EINSTEIN-CARTAN-MAXWELL-BIANCHI COSMOLOGIES

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The Einstein-Cartan-Maxwell equations are studied for Bianchi types I, VII_0, VIII and IX cosmological models. A non-singular solution of Bianchi type I is given. It is shown that electromagnetic solutions of Bianchi types VIII and IX exist.

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1. Introduction

The Einstein-Cartan theory (ECT) of gravitation provides a specific spin-spin interaction of matter which opposes the usual gravitational attraction and thus may avert the singularities that characterize the theory of general relativity (GRT). Cosmological models constructed on the basis of ECT (Kopczyński 1972, 1973; Tafel 1973, 1975; Stewart, Hájiček 1973; Kuchowicz 1975a–e, 1976a–d, 1978; Raychaudhuri 1975, 1979; Kerlick 1975, 1976; Tsoubelis 1979a, 1981; Lorenz 1981c) have given theoretical support to this so-called Trautman conjecture (Trautman 1973b–d). In most of these models, the spin of the cosmic fluid is assumed to be aligned along a particular direction (see Kuchowicz 1976a).

It has been suggested that the alignment might be brought about by a high-enough magnetic field in the early phase of the universe (Trautman 1973c, d). The idea of a universe with a homogeneous primordial magnetic field was proved to be very successful in flat GRT-Bianchi type I models (Zel'dovich and Novikov (1975)). However, since Bianchi type I models are a very special subset of spatially homogeneous models, one should consider more general situations, in order to check what implications large-scale primordial magnetic fields would have on the dynamics of the Universe. The most general sets of homogeneous models are Bianchi types VI, VII, VIII and IX (Collins and Hawking (1973)). However, the basic work of Hughston and Jacobs (1970) (see also Jacobs (1977) and Tsoubelis (1979b)) has shown that the existence of a homogeneous primordial magnetic

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field in GRT-models is limited to Bianchi types I, II, III, VI ($h = -1$), or VII ($h = 0$). These results also hold for pure electric fields.

Since the Maxwell field does not couple to torsion (see, however, Novello (1976), De Sabbata and Gasperini (1980)) the sourceless Maxwell equations are the same in ECT and GRT (Hehl (1974), Prasanna (1975)). Thus one is forced to consider models with both a magnetic and an electric field. In this paper we solve the Einstein-Cartan-Maxwell equations for Bianchi type I, VII_0, VIII and IX models. We investigate universes containing electromagnetic fields obeying the sourceless Maxwell equations and matter, with a "stiff" equation of state. We restrict ourselves to the "classical description of spin" based on the special relativistic treatment of the intrinsic spin angular momentum.

For the most part we use Cartan's calculus of differential forms. The notation and conventions of Trautman (1972a–c, 1973a–d) are utilized.

2. *Derivation of the curvature*

In choosing a local orthonormal basis σ^μ, we can put the metric of space-time in the form

$$ds^2 = \eta_{\mu\nu}\sigma^\mu\sigma^\nu, \tag{1}$$

where $\eta_{\mu\nu}$ is the Minkowski metric tensor. For a spatially homogeneous model, we take

$$\sigma^0 = \theta^0 = dt, \qquad \sigma^i = R_i\theta^i \text{ (no sum)}, \tag{2}$$

where θ^μ are the time-independent differential one-forms and where, due to homogeneity, the R_i are functions of t only. (Here and henceforth Latin indices assume the values 1, 2, 3 whereas Greek indices will assume the values 0, 1, 2, 3.)

