HEXADECAPOLE NUCLEAR POTENTIAL FOR NON-AXIAL SHAPES*

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Parametrization of the hexadecapole deformation is generalized to non-axial shapes. It is given in a form directly applicable to a construction of the Woods-Saxon and the Nilsson single-particle potentials corresponding to such shapes. The Nilsson potential is given explicitly.

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1. Introduction

Up to now, mainly axially symmetric shapes of nuclei have been considered. This is because of the fact that well deformed nuclei are axially symmetric in their ground state and a cosideration of such deformations is sufficient for description of most of the properties of these nuclei.

However, a consideration of dynamic properties of any nucleus and of any property of transitional nuclei which are soft to various kinds of deformation, in particular to non-axial deformations (cf. e.g. [1]), requires a consideration of non-axial shapes. Such shapes are also significant in a study of fission (cf. [2]).

Up to now, the non-axial shapes have only been investigated for the quadrupole component of the deformation [3] and only this component has been taken into account in the study of collective states [4, 5].

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The aim of the present paper is to generalize the hexadecapole deformation (which is, in importance, the next after the quadrupole deformation) to non-axial shapes. There was an earlier attempt of such generalization [6]. However, only very particular case was considered and, besides, the description of the shapes did not satisfy the proper invariance conditions, as discussed below. Particular case of the hexadecapole tensor, constructed as a product of two quadrupole tensors, was studied in [7]. The non-axial hexadecapole shapes described by the tensor were then applied [8] to an explanation of enormously large Q_4 values in the region of heavy Hf, W and Os isotopes.

2. *Description of the non-axial quadrupole and hexadecapole deformations*

In description of a surface (of constant density of a nucleus or equipotential surface), it is important and convenient to find such parametrization for which there is one-to-one correspondence between the surface and the values of the parameters. In particular, such parametrization implies independence of the description of the choice of the coordinate system. In a general non-axial case, a parametrization of this kind was given up to now only for the quadrupole deformation [3].

As the hexadecapole deformation is always added to the quadrupole deformation and as we parametrize it in a similar way as the quadrupole deformation, we consider here both deformations together.

Usual, explicitly invariant description of a surface is an expansion of the radius $R(\theta, \phi)$, in the laboratory coordinate system, in spherical harmonics

$$R(\theta, \phi) = R_0[1+\sum_{\lambda\mu} \alpha_{\lambda\mu} Y_{\lambda\mu}(\theta, \phi)], \tag{1}$$

where $\alpha_{\lambda\mu}$ are components of the spherical tensor of rank λ. In the intrinsic system, the expansion becomes

$$R(\vartheta, \varphi) = R_0[1+\sum_{\lambda\mu} a_{\lambda\mu} Y_{\lambda\mu}(\vartheta, \varphi)]. \tag{2}$$

We define the intrinsic system as that of the quadrupole part ($\lambda = 2$) of the surface (1) and restrict ourselves to the surfaces which are symmetric with respect to the reflections in all the planes of this system.

The requirement of the reflection symmetry with respect to the (y, z)-plane leads to

$$a_{\lambda\mu} = a_{\lambda-\mu}. \tag{3}$$

The demand of the symmetry with respect to the (x, z)-plane, combined with the result (3), implies

$$a_{\lambda\mu} = 0 \quad \text{for odd } \mu, \tag{4}$$

and the symmetry with respect to the (x, y)-plane, combined with Eq. (4), gives

$$a_{\lambda\mu} = 0 \quad \text{for odd } \lambda. \tag{5}$$

Restricting to the quadrupole ($\lambda = 2$) and hexadecapole ($\lambda = 4$) shapes only, the surface is

$$R(\vartheta, \varphi) = R_0\{1+[a_{20}Y_{20}+a_{22}(Y_{22}+Y_{2-2})]$$
$$+[a_{40}Y_{40}+a_{42}(Y_{42}+Y_{4-2})+a_{44}(Y_{44}+Y_{4-4})]\}. \tag{6}$$

Thus, the quadrupole part has two free parameters and the hexadecapole part three. The form (6) is completely general for the quadrupole part as it fulfils the symmetry conditions imposed by us in a general case and it should just have $5-3 = 2$ free intrinsic parameters; the hexadecapole part is, however, restricted by these conditions, as in a general case it should have $9-3 = 6$ parameters.

