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INTERNAL FIELDS OF THE LYTTLETON-BONDI SPHERE

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We obtain a solution of the relativistic field equations for a static Lyttleton–Bondi sphere. We establish the following results: A static Lyttleton–Bondi sphere is a charged dust. In particular we demonstrate that under certain conditions, the sphere cannot exist.

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1. *Introduction*

In 1948 Bondi and Gold (Bondi and Gold, 1948) suggested that the Universe obeys "a perfect Cosmological Principle" — that the Universe looks the same not only at all points and in all directions, but at all times. To explain the observed expansion of the Universe Lyttleton and Bondi (Lyttleton and Bondi, 1959) modified the 1948 theory slightly by suggesting the possibility of a general excess of charge in the Universe. This excess of charge may arise from the difference in magnitude of the charge of the proton and that of the electron, or from the difference in the number of protons as compared to the number of electrons.

Further, if the charge excess exceeds a certain critical value, expansion would result. If this expansion alone occurred, the space density of material would steadily diminish and the acceleration of expansion will decrease with it. To off-set the decrease in density implied by the expansion, Lyttleton and Bondi postulated the creation of matter, and also necessarily charge, everywhere in space. This creation process will then keep the density invariable. Since charge is not then conserved in the strict sense some modification of the Maxwell equations must be made to permit the resulting breach of conservation of charge. This is what is normally called "the steady state model of the Universe" in the literature.

However, through the work of Penzias and Wilson (Penzias and Wilson, 1965) it is now generally believed that the Universe has evolved to its present state from an initially dense and hot configuration. This observation confirms the "Big Bang" origin and "destroys" the "steady state model" of the Universe.

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In a recent paper (Nduka, 1979) we re-examined the assumptions of the steady state model and modified them to conform to experimental observations of Penzias and Wilson. Specifically we rejected the "perfect cosmological principle" assumption and replaced it with the usual "cosmological principle" and then retained the other assumptions of Lyttleton and Bondi. The implication is that we can now use Einstein's field equations. Thus we investigated the Lyttleton Bondi sphere, in the circumstance we allow creation of matter and charge, on the basis of Einstein's field equations. We established that if the charge distribution throughout the sphere attains a certain critical value, the creation rate vanishes and the sphere becomes static, otherwise the sphere expands indefinitely.

In this paper we investigate the nature of the internal fields of our modified Lyttleton–Bondi cosmology for the case of a sphere which has evolved to its final static state — and hence its expansion halted. It is shown that the only possible final state of such a sphere is characterized by the condition $p = 0$ so that the final state is characterized by the requirement that the attractive gravitational field of force is balanced by the repulsive electrostatic field of force — a charged dust. In the extreme case that the evolution of the sphere is such that the static state is characterized by a Lorentzian metric, we find that $\varrho = p = E = \sigma = 0$ (see below) throughout the sphere; which means non-existence of the sphere itself.

2. *Field equations*

The appropriate metric form to describe a static, spherically symmetric space-time considered in our investigation is given by

$$ds^2 = e^{\nu}dt^2 - e^{\gamma}dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{2.1}$$

where ν and γ are functions of r only. Throughout this paper units are chosen so that G and C are each unity.

The static condition is characterized by $u^i = e^{-\nu/2}\delta^i{}_0$. The Maxwell field is restricted by spherical symmetry to be a radial electrostatic field. Thus the only non-vanishing components of F^{ij} are $F^{01} = -F^{10}$.

The Einstein–Maxwell equations are (Lyttleton and Bondi, 1959)

$$G^i{}_k = -8\pi T^i{}_k, \tag{2.2}$$

$$F_{ik} = A_{i;k} - A_{k;i}, \tag{2.3}$$

$$F^{ik}{}_{;k} = 4\pi \mathcal{J}^i, \tag{2.4}$$

$$J^i{}_{;i} = q. \tag{2.5}$$

Here $T^i{}_k$ is the energy-momentum tensor for the system and we express it in the form

$$T^i{}_k = M^i{}_k + E^i{}_k, \tag{2.6}$$

where $M^i{}_k$ is the energy-momentum tensor of perfect fluid:

$$M^i{}_k = (\varrho + p)u^i u_k - p\delta^i{}_k \tag{2.7}$$

and E^i_k is the electromagnetic stress-energy tensor:

$$4\pi E^i_k = -F^{il}F_{kl}+\tfrac{1}{4}\,\delta^i_k F^{hl}F_{nl}+\lambda(A^iA_k-\tfrac{1}{2}\,\delta^i_k A_l A^l). \tag{2.8}$$

The quantity q is the rate of creation of charge per unit volume and $\mathscr{J}^i$ is defined by

$$\mathscr{J}^i = \sigma u^i - \frac{\lambda}{4\pi}A^i. \tag{2.9}$$

In these equations p is the internal pressure, ϱ, σ are the densities of matter and charge respectively and $\lambda^{-1/2}$ is a physical constant of the order of the radius of the Universe. We note from $J^i = \sigma u^i$, $u^i = e^{-\nu/2}\delta^i_0$ and equation (2.5) that $q \equiv 0$, in agreement with our previous conclusion (Nduka, 1979).