The one-forms σ^μ, θ^μ obey the relations

$$d\sigma^\mu = -\gamma^\mu{}_{\lambda\nu}\sigma^\nu \wedge \sigma^\lambda + \tfrac{1}{2}\, Q^\mu{}_{\alpha\beta}\sigma^\alpha \wedge \sigma^\beta, \tag{3a}$$

$$d\theta^0 = 0, \qquad d\theta^i = -\tfrac{1}{2}\, C_{kl}{}^i\theta^k \wedge \theta^l \tag{3b}$$

where the $\gamma^\mu{}_{\lambda\nu}$ are the connections coefficients, $Q^\mu{}_{\alpha\beta}$ are the components of the torsion tensor, $C_{kl}{}^i$ are the structure constants, $\wedge$ denotes the exterior product and d is the exterior derivative operator. The structure constants for the Bianchi types I, VII_0, VIII and IX can be written as

$$C_{ik}{}^l = -\varepsilon_{ikl}n_l, \tag{4}$$

where ε_{ikl} is the totally antisymmetric Levi-Civita pseudotensor and

$$\begin{aligned} &n_l = 0, \quad \forall_l && \text{type 1}\\ &n_1 = n_2 = 1,\ n_3 = 0, && \text{type VII}_0\\ &n_1 = n_2 = -n_3 = 1, && \text{type VIII}\\ &n_1 = n_2 = n_3 = 1, && \text{type IX} \end{aligned} \tag{5}$$

The exterior derivatives of the orthonormal basis one-forms σ^μ are readily found by use of Eq. (2) and substitution of Eq. (3)

$$d\sigma^0 = 0, \tag{6a}$$

$$d\sigma^i = H_i\sigma^0 \wedge \sigma^i + \tfrac{1}{2}\,\varepsilon_{ikl}n_l(R_i/R_kR_l)\sigma^k \wedge \sigma^l, \tag{6b}$$

where $H_i := \dot{R}_i/R_i$ are the Hubble parameters. (A dot denotes differentiation with respect to time.)

Assuming a classical description of spin (see below) and that the spins of the individual particles composing the fluid are all aligned along the σ^3 direction, one finds from the spin-torsion equation (17) that the only nonvanishing components of the torsion tensor are

$$Q^0{}_{12} = -Q^0{}_{21} =: 2s = 2s(t). \tag{7}$$

Comparison of Eq. (6) with the relationships (3) provides immediately the connection coefficients

$$\begin{aligned} \gamma_{012} &= \gamma_{201} = \gamma_{210} = s, \\ \gamma_{i0i} &= H_i, \\ \gamma_{ikl} &= \tfrac{1}{2}\,\varepsilon_{ikl}\left(\frac{n_iR_i}{R_kR_l} + \frac{n_kR_k}{R_iR_l} - \frac{n_lR_l}{R_iR_k}\right). \end{aligned} \tag{8}$$

These quantities enter into the formula

$$\sigma^\mu{}_\nu = \gamma^\mu{}_{\nu\lambda}\sigma^\lambda \tag{9}$$

to provide six connection two-forms $\sigma_{\mu\nu}$ (to lower or raise an index use $\eta_{\mu\nu}$). The results are

$$\begin{aligned} \sigma_{0i} &= \varepsilon_{ik3}s\sigma^k - H_i\sigma^i, \\ \sigma_{ik} &= -\varepsilon_{ik3}s\sigma^0 + \tfrac{1}{2}\,\varepsilon_{ikl}\left(\frac{n_iR_i}{R_kR_l} + \frac{n_kR_k}{R_iR_l} - \frac{n_lR_l}{R_iR_k}\right)\sigma^l. \end{aligned} \tag{10}$$

Equation (10) implies now

$$\begin{aligned} d\sigma_{0i} &= \varepsilon_{ik3}(\dot{s}+sH_k)\sigma^0 \wedge \sigma^k - (\dot{H}_i+H_i^2)\sigma^0 \wedge \sigma^i \\ &\quad + \varepsilon_{nmk}\varepsilon_{ik3}\,\frac{sn_kR_k}{R_nR_m}\,\sigma^n \wedge \sigma^m - \varepsilon_{lmi}\,\frac{n_iR_iH_i}{R_lR_m}\,\sigma^l \wedge \sigma^m, \\ d\sigma_{ik} &= \tfrac{1}{2}\,\varepsilon_{ikl}\left(\left(\frac{n_iR_i}{R_kR_l} - \frac{n_kR_k}{R_iR_l}\right)(H_i-H_k) + \frac{n_lR_l}{R_iR_k}(H_i+H_k-2H_l)\right)\sigma^0 \wedge \sigma^l \\ &\quad + \tfrac{1}{2}\,\varepsilon_{ikl}n_l\left(\frac{n_i}{R_k^2} + \frac{n_k}{R_i^2} - n_l\left(\frac{R_l}{R_iR_k}\right)^2\right)\sigma^i \wedge \sigma^k. \end{aligned} \tag{11}$$

