A MODEL OF MAGNETOVISCOUS FLUID IN GENERAL RELATIVITY

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A model of magnetoviscous fluid has been derived which is of non-degenerate Petrov type I. The model for viscous fluid, magnetofluid and perfect fluid have also been obtained in particular cases. Various physical and geometrical properties of the model have been discussed.

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1. *Introduction*

In our previous papers ([1–3]) we have obtained some viscous fluid cosmological models and anisotropic magnetohydrodynamic cosmological models. Recently Roy and Raj Bali [4] have obtained a magnetoviscous fluid cosmological model of type D. But the main drawback of this model [4] is that the models for magnetofluid, viscous fluid and perfect fluid cannot be obtained from it in the absence of viscosity, magnetic field or both respectively, in particular cases. It is therefore of interest to derive a magnetoviscous fluid model in general and obtain models for viscous fluid, magnetofluid and perfect fluid respectively, in particular cases. It is well known that there exists a certain degree of anisotropy in the actual universe. We therefore choose the metric for the cosmological model to be cylindrically symmetric. In this paper a cylindrically symmetric cosmological model of magnetoviscous fluid has been derived which is of non-degenerate Petrov type I. It represents an expanding and shearing but non-rotating fluid flow. The expression for the generalized Doppler effect in the model has been obtained. The models for viscous fluid, magnetofluid and perfect fluid have also been obtained in particular cases.

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2. Derivation of the line element

The general cylindrically symmetric metric is considered in the form given by [5]

$$ds^2 = A^2(dx^2 - dt^2) + B^2 dy^2 + C^2 dz^2, \tag{1}$$

where A, B, C are functions of t-alone. The energy-momentum tensor for a magneto-viscous fluid distribution is given by ([6, 7])

$$T_i^k = (\varepsilon + p) v_i v^k + p\delta_i^k - \eta(v_{i,}{}^k + v^k{}_{,i} + v^k v^l v_{i,l} + v_i v^l v^k{}_{,l})$$
$$-(\varrho - \tfrac{2}{3}\eta) v^l{}_{,l}(\delta_i^k + v_i v^k) + \mu\{|h|^2 (v_i v^k + \tfrac{1}{2}\delta_i^k) - h_i h^k\} \tag{2}$$

p being the isotropic pressure, ε the density, η and ϱ the two coefficients of viscosity, μ the constant magnetic permeability and a comma indicates covariant differentiation. The magnetic field vector h_i is defined by [7]

$$h_i = \frac{1}{\mu}\, {}^*F_{ji} v^j. \tag{3}$$

The dual electromagnetic field tensor is defined by [8]

$${}^*F_{ij} = \tfrac{1}{2}\eta_{ijkm} F^{km} \tag{4a}$$

and the permutation tensor

$$\eta_{ijkm} = -\sqrt{-g}\,\varepsilon_{ijkm}. \tag{4b}$$

F_{ij} is the electromagnetic field tensor, ε_{ijkm} is the numerical permutation symbol and v^i is the flow vector satisfying

$$g_{ij} v^i v^j = -1. \tag{5}$$

The coordinates are assumed to be comoving so that $v^1 = v^2 = v^3 = 0$ and $v^4 = 1/A$. We take the incident magnetic field to be in the direction of x-axis so that F_{23} is the only non-vanishing component of the electromagnetic field tensor F_{ij}. The first set of Maxwell's equation lead to F_{23} being a constant, say, H. We also notice that the components of the electromagnetic field tensor F_{ij} are constant in the present coordinate system but the tensor is not covariant constant. We therefore do not call it a constant magnetic field. The field equations

$$R_i^k - \tfrac{1}{2} R\delta_i^k + \Lambda\delta_i^k = -8\pi T_i^k \tag{6}$$

for the line-element (1) are as follows:

$$\frac{1}{A^2}\left[-\frac{B_{44}}{B} - \frac{C_{44}}{C} - \frac{B_4C_4}{BC} + \frac{A_4B_4}{AB} + \frac{A_4C_4}{AC}\right] - \Lambda$$
$$= 8\pi\left[p - 2\eta\frac{A_4}{A^2} - (\varrho - \tfrac{2}{3}\eta) v^l{}_{,l} - \frac{H^2}{2\mu B^2 C^2}\right], \tag{7}$$