The parameters $a_{\lambda\mu}$ are not uniquely determined by the surface. They also depend on the designation of the intrinsic axes: x, y, z and on the choice of positive direction for them. There exist 24 different possibilities for designations and directions of the axes (if we restrict ourselves to only e.g. right-hand systems). Each of them may be obtained from another by a superposition of the three basic rotations R_i ($i = 1, 2, 3$), well known in the literature [3, 9] ($R_1 = R(\pi, \pi, 0)$, $R_2 = R(0, 0, \pi/2)$, $R_3 = R(0, \pi/2, \pi/2)$, where the arguments are the Euler angles).

The rotations R_i result in the following transformations of $a_{2\mu}$ and $a_{4\mu}$:

$$R_1: \quad a'_{2\mu} = a_{2\mu}, \qquad a'_{4\mu} = a_{4\mu}, \tag{7}$$

$$R_2: \quad a'_{20} = a_{20}, \qquad a'_{22} = -a_{22}, \tag{8a}$$

$$a'_{40} = a_{40}, \qquad a'_{42} = -a_{42}, \qquad a'_{44} = a_{44}, \tag{8b}$$

$$R_3: \quad a'_{20} = -\tfrac{1}{2}a_{20}+\sqrt{\tfrac{3}{2}}\,a_{22}, \qquad a'_{22} = -\tfrac{1}{2}\sqrt{\tfrac{3}{2}}\,a_{20}-\tfrac{1}{2}a_{22}, \tag{9a}$$

$$a'_{40} = \tfrac{3}{8}a_{40}-\frac{\sqrt{10}}{4}a_{42}+\frac{\sqrt{70}}{8}a_{44},$$

$$a'_{42} = \frac{\sqrt{10}}{8}a_{40}-\tfrac{1}{2}a_{42}-\frac{\sqrt{7}}{4}a_{44},$$

$$a'_{44} = \frac{\sqrt{70}}{16}a_{40}+\frac{\sqrt{7}}{4}a_{42}+\tfrac{1}{8}a_{44}. \tag{9b}$$

It is convenient to express $a_{2\mu}$ and $a_{4\mu}$ in terms of parameters which have simpler transformation rules than those of Eqs. (7)–(9).

2.1. Quadrupole deformation

One introduces [3] the parameters β, γ

$$a_{20} = \beta\cos\gamma, \qquad \sqrt{2}\,a_{22} = \beta\sin\gamma, \tag{10}$$

where $\beta \geqslant 0$ and $-\pi < \gamma \leqslant \pi$. The relation (10) may be interpreted as the transformation from the rectangular a_{20}, a_{22} to the polar β, γ coordinates.

Due to the relation

$$\beta^2 = a_{20}^2+2a_{22}^2 = \sum_{\mu} \alpha_{2\mu}\alpha_{2\mu}^*, \tag{11}$$

the parameter β is an invariant of all rotations of the coordinate system, in particular of the rotations R_i, and thus is uniquely determined by the surface.

The parameter γ transforms in the following, rather simple, way under the rotations R_i

$$R_1: \ \gamma' = \gamma, \quad R_2: \ \gamma' = -\gamma, \quad R_3: \ \gamma' = \gamma - 2\pi/3. \tag{12}$$

Due to this, to get uniqueness in the determination of γ by the surface, it is sufficient to restrict the variation of γ to the region $0 \leqslant \gamma \leqslant \pi/3$.

One may also mention that $\gamma = 0$ ($a_{22} = 0$) corresponds to the shapes which are symmetric with respect to the z-axis and are prolate, while $\gamma = \pi/3$ ($a_{22} = \sqrt{3/2}a_{20}$) corresponds to the shapes which are symmetric with respect to the y-axis and are oblate.