The torsion and curvature two-forms are respectively given by

$$Q^{\mu} = d\sigma^{\mu} + \sigma^{\mu}{}_{\nu} \wedge \sigma^{\nu}, \tag{12a}$$

$$\Omega^{\mu}{}_{\nu} = d\sigma^{\mu}{}_{\nu} + \sigma^{\mu}{}_{\lambda} \wedge \sigma^{\lambda}{}_{\nu} \tag{12b}$$

and can be readily computed by use of Eqs (6), (11) and the compatibility equation

$$\sigma_{\mu\nu} + \sigma_{\nu\mu} = 0. \tag{13}$$

Out of this calculation, one reads the individual components of the torsion and the curvature tensors by using the Cartan equations

$$Q^{\mu} = \tfrac{1}{2}\, Q^{\mu}{}_{\lambda\nu}\sigma^{\lambda} \wedge \sigma^{\nu}, \tag{13a}$$

$$\Omega^{\mu}{}_{\nu} = \tfrac{1}{2}\, R^{\mu}{}_{\nu\lambda\kappa}\sigma^{\lambda} \wedge \sigma^{\kappa} \tag{13b}$$

as identification schemes. The results are

$$\begin{aligned}
Q^{0}{}_{12} &= 2s,\\
R^{0}{}_{i0i} &= \dot{H}_i + H_i^2 - s^2\varepsilon_{ik3}^2,\\
R^{0}{}_{i0k} &= -\varepsilon_{ik3}(\dot{s} + 2sH_k),\\
R^{0}{}_{ilm} &= -\tfrac{1}{2}\,\varepsilon_{ilm}\varepsilon_{il3}s\left(\frac{n_iR_i}{R_cR_m} - \frac{n_lR_l}{R_iR_m} - \frac{n_mR_m}{R_iR_l}\right),\\
R^{0}{}_{ilm} &= -\tfrac{1}{2}\,\varepsilon_{ilm}\left(\frac{n_iR_i}{R_lR_m}(H_l + H_m - 2H_i) + \left(\frac{n_lR_l}{R_iR_m} - \frac{n_mR_m}{R_iR_l}\right)(H_l - H_m)\right),\\
R^{i}{}_{kik} &= \tfrac{1}{2}\,\varepsilon_{ikl}\left(\frac{n_in_l}{R_k^2} + \frac{n_ln_k}{R_i^2} - \frac{n_in_k}{R_l^2}\right) + H_iH_k + s^2\varepsilon_{ik3}\\
&\quad + \tfrac{1}{4}\,\varepsilon_{ikl}\left(\left(\frac{n_iR_i}{R_kR_l}\right)^2 + \left(\frac{n_kR_k}{R_iR_l}\right)^2 - 3\left(\frac{n_lR_l}{R_iR_k}\right)^2\right),\\
R^{i}{}_{klk} &= -s\varepsilon_{il3}H_k.
\end{aligned} \tag{14}$$

Thus we can easily calculate the Ricci tensor $R^{\mu}{}_{\nu} = R^{\lambda\mu}{}_{\lambda\nu}$. The nonvanishing components are

$$\begin{aligned}
R_{00} &= -(3\dot{H} + H_1^2 + H_2^2 + H_3^2 - 2s^2),\\
R_{ii} &= \dot{H}_i + 3HH_i + \frac{1}{2(R_1R_2R_3)^2}((n_iR_i^2)^2 - (n_kR_k^2) - (n_lR_l^2)^2 + 2n_kn_l(R_kR_l)^2),\\
R_{12} &= -(\dot{s} + s(2H_2 + H_3)),\\
R_{21} &= \dot{s} + s(2H_1 + H_3),\\
R_{03} &= -\tfrac{1}{2}\,s\left(\frac{n_1R_1}{R_2R_3} - \frac{n_2R_2}{R_1R_3} - \frac{n_3R_3}{R_1R_2}\right),
\end{aligned} \tag{15}$$

where $H = \frac{1}{3}\sum_{i=1}^{3} H_i$ is the average Hubble parameter.