$$\frac{1}{A^2}\left[-\frac{C_{44}}{C}-\frac{A_{44}}{A}+\frac{A_4^2}{A^2}\right]-\Lambda$$

$$= 8\pi\left[p-2\eta\frac{B_4}{AB}-(\varrho-\tfrac{2}{3}\eta)v^l{}_{;l}+\frac{H^2}{2\mu B^2C^2}\right], \tag{8}$$

$$\frac{1}{A^2}\left[-\frac{B_{44}}{B}-\frac{A_{44}}{A}+\frac{A_4^2}{A^2}\right]-\Lambda$$

$$= 8\pi\left[p-2\eta\frac{C_4}{AC}-(\varrho-\tfrac{2}{3}\eta)v^l{}_{;l}+\frac{H^2}{2\mu B^2C^2}\right], \tag{9}$$

$$\frac{1}{A^2}\left[\frac{A_4B_4}{AB}+\frac{B_4C_4}{BC}+\frac{C_4A_4}{CA}\right]+\Lambda = 8\pi\left[\varepsilon+\frac{H^2}{2\mu B^2C^2}\right]. \tag{10}$$

The suffix 4 after the symbols A, B, C denotes ordinary differentiation with respect to t. These are four equations in five unknowns A, B, C, ε and p. The coefficients of viscosity η and ϱ are taken as constants. Equations (7)–(10) are not independent, but they are related by the contracted Bianchi identities. In the present case they lead to the single condition

$$\frac{d\varepsilon}{dt}+\frac{d}{dt}\left(\frac{H^2}{2\mu B^2C^2}\right)+(\varepsilon+p)\frac{d}{dt}\log(ABC)-(\varrho-\tfrac{2}{3}\eta)\frac{1}{A}\left\{\frac{d}{dt}\log(ABC)\right\}^2$$

$$-\frac{2\eta}{A}\left(\frac{A_4^2}{A^2}+\frac{B_4^2}{B^2}+\frac{C_4^2}{C^2}\right)-\frac{H^2}{2\mu B^2C^2}\left(\frac{A_4}{A}-\frac{B_4}{B}-\frac{C_4}{C}\right)=0. \tag{11}$$

For complete solution of equations (7)–(10), we need an extra condition. An obvious one is the imposition of an equation of state. However, in this paper we take

$$A = BC \tag{12}$$

as a supplementary condition for the complete determination of this set. From equations (7), (8) and (12) we get

$$\frac{C_{44}}{C}+\frac{B_4C_4}{BC} = -\left(16\pi\eta BC_4+\frac{8\pi H^2}{\mu}\right). \tag{13}$$

From equations (8), (9) and (12) we get

$$\frac{B_{44}}{B}-\frac{C_{44}}{C} = 16\pi\eta BC\left(\frac{C_4}{C}-\frac{B_4}{B}\right). \tag{14}$$

Putting $BC=\alpha$ and $B/C=\beta$ in equations (13) and (14) we obtain

$$\alpha_{44}-(\alpha\beta_4/\beta)_4+16\pi\eta\alpha\left(\alpha_4-\frac{\alpha\beta_4}{\beta}\right)+\frac{16\pi H^2\alpha}{\mu}=0 \tag{15}$$

and

$$(\alpha\beta_4/\beta)_4 = -16\pi\eta\alpha^2\beta_4/\beta, \tag{16}$$

respectively. From equations (15) and (16) we get

$$\alpha_{44}+16\pi\eta\alpha\alpha_4+16\pi Il^2\alpha/\mu = 0. \tag{17}$$

Let us now define $F(\alpha) = \dfrac{d\alpha}{dt}$. Equation (17) then leads to

$$\frac{FdF}{(16\pi\eta F+l^2)} = -\alpha d\alpha, \tag{18}$$

which on integration gives

$$\alpha^2 = \left[\frac{n^2}{P+l^2} - \frac{1}{P}\left\{F+\frac{l^2}{2P}\log\left(l^2/2PF+l^2\right)\right\}\right], \tag{19}$$

where $l^2 = 16\pi H^2/\mu$, $P = 8\pi\eta$ and n is an arbitrary constant. Equations (16) and (18) lead to

$$\frac{d}{d\alpha}(\log\beta) = (16\pi\eta F+l^2)K/Q, \tag{20}$$

where $Q = (16\pi\eta n+l^2)$ and K is a constant of integration. From equations (18), (19) and (20) we get