The field equations that follow drom the above are

$$e^{-\gamma}\left(\frac{\gamma'}{r}-\frac{1}{r^2}\right)+\frac{1}{r^2} = 8\pi\varrho+E+\lambda\phi^2e^\nu, \tag{2.10}$$

$$e^{-\gamma}\left(\frac{\nu'}{r}+\frac{1}{r^2}\right)-\frac{1}{r^2} = 8\pi p-E+\lambda\phi^2e^\nu, \tag{2.11}$$

$$e^{-\gamma}\left(\frac{\nu''}{2}+\frac{\nu'^2}{4}-\frac{\nu'\gamma'}{4}+\frac{\nu'-\gamma'}{2r}\right) = 8\pi p+E+\lambda\phi^2e^\nu, \tag{2.12}$$

$$4\pi\sigma = \left[\frac{dF^{01}}{dr}+\frac{2}{r}F^{01}+\frac{\gamma'+\nu'}{2}F^{01}+\lambda\phi\right]e^{\nu/2} \tag{2.13}$$

and

$$(\phi e^\nu)' = F_{01}, \tag{2.14}$$

where $\phi = A^0$ and

$$E = -F^{01}F_{01}. \tag{2.15}$$

It has already been established (Nduka, 1979) that the static condition is characterized by the requirement that

$$\phi = \Gamma_0 e^{\nu/2}, \tag{2.16}$$

where Γ_0 is a constant. It then follows from equations (2.14) and (2.15) that

$$E = \left(\frac{3\Gamma_0}{2}\nu'\right)^2 e^{2\nu-\gamma}. \tag{2.17}$$

Equations (2.10), (2.11) and (2.12) may be rewritten in the forms

$$8\pi(\varrho+p) = e^{-\gamma}\left(\frac{\nu'}{r}+\frac{\gamma'}{r}\right)-2\lambda\Gamma_0^2e^{2\nu}, \tag{2.18}$$

$$8\pi(\varrho - p) = e^{-\gamma}\left(\frac{\nu'\gamma'}{4} - \frac{\nu'}{2r} + \frac{3\gamma'}{2r} - \frac{\nu''}{2} - \frac{\nu'^2}{4} - \frac{1}{r^2}\right) + \frac{1}{r^2}, \tag{2.19}$$

$$2E = e^{-\gamma}\left[\frac{\nu''}{2} + \frac{\nu'^2}{4} - \frac{\nu'\gamma'}{4} - \frac{\nu'}{2r} - \frac{\gamma'}{2r} - \frac{1}{r^2}\right] + \frac{1}{r^2} \tag{2.20}$$

and

$$4\pi\sigma = \left[e^{-\gamma}\left(\frac{3\nu'\gamma'}{4} - \frac{3\nu'^2}{2} - \frac{3\nu''}{2} - \frac{3\nu'}{r}\right) + \lambda\Gamma_0\right] e^{\nu/2}. \tag{2.21}$$

3. Solution of the field equations

Equations (2.18)–(2.21) constitute a system of four equations for the six unknowns $\varrho, p, \Gamma_0, \nu, \gamma$, and σ. We solve them subject to the additional constraint that ϱ is a constant. Two of these variables are completely arbitrary. We take ν and γ as the arbitrary variables, and we choose them in the form

$$\gamma(r) = \alpha r^2, \tag{3.1}$$

$$\nu(r) = \beta r^2, \tag{3.2}$$

where α and β are arbitrary constants. Then from equations (2.18)–(2.21), (3.1) and (3.2) we obtain

$$16\pi\varrho = e^{-\alpha r^2}\left[\beta(\alpha-\beta)r^2 + 5\alpha - \frac{1}{r^2}\right] + \frac{1}{r^2} - 2\lambda\Gamma_0^2 e^{2\beta r^2}, \tag{3.3}$$