3. *Field equations*

The Einstein-Cartan equations considered here are

$$G_{\mu\nu} = t_{\mu\nu} + E_{\mu\nu}, \tag{16}$$

$$Q^{\mu}{}_{\nu\lambda} - \delta^{\mu}{}_{\nu} Q^{\kappa}{}_{\kappa\lambda} - \delta^{\mu}{}_{\lambda} Q^{\kappa}{}_{\nu\kappa} = s^{\mu}{}_{\nu\lambda}, \tag{17}$$

where

$$t_{\mu\nu} = (\varepsilon + p) u_{\mu} u_{\nu} + p \eta_{\mu\nu} + \nabla_{\kappa}(u^{\kappa} S_{\mu\lambda}) u^{\lambda} u_{\nu} \tag{18}$$

is the canonical energy-momentum tensor of matter,

$$E_{\mu\nu} = F_{\mu\alpha} F_{\nu}{}^{\alpha} - \tfrac{1}{4} \eta_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \tag{19}$$

is the electromagnetic stress-energy tensor, $s^{\mu}{}_{\nu\lambda}$ is the canonical spin angular momentum tensor,

$$\varepsilon = t_{\mu\nu} u^{\mu} u_{\nu} \tag{20}$$

is the energy density and p is the pressure of the so-called Weyssenhoff fluid (a perfect fluid with spin). The fluid is characterized by the equation of state

$$p = (\gamma - 1)\varepsilon, \quad 1 \leqslant \gamma \leqslant 2. \tag{21}$$

u^{μ} denotes the velocity four-vector and ∇_{μ} the covariant derivation operator.

In general, it is possible to study space-times with 24 independent components of the torsion tensor $Q^{\mu}{}_{\nu\lambda}$. In this case, however, a physical interpretation of the 24 components of the spin angular momentum tensor $s^{\mu}{}_{\nu\lambda}$ is rendered very difficult. It appears therefore to be sufficient if one uses the "classical description of spin", based on the special relativistic treatment of the intrinsic angular momentum (see e.g. Kuchowicz 1976a). In this description, the spin angular momentum tensor $s^{\mu}{}_{\nu\lambda}$ is decomposed as

$$s^{\mu}{}_{\nu\lambda} = u^{\mu} S_{\nu\lambda}, \tag{22}$$

where $S_{\nu\lambda}$ is the antisymmetric tensor of the density of spin, which obeys the orthogonality condition

$$u^{\mu} S_{\nu\mu} = 0. \tag{23}$$

In the local inertial frame determined by (1), an observer comoving with the fluid is assumed to have four-velocity $u^{\mu} = \delta^{\mu}{}_{0}$. Such models are called orthogonal universes (see e.g. Collins and Ellis (1979)), in which the matter moves orthogonally to the hypersurfaces of homogeneity. However, it must be noted that in general the fluid flow vector $\vec{u}$ is not normal to the hypersurfaces of homogeneity. These are the so-called tilted models (King and Ellis 1973). Exact solutions for tilted electromagnetic GRT-Bianchi type I, II, III and VI have been found by Dunn and Tupper (1978, 1980) and for type I and II by Lorenz (1981a). The incorporation of an electromagnetic field into a locally rotationally tilted GRT-Bianchi type V model has been given by us in a previously published paper (Lorenz 1981b). The corresponding non-tilted ECT-Bianchi type V model has been found recently (Lorenz 1981c). So far, no tilted ECT-Bianchi model has been constructed.