2.2. Hexadecapole deformation

Let us notice that the quantities

$$b_4 \equiv \sqrt{5/12}\, a_{40} - \sqrt{7/6}\, a_{44}, \qquad c_4 \equiv -\sqrt{2}\, a_{42} \tag{13}$$

have the same transformation rules under R_i as the coordinates a_{20} and a_{22}. Due to this, we can parametrize them in the same way, i.e.

$$b_4 = \varrho_4 \cos \gamma_4, \qquad c_4 = \varrho_4 \sin \gamma_4, \tag{14}$$

where ϱ_4 ($\varrho_4 \geqslant 0$) is invariant under R_i and γ_4 ($-\pi < \gamma_4 \leqslant \pi$) has the same transformation rules as γ, i.e.

$$R_1: \ \gamma_4' = \gamma_4, \quad R_2: \ \gamma_4' = -\gamma_4, \quad R_3: \ \gamma_4' = \gamma_4 - 2\pi/3. \tag{12a}$$

Let us further notice that the quantity

$$a_4 \equiv \sqrt{7/12}\, a_{40} + \sqrt{5/6}\, a_{44} \tag{15}$$

is invariant under R_i and that we have

$$a_4^2+b_4^2+c_4^2 = a_{40}^2+2a_{42}^2+2a_{44}^2 = \sum_{\mu} \alpha_{4\mu}\alpha_{4\mu}^* \equiv \beta_4^2. \tag{16}$$

Thus, the parameter β_4 defined in Eq. (16) and being a measure of the total hexadecapole deformation is invariant under all rotations. Eqs. (14) and (16) allow us to treat β_4, δ_4 ($\varrho_4 \equiv \beta_4 \sin \delta_4$) and γ_4 as the spherical coordinates of a point specified by the rectangular coordinates a_4, b_4 and c_4, i.e.

$$a_4 = \beta_4 \cos \delta_4, \qquad b_4 = \beta_4 \sin \delta_4 \cos \gamma_4, \qquad c_4 = \beta_4 \sin \delta_4 \sin \gamma_4, \tag{17}$$

where $0 \leqslant \delta_4 \leqslant \pi$. One can see from Eqs. (13), (15) and (17) that there exists one-to-one correspondence between the coordinates

$$a_{40} = \beta_4\left(\sqrt{\tfrac{7}{12}} \cos \delta_4 + \sqrt{\tfrac{5}{12}} \sin \delta_4 \cos \gamma_4\right),$$

$$\sqrt{2}\, a_{42} = -\beta_4 \sin\delta_4 \sin\gamma_4,$$

$$\sqrt{2}\, a_{44} = \beta_4(\sqrt{\tfrac{5}{12}} \cos\delta_4 - \sqrt{\tfrac{7}{12}} \sin\delta_4 \cos\gamma_4) \tag{18}$$

and the parameters (β_4, δ_4, γ_4). According to the transformation rules (12a), to get one-to-one correspondence between a surface and the describing it parameters ($\beta_4, \delta_4, \gamma_4$), it is sufficient to restrict the region of variation of γ_4 to $0 \leqslant \gamma_4 \leqslant \pi/3$. All this indicates a significant similarity between the parametrization ($\beta_4, \delta_4, \gamma_4$) of the hexadecapole and the parametrization (β, γ) of the quadrupole shapes. The total deformation parameter β_4 is a close analogue of β and the non-axiality parameters δ_4 and γ_4 are rather natural generalization of γ. They commonly describe two kinds of non-axiality: of the "quadrupole" type ($\cos 2\varphi$ or $\cos 2\vartheta$ terms appearing in the quadrupole or higher multipolarity shapes) and of the "hexadecapole" type ($\cos 4\varphi$ or $\cos 4\vartheta$ terms appearing in the hexadecapole or higher multipolarity shapes).

The axial symmetry of the hexadecapole shape with respect to the z-axis ($a_{42} = a_{44} = 0$) is obtained for

$$\gamma_4 = 0, \qquad \delta_4 = \delta_4^0, \tag{19}$$

where $\cos\delta_4^0 = \sqrt{7/12}$, and with respect to the y-axis ($a_{42} = (\sqrt{10}/3)a_{40}$, $a_{44} = (\sqrt{70}/6)a_{40}$) for

$$\gamma_4 = \pi/3, \qquad \delta_4 = \pi - \delta_4^0. \tag{20}$$

In Ref. [6], the components a_{40}, a_{42}, a_{44} were assumed to be simple functions of $\cos 3\gamma$. Thus, instead of being components of a tensor, they were scalars with respect to the rotations R_i. Consequently, the surface described by them rotated rigidly together with the intrinsic coordinate system, under the rotations R_i.