$$16\pi p = e^{-\alpha r^2}\left[\beta(\beta-\alpha)r^2 + 4\beta - \alpha + \frac{1}{r^2}\right] - \frac{1}{r^2} - 2\lambda\Gamma_0^2 e^{2\beta r^2}, \tag{3.4}$$

$$2E = e^{-\alpha r^2}\left[\beta(\beta-\alpha)r^2 - \alpha - \frac{1}{r^2}\right] + \frac{1}{r^2}, \tag{3.5}$$

$$4\pi\sigma = \{e^{-\alpha r^2}[3\beta(\alpha-2\beta)r^2 - 9\beta] + \lambda\}\Gamma_0 e^{(\beta/2)r^2}. \tag{3.6}$$

The requirement that ϱ, defined by equation (3.3), be a constant dictates the possible values the constants α and β can take. As was stated earlier the model sphere we are dealing with is one which has evolved to its present static state from an initially dense configuration. Thus as the sphere expands we expect that both ν and γ tend to zero as r tends to infinity. This is quite reasonable because we expect the metric to be Lorentzian at infinity. Equation (3.3) shows that ϱ is a constant only if both α and β are small quantities. Thus to first order in the small quantities α and β, equations (3.3), (3.4) and (3.6) yield:

$$8\pi\varrho = 3\alpha - \lambda\Gamma_0^2, \tag{3.7}$$

$$8\pi p = (2\beta - \alpha) - \lambda\Gamma_0^2, \tag{3.8}$$

$$4\pi\sigma = (\lambda - 9\beta)\Gamma_0. \tag{3.9}$$

4. Boundary conditions

We note from equations (3.7) and (3.9) that ϱ and σ are positive provided that

$$3\alpha \geqslant \lambda \Gamma_0^2, \tag{4.1}$$

$$\lambda \geqslant 9\beta. \tag{4.2}$$

Next, we fix the constants of our problem in terms of the physical constants of mass, charge and radius of the sphere. To achieve this we impose the usual conditions at the boundary $r = r_0$ of the sphere:

(*i*) $p(r_0) = 0$. From equation (3.8) we conclude that

$$2\beta - \alpha = \lambda \Gamma_0^2. \tag{4.3}$$

(*ii*) $E(r_0) = Q^2/r_0^4$, where Q is the total charge of the sphere. Equation (2.17) then gives for $r = r_0$

$$3\beta\Gamma_0 = Q/r_0^3. \tag{4.4}$$

We also establish from equations (2.17) and (3.5) that

$$3\beta\Gamma_0 = \alpha/2. \tag{4.5}$$

Equations (4.4) and (4.5) lead to

$$\alpha = 2Q/r_0^3. \tag{4.6}$$

(*iii*) The mass distribution is defined by (Nduka, 1979)

$$8m(r) = 4e^{\mu/2} - \mu'^2 e^{3\mu/2-\gamma}, \tag{4.7}$$

where in the present case $e^\mu = r^2$. Putting $m(r_0) = M$, we find that

$$\alpha = 2M/r_0^3. \tag{4.8}$$

Use can now be made of equations (4.3) and (4.4) to obtain

$$\Gamma_0 = (2Q/3\lambda r_0^3)^{1/3}, \quad \beta = (\lambda Q^2/6r_0^6)^{1/3}. \tag{4.9}$$

Equations (3.7) and (3.9) then give for the charge and matter densities the expressions

$$4\pi\sigma = (2Q\lambda^2/3r_0^3)^{1/3} - (3Q/r_0^3), \tag{4.10}$$

$$4\pi\varrho = (3Q/r_0^3) - \frac{\lambda}{2}(2Q/3\lambda r_0^3)^{2/3}. \tag{4.11}$$

Equations (4.6) and (4.8) show that $M = Q$, so that the total charge of the sphere equals its total mass. We note further that the first term of equation (4.10) and the second term of equation (4.11) are both small compared to $(3Q/r_0^3)$ and hence may be dropped. We then reach the usual conclusion for charged dust that the ratio of matter to charge density is unity. It is to be noted from equation (4.10) that the charge density is negative so that the condition (4.2) is violated and must be replaced by the requirement $\lambda \leqslant 9\beta$

instead. In the extreme case that the radius of the sphere is infinite, we find from equations (4.6) and (4.9) that $\alpha = \beta = \Gamma_0 = 0$, so that the metric becomes Lorentzian. For this special case the solutions of equations (3.3) to (3.6) give $\varrho = p = \sigma = E = 0$, implying that the sphere no longer exists.

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