Assuming that the spins of the individual particles composing the fluid are all aligned along the σ^3 direction, one finds from the spin-torsion equation (17) that

$$Q^0{}_{12} = -Q^0{}_{21} = :2s = 2s(t). \tag{24}$$

The canonical energy-momentum tensor takes thus the simple form

$$t_{\mu\nu} = \mathrm{diag}\,(\varepsilon, p, p, p). \tag{25}$$

The field equations (16, 17) combined with the Bianchi identities

$$DQ^\mu = \Omega^\mu{}_\nu \wedge \sigma^\nu, \qquad D\Omega^\mu{}_\nu = 0 \tag{26}$$

where D denotes the exterior covariant derivation, give rise to the conservation laws for the spin and the energy density:

$$\nabla_\mu(2su^\mu) = 0, \tag{27a}$$

$$u^\mu\nabla_\mu\varepsilon+(\varepsilon+p)\nabla_\mu u^\mu = 0. \tag{27b}$$

From the covariant derivative formula

$$\nabla_\mu e_\nu = \gamma^\lambda{}_{\nu\mu}e_\lambda, \tag{28}$$

where e_ν is an arbitrary basis vector field, we obtain with the aid of Eq. (8) the following conservation equations

$$\dot{\varepsilon}+3(\varepsilon+p)H = 0, \tag{29a}$$

$$\dot{s}+3sH = 0. \tag{29b}$$

From (29a) and (29b) it follows that

$$\varepsilon = \frac{\varepsilon_0^2}{(R_1R_2R_3)^2}, \qquad \varepsilon_0^2 = \text{const.} \tag{30a}$$

and

$$s = \frac{s_0}{R_1R_2R_3}, \qquad s_0 = \text{const.} \tag{30b}$$

We now turn to the Maxwell equations. The source-free Maxwell equations are

$$dF = d{*F} = 0, \tag{31}$$

where the two-form F represents the electromagnetic field and $*F$ is its dual. In the basics $\sigma^\mu \wedge \sigma^\nu$ we have

$$F = E_i\sigma^i \wedge \sigma^0+\tfrac{1}{2}\, B_i\varepsilon_{ijk}\sigma^j \wedge \sigma^k, \tag{32a}$$

$$*F = -B_i\sigma^i \wedge \sigma^0+\tfrac{1}{2}\, E_i\varepsilon_{ijk}\sigma^j \wedge \sigma^k. \tag{32b}$$

Owing to homogeneity, the electric field E_i and the magnetic field B_i depend only on t. By using Eqs (2), (3a) and (4) the sourceless Maxwell equations (31) become

$$E_i R_i n_i - \partial_t (B_i R_j R_k) = 0, \tag{33a}$$

$$B_i R_i n_i + \partial_t (E_i R_j R_k) = 0, \tag{33b}$$

where $\partial_t := \frac{\partial}{\partial t}$. It is convenient to introduce the variables $dt_i := n_i(R_i/R_j R_k)dt$ for Bianchi types VII_0, VIII and IX. The general solutions of Eqs (33a, b) are

$$E_i = \frac{e_i}{R_j R_k}, \quad B_i = \frac{b_i}{R_j R_k}; \quad e_i, b_i = \text{const.}, \tag{34a}$$

where $i = 1, 2, 3$ for type I and $i = 3$ for type VII_0 and

$$E_i = \frac{a_i}{R_j R_k} \cos(t_i + \tau_i), \quad B_i = \frac{a_i}{R_j R_k} \sin(t_i + \tau_i), \quad a_i, \tau_i = \text{const.} \tag{34b}$$

where $i = 1, 2, 3$ for types VIII and IX and $i = 1, 2$ for type VII_0.

The reason why we have restricted ourselves to the simple Bianchi type VII_0 model instead of the more general type VII_h model is because in this case the Maxwell equations are difficult to solve (see Jacobs 1977). Solutions of a GRT-Bianchi type VII_h model with dust ($\gamma = 1$) and an electromagnetic field have been considered by Melvin (1975)[1] using the dyadic formalism of Estabrook and Wahlquist (1964, 1968).