3. Construction of the single-particle potential

3.1. Woods-Saxon potential

Inserting Eq. (18) to Eq. (6), we express the radius $R(\vartheta, \varphi)$ of Eq. (6) in terms of the parameters $\beta_4, \delta_4, \gamma_4$. The application of this expression to the generalization of the Woods-Saxon potential to the shapes described by it is presently a rather standard procedure (e.g. [10–12]). Some care should be taken here for the desired behaviour of the surface thickness (to have it e.g. constant) over the surface.

It is easy to see that our expression for $R(\vartheta, \varphi)$ has the proper, standard form

$$R(\vartheta, \varphi) = R_0(1 + \beta Y_{20} + \beta_4 Y_{40}) \tag{21}$$

in the limit of axial symmetry around the z-axis.

3.2. Nilsson potential

For the Nilsson potential, a generalization to the shapes parametrized according to Eqs. (10) and (18) is direct. The potential (more exactly, its central part, as the spin-orbit and correction parts are usually assumed to be independent of deformation and thus are not interesting for the present considerations) is

$$V(\varepsilon, \gamma, \varepsilon_4, \delta_4, \gamma_4) = \tfrac{1}{2}\hbar\omega_0\varrho^2$$

$$\times \left\{1-\tfrac{2}{3}\varepsilon\sqrt{\frac{4\pi}{5}}\left[\cos\gamma Y_{20}+\frac{\sin\gamma}{\sqrt{2}}(Y_{22}+Y_{2-2})\right]\right.$$

$$+2\varepsilon_4\sqrt{\frac{4\pi}{9}}\left[(\sqrt{\tfrac{7}{12}}\cos\delta_4+\sqrt{\tfrac{5}{12}}\sin\delta_4\cos\gamma_4)Y_{40}-\frac{1}{\sqrt{2}}\sin\delta_4\sin\gamma_4(Y_{42}+Y_{4-2})\right.$$

$$\left.\left.+(\sqrt{\tfrac{5}{24}}\cos\delta_4-\sqrt{\tfrac{7}{24}}\sin\delta_4\cos\gamma_4)(Y_{44}+Y_{4-4})\right]\right\}, \tag{22}$$

where the radius ϱ and the angles in the arguments of $Y_{\lambda\mu}$ are in the stretched coordinate system [13], characteristic for the Nilsson potential.

To get the potential (22), we have normalized both quadrupole and hexadecapole parts of it in such a way as to obtain the correspondence with the limit case $\gamma_4 = 0, \delta_4 = \delta_4^0$, used up to now [13, 14].

The quadrupole part of the potential (22) coincides with the potential used by Larsson [14] with the only difference in sign of the $(Y_{22}+Y_{2-2})$-term. The difference comes from the fact that we follow the original definition of γ, Eq. (10), given by Bohr [3], while γ of Larsson has the opposite sign.

4. *Reduction of the number of parameters*

We are always interested in a reduction of the number of parameters. Usually, some of the free parameters are eliminated by minimization of the potential energy with respect to them. Without that, we may also obtain the reduction, in a preliminary way, relating the hexadecapole shapes with the quadrupole ones. For example, a simple choice

$$\gamma_4 = \gamma, \quad \cos\delta_4 = \sqrt{7/12}\cos 3\gamma \tag{23}$$

ensures a simultaneous axial symmetry of both shapes with respect to the z-axis ($\gamma = 0$) as well as to the y-axis ($\gamma = \pi/3$). With this choice, we definitely get more realistic shapes than those obtained with the axially symmetric hexadecapole shapes, discussed up to now, while having the same number of parameters (three parameters: $\varepsilon, \gamma, \varepsilon_4$). This partly illustrates the profit of the described generalization.

More details on the presented parametrization, discussion of its particular cases, illustrations of the involved shapes as well as the results of the study of the energy surface corresponding to these shapes will be given elsewhere [15].

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