$$\beta = \frac{b2^{K/n}}{(2n\sqrt{P}+l)^{K/n}}\exp\left(-\int\frac{KdF}{Q\alpha^2}\right), \tag{21}$$

where b is a constant. By a suitable transformation of coordinates the metric of this model can be put into the form

$$ds^2 = \psi(T)dX^2 - \frac{dT^2}{(2PT+l^2)^2} + [\psi(T)]^{1/2}\,[\beta dY^2+\beta^{-1}dZ^2], \tag{22}$$

where

$$\psi(T) = \frac{n^2}{P+l^2} - \frac{1}{P}\left\{T+\frac{l^2}{2P}\log\left(l^2/2PT+l^2\right)\right\},$$

and

$$\beta = \frac{b2^{K/n}}{(2n\sqrt{P}+l)^{K/n}}\exp\left(\int\frac{KdT}{Q\psi(T)}\right).$$

In the absence of viscosity the metric (22) after suitable coordinates transformation reduces to

$$ds^2 = \frac{n^2 \sin^2 lT}{l^2} dX^2 - \frac{n^2 \sin^2 lT}{l^2} dT^2$$

$$+ \frac{n \sin lT}{l} \left[b(2/l)^{K/n} \left(\frac{1+\cos T}{1-\cos T} \right)^{-K/2n} dY^2 + (1/b)(l/2)^{K/n} \left(\frac{1+\cos T}{1-\cos T} \right)^{K/2n} dZ^2 \right], \tag{23}$$

representing a magnetohydrodynamic cosmological model. Also in the absence of the magnetic field the metric (22) after suitable coordinates transformation can be put into the form

$$ds^2 = \frac{n^2 \sin^2 (\sqrt{P}\, T)}{P} dX^2 - \frac{\sin^2 2(\sqrt{P}\, T)}{4P \cos^4 (\sqrt{P}\, T)} dT^2$$

$$+ \frac{n \sin \sqrt{P}\, T}{\sqrt{P}} \left[\frac{b \sin^{K/n} (\sqrt{P}\, T)}{P^{K/2n}} dY^2 + \frac{1}{b} P^{K/2n} \operatorname{cosec}^{K/n} (\sqrt{P}\, T) dZ^2 \right], \tag{24}$$

representing a viscous fluid distribution. The metrics (23) and (24) reduce to

$$ds^2 = T^2(dX^2 - dT^2) + T^{1+\gamma} dY^2 + T^{1-\gamma} dZ^2 \tag{25}$$

in the absence of magnetic field and viscosity respectively which represents a perfect fluid distribution, γ being an arbitrary constant.

3. Some physical and geometrical features

The pressure and density in the model (22) are given by

$$8\pi p = \frac{1}{\psi(T)} \left[\frac{5T^2}{4\psi(T)} + \tfrac{5}{12}(4PT + 3l^2) - \frac{K^2(2PT+l^2)^2}{4Q^2 \psi(T)} + \frac{KP(2PT+l^2)}{Q} + 16\pi \varrho T \right] - \Lambda \tag{26}$$

and

$$8\pi \varepsilon = \frac{1}{\psi(T)} \left[\frac{5T^2}{4\psi(T)} - \frac{K^2(2PT+l^2)^2}{4Q^2 \psi(T)} - \frac{l^2}{4} \right] + \Lambda. \tag{27}$$

The model has to satisfy the reality conditions [9]

(*i*) $(\varepsilon + p) > 0,$

(*ii*) $(\varepsilon + 3p) > 0,$

which require

$$R > \max\left(S/2, \frac{2\Lambda+S}{3}\right),$$

and

$$\Lambda < (3R-S)/2, \tag{29}$$

where

$$R = \tfrac{1}{6}(10PT+9l^2)+\frac{PK(2PT+l^2)}{Q}+16\pi\varrho T, \tag{30}$$

and

$$S = l^2+\frac{K^2(2PT+l^2)^2}{Q^2\psi(T)}-\frac{5T^2}{\psi(T)}. \tag{31}$$

Following the method outlined by Tolman [10] the red shift in the model (22) is given by

$$\frac{\lambda+\delta\lambda}{\lambda} = \frac{(2PT_2+l^2)\,[\psi(T_2)]^{1/4}\left[\exp\left\{\int\frac{2KdT_1}{Q\psi(T_1)}\right\}+(2PT_1+l^2)\,(\psi(T_1))^{1/4}U_Z\right]}{(2PT_1+l^2)\,[\psi(T_1)]^{1/4}\,[1-(2PT+l^2)^2U^2]^{1/2}\exp\left\{\int\frac{2KdT_2}{Q\psi(T_2)}\right]}, \tag{32}$$

where U is the velocity of the source at the time of emission and U_Z is the Z-component of the velocity.