From Eq. (34) we can immediately calculate the corresponding components of the electromagnetic stress energy tensor $E_{\mu\nu}$. Using the expressions for the Ricci tensor components the field equations take the final form

$$3\dot{H} + H_1^2 + H_2^2 + H_3^2 - 2s^2 = -\frac{\varepsilon}{2}(3\gamma - 2) - E_{00}, \tag{a}$$

$$\dot{H}_i + H_c^2 + \frac{1}{2(R_1 R_2 R_3)^2}((n_i R_i^2)^2 - (n_k R_k^2)^2 - (n_l R_l^2)^2 + 2n_k n_l (R_k R_l)^2) = \frac{\varepsilon}{2}(2 - \gamma) + E_{ii}, \tag{b}$$

$$-(\dot{s} + s(2H_2 + H_3)) = E_{12}, \tag{c}$$

$$\dot{s} + s(2H_1 + H_3) = E_{21}, \tag{d}$$

$$-\frac{s}{2}\left(\frac{n_1 R_1}{R_2 R_3} - \frac{n_2 R_2}{R_1 R_3} - \frac{n_3 R_3}{R_1 R_2}\right) = E_{03}. \tag{e)(35}$$

[1] Dr. M. A. H. MacCallum kindly points out to me that Melvin's solution, although he has found it as a type-VII_h solution, is better referred to as a type-V solution because the solution is rotationally symmetric (see also MacCallum 1979).

4. *Bianchi type I and type VII$_0$ models*

We consider first the Bianchi type I ($n_i = 0$) and type VII$_0$ ($n_3 = 0$) models. From Eq. (35e) it follows that $E_{03} = 0$ for type I. However, for the Bianchi type VII$_0$ model we have

$$-\frac{s}{R_1R_2R_3}(R_1^2-R_2^2) = E_{03}. \tag{36}$$

(This relation has been overlooked by Tsoubelis (1979a, 1981), who considered the Bianchi types I, II, III, VI$_0$ and VII$_0$, setting $E_{03} = 0$.) Assuming $R_1 = R_2$ we find from the field equations (35d, e) and the conservation equation (30b) that $E_{12} = E_{21} = 0$. The Ricci tensor turns out to be diagonal ($R_{\mu\nu} = 0$ for $\mu \neq \nu$) and the electromagnetic stress tensor must be diagonal too. The off-diagonal components are

$$\begin{aligned} F_{02}F_{12}+F_{03}F_{13} &= 0, \quad & F_{01}F_{12}-F_{03}F_{23} &= 0,\\ F_{01}F_{13}+F_{02}F_{23} &= 0, \quad & F_{01}F_{02}-F_{13}F_{23} &= 0,\\ F_{01}F_{03}+F_{12}F_{23} &= 0, \quad & F_{02}F_{03}-F_{12}F_{13} &= 0, \end{aligned} \tag{37}$$

which leads to three possible cases:

$$\begin{aligned} &(i) \quad F_{02} = F_{03} = F_{12} = F_{13} = 0, \quad F_{01}, F_{23} \neq 0,\\ &(ii) \quad F_{01} = F_{03} = F_{12} = F_{23} = 0, \quad F_{02}, F_{13} \neq 0,\\ &(iii) \quad F_{01} = F_{02} = F_{13} = F_{23} = 0, \quad F_{03}, F_{12} \neq 0. \end{aligned} \tag{38}$$

Without loss of generality, we may consider only case (*iii*). We note that the electric and the magnetic fields must be parallel and point in the direction of the σ^3-axis. The nonvanishing components of the electromagnetic stress-energy tensor are

$$E_{00} = E_{11} = E_{22} = -E_{33} = \tfrac{1}{2}(E_3^2+B_3^2) = \frac{a^2}{R^4}, \tag{39}$$

where $a^2 := \frac{1}{2}(e_3^2+b_3^2)$ and $R := R_1 = R_2$.

From Eq. (35b) it can be seen that in the case $R_1 = R_2$ we have

$$R_{ii} = \dot{H}_i+3HH_i \tag{40}$$

so that the field equations of type VII$_0$ are the same as for type I. We take the stiff ($\gamma = 2$) equation of matter. The possible relevance of the equation of state $p = \varepsilon$ as regards the matter content of the universe in its early stages has been discussed by a number of authors, since is was first proposed by Zel'dovich (1961, 1970). We refer to the recent paper of Barrow (1978).

