Ln}\,(S)\,\text{LnLn}\,(S)\,\ldots]\}^{\frac{1}{N+1}} . \tag{11}$$

In summary, we have shown that if the elastic scattering amplitude satisfies a twice subtracted dispersion relation and has an N-th derivative continuous in the interval $t_1 \leqslant t \leqslant 0$, where t_1 is a negative constant, then the corresponding total cross section is bounded at asymptotic energies by the inequality of Eq. (11). This result is an improvement over the bound of Ref. [1] which is given by Eq. (1); in addition, fewer assumptions are required to obtain this bound [7].

Finally, we note that the bound of this paper may be of use in the investigation of the properties of scattering amplitudes constructed to describe weak and electromagnetic processes. In general, the scattering amplitudes for these processes are not analytic in t at $t = 0$ [8, 9]; however, the result of this paper still allows an upper bound to be obtained on the correspooding total cross section.

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