The expressions for expansion θ, rotation ω and shear σ_{ij} calculated for the flow vector v^i are given by

$$\theta = \frac{2T}{\psi(T)},$$

$$\omega = 0,$$

and

$$\sigma_{11} = T/3,$$

$$\sigma_{22} = -\frac{\beta}{6[\psi(T)]^{1/2}}\left[T-\frac{3K(2PT+l^2)}{Q}\right],$$

$$\sigma_{33} = -\frac{1}{6\beta[\psi(T)]^{1/2}}\left[T+\frac{3K(2PT+l^2)}{Q}\right], \tag{33}$$

the other components of the shear tensor σ_{ij} being zero. The non-vanishing components of the conformal curvature tensor are

$$C^{34}_{34} = C^{12}_{12} = -\frac{1}{\psi(T)}\left[\tfrac{1}{2}(2PT+l^2)+\frac{T^2}{2\psi(T)}+\frac{3PK(2PT+l^2)}{Q}\right.$$
$$\left.+\frac{3KT(2PT+l^2)}{Q\psi(T)}+\frac{K^2(2PT+l^2)^2}{2Q^2\psi(T)}\right],$$

$$C^{24}_{24} = C^{13}_{13} = -\frac{1}{\psi(T)}\left[\frac{1}{2}(2PT+l^2)+\frac{T^2}{2\psi(T)}-\frac{3PK(2PT+l^2)}{Q}\right.$$
$$\left.-\frac{3KT(2PT+l^2)}{Q\psi(T)}+\frac{K^2(2PT+l^2)^2}{2Q^2\psi(T)}\right],$$
$$C^{14}_{14} = C^{23}_{23} = \frac{1}{\psi(T)}\left[(2PT+l^2)+\frac{T^2}{\psi(T)}+\frac{K^2(2PT+l^2)^2}{Q^2\psi(T)}\right]. \quad (34)$$

Obviously the space-time (22) is non-degenerate Petrov type I and becomes degenerate Petrov type I when K is zero. Thus, the metric (22) represents an expanding, shearing but non-rotating, non-degenerate Petrov type I magnetoviscous fluid cosmological model which tends to type D for large values of time in the absence of viscosity.

The distribution in the model (25) are given by

$$8\pi p_0 = \frac{1}{T^2}\left[\frac{5}{4T^2}-\frac{\gamma^2}{4T^2}\right]-\Lambda, \quad (35)$$

and

$$8\pi\varepsilon_0 = \frac{1}{T^2}\left[\frac{5}{4T^2}-\frac{\gamma^2}{4T^2}\right]+\Lambda. \quad (36)$$

The reality conditions require that

$$\gamma^2 < \min\left[5, (5-2\Lambda T^4)\right],$$

and

$$\Lambda < (5-\gamma^2)/2T^4. \quad (37)$$

The red shift in the model (25) is given by

$$\frac{\lambda+\delta\lambda}{\lambda} = \frac{T\left[T_1^{\left(\frac{1+\gamma}{2}\right)}+U_Z\right]}{T_2^{\left(\frac{1+\gamma}{2}\right)}\left[T^2-U^2\right]^{1/2}}. \quad (38)$$

The non-vanishing components of the Weyl's conformal curvature tensor are given by

$$C^{34}_{34} = C^{12}_{12} = -\frac{1}{6T^2}\left[\frac{1}{2T^2}+\frac{\gamma^2}{2T^2}+\frac{3\gamma}{T^2}\right],$$
$$C^{24}_{24} = C^{13}_{13} = -\frac{1}{6T^2}\left[\frac{1}{2T^2}+\frac{\gamma^2}{2T^2}-\frac{3\gamma}{T^2}\right],$$
$$C^{14}_{14} = C^{23}_{23} = \frac{1}{6T^2}\left[\frac{\gamma^2}{T^2}+\frac{1}{T^2}\right], \quad (39)$$

showing that the model (25) is in general non-degenerate Petrov type I and becomes degenerate when $\gamma = 0$.

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