It is convenient to solve the equation $R_{11}+R_{22} = 0$. This can be expressed in the following form

$$\frac{1}{R^2 S}(R(RS)^{\cdot})^{\cdot} = 0, \qquad S := R_3. \tag{41}$$

Introducing a new time variable τ by $dt = R^2 S d\tau$ we obtain

$$(\ln(RS))'' = 0, \quad (\ )' := \frac{\partial}{\partial\tau} \tag{42}$$

with the general solution

$$RS = \exp(\tilde{k}\eta), \ \eta = \tau + \tau_0, \ \tilde{k}\tau_0 = \text{const.} \tag{43}$$

Substitution of (43) into the linear combination of (35a, b) gives

$$(\ln S^2)'^2 = 4(k^2 - a^2 S^2), \quad k^2 := \tilde{k}^2 - \varepsilon_0^2 + s_0^2 \tag{44}$$

with the general solution

$$S^2 = \frac{2k^2}{a^2(1+\cosh 2(k+q))}, \quad 2q := \ln a^2 \tag{45}$$

We note that the solutions (43), (45) are the same as in the corresponding locally rotationally symmetric GRT-Bianchi type I model (see Lorenz 1980a). Eq. (45) can be put into the equivalent form

$$S = \frac{2ke^{-k\eta}}{e^{-2k\eta} + a^2}. \tag{46}$$

We now examine the behaviour of this model in some more detail. With $V := R^2 S$ we have

$$V = \frac{1}{2k}(e^{(2\tilde{k}-k)\eta} + a^2 e^{(2\tilde{k}+k)\eta}). \tag{47}$$

The necessary condition for a minimum value of V is

$$2\tilde{k} - k + a^2(2\tilde{k}+k)e^{2k\eta} = 0. \tag{48}$$

The case $2\tilde{k} - k = 0$ can be excluded ($a, k \neq 0$). For $2\tilde{k} - k \neq 0$ we obtain

$$e^{2k\eta} = -\frac{2\tilde{k}-k}{a^2(2\tilde{k}+k)} > 0 \tag{49}$$

which implies

$$\begin{aligned} &(i) \quad 2\tilde{k} - k > 0, \quad 2\tilde{k} + k < 0, \\ &(ii) \quad 2\tilde{k} - k < 0, \quad 2\tilde{k} + k > 0. \end{aligned} \tag{50}$$

The sufficient condition gives

$$V''_{\min} = -e^{(2\tilde{k}-k)\eta}(2\tilde{k}-k) > 0, \tag{51}$$

which shows that the case (i) has to be excluded, if ε_0^2 is assumed positive definite. The second case gives

$$V_{\min} = \frac{a^2}{k-2\tilde{k}} \left(\frac{k-2\tilde{k}}{a^2(k+2\tilde{k})} \right)^{(2\tilde{k}+k)/2k} \tag{52}$$

With the aid of inequalities (*ii*) we obtain $\varepsilon_0^2 < s_0^2$, which is impossible in the case of vanishing spin. We would like to point out that a similar analysis of the case $\gamma = 2$ has been given by Raychaudhuri (1975, 1979).

It has been stated by Kuchowicz (1976a, 1976d) that in the case $\gamma = 2$ the spin-spin interaction is unable to prevent a singularity in them. Our solution is thus a counterexample.

5. *Bianchi type VIII and type IX models*

From Eq. (35e) we have the constraint equation

$$-\frac{s_0}{2(R_1R_2R_3)^2}(R_1^2-R_2^2-\delta R_3^2) = E_{03}, \quad \delta := \begin{cases} -1, & \text{type VIII} \\ 1, & \text{type IX} \end{cases} \tag{53}$$

For the case of vanishing spin ($s_0 = 0$) we have $E_{03} = 0$ and no restriction upon the cosmic scale factors R_i is given. In the locally rotationally symmetric case $R_1 = R_2$ with $\gamma = 2$ we can easily obtain exact solutions of the GRT-Bianchi type VIII and type IX models (see Lorenz 1980b). However, in the case considered here ($s_0 \neq 0$) and assuming $R_1 = R_2$, $E_{03} = 0$ we have $R_3 = 0$. This result is in accordance with the result obtained by Tafel (1975), that ECT-Bianchi type VIII (with $n_1 = n_2 =: n \neq n_3$) and type IX (with $n_1 = n_2 = n_3$) models cannot be reconciled with the presence of spin (Tafel considers only the case $E_{\mu\nu} = 0$).

It can easily be seen that Eqs. (35b) do not turn into each other under any permutation of the indices i, j, k for type VIII, whereas for type IX the intrinsic geometry of three-space does not privilege any direction of space. For type VIII we can equate only R_1 with R_2 obtaining a symmetry about the third axis.

The main difference between type VIII and type IX is the sign of curvature. For type VIII this is always negative, whereas for type IX it can be positive as well as negative depending on the relations between the cosmic scale factors R_i. In cosmology "closed" meaning "having compact spatial three-surfaces of homogeneity" has been taken as synonymous with "having spatial three-surfaces of homogeneity of positive curvature". The latter two phrases are inequivalent. The Bianchi type IX model with compact three--surfaces of homogeneity will not in general have positive spatial curvature at all time.

This clarification is of some importance since it has been stated that a homogeneous distribution of polarized spin is incompatible with spatial closure in the case of Bianchi type IX models (Kerlick 1975, 1976; Kuchowicz 1975d, 1976d). (The exceptional case S considered by Kuchowicz (1976c) with a positive scalar curvature belongs to the Kantowski--Sachs models (see e.g. Collins 1977), for which the formalism used here cannot be applied.) Tafel (1975) has shown that the only type VIII and type IX solutions with non-vanishing spin are those described by $n_3 = n_1+n_2+2c|n_1n_2|^{1/2}$, $c = \text{const}$.

The nonvanishing components of the electromagnetic stress-energy tensor are

$$E_{00} = \tfrac{1}{2}\left(\left(\frac{a_1}{R_2R_3}\right)^2 + \left(\frac{a_2}{R_1R_3}\right)^2 + \left(\frac{a_3}{R_1R_2}\right)^2\right), \tag{a}$$

$$E_{11} = \tfrac{1}{2}\left(-\left(\frac{a_1}{R_2R_3}\right)^2 + \left(\frac{a_2}{R_1R_3}\right)^2 + \left(\frac{a_3}{R_1R_2}\right)^2\right), \qquad \text{(b)}$$

$$E_{22} = \tfrac{1}{2}\left(-\left(\frac{a_2}{R_1R_3}\right)^2 + \left(\frac{a_1}{R_2R_3}\right)^2 + \left(\frac{a_3}{R_1R_2}\right)^2\right), \qquad \text{(c)}$$

$$E_{33} = \tfrac{1}{2}\left(-\left(\frac{a_3}{R_1R_2}\right)^2 + \left(\frac{a_1}{R_2R_3}\right)^2 + \left(\frac{a_2}{R_1R_3}\right)^2\right), \qquad \text{(d)}$$

$$E_{12} = E_{21} = -\frac{a_1a_2}{R_1R_2R_3^2}(\cos(t_1+\tau_1)\cos(t_2+\tau_2) - \sin(t_1+\tau_1)\sin(t_2+\tau_2)), \qquad \text{(e)}$$

$$E_{03} = -\frac{a_1a_2}{R_1R_2R_3^2}(\cos(t_1+\tau_1)\sin(t_2+\tau_2) - \cos(t_2+\tau_2)\sin(t_1+\tau_1)). \qquad \text{(f) (54)}$$

Thus in general we can deduce only the relation

$$R_1^2 - R_2^2 - \delta R_3^2 = \frac{2a_1a_2}{s_0} R_1R_2(\cos(t_1+\tau_1)\sin(t_2+\tau_2) - \cos(t_2+\tau_2)\sin(t_1+\tau_1)). \qquad (55)$$

We have therefore shown that in the electromagnetic case considered here there exist ECT-solutions of the Bianchi types VIII and IX. We finally would like to point out that in addition ECT-solution of Bianchi type II exist (Lorenz 1981d, 1981e), contrary to the claims in the literature on the subject (see e.g. Tafel 1975, Tsoubelis 1981).

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