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A SIMPLE MODEL EXHIBITING THE THIRD ORDER PHASE TRANSITION OBSERVED IN SU($N \to \infty$) GAUGE THEORIES

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Third order phase transitions are studied in a model derived from quantum mechanics, but closely related to lattice gauge theories. The transitions are found to result from changes in the evolution of the range available to the eigenvalues of the matrix built from "gluon" fields. Without changing the continuum limit of the model it is possible to introduce an arbitrary number of such phase transitions, or to eliminate all of them. One can also introduce phase transitions in such a way that an analytic continuation past them is impossible.

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1. Introduction

Among indirect approaches to the confinement problem in QCD, studies of SU(N) gauge theories in the limit $N \to \infty$, $g \to 0$, $\lambda = g^2N$ fixed have been recently popular. The reason is that it is widely believed that this theory confines, with the confinement mechanism essentially similar to that in QCD (cf. e.g. [1]); on the other hand it seems to be much easier than QCD, for instance in the perturbative expansion it has only planar diagrams and no baryons [2]. In order to simplify further the large N theory, it has been reformulated on a lattice following the now standard approach developed for QCD by Wilson [3]. On a lattice the strong coupling limit $\lambda \to \infty$ is easy and gives confinement. The problem is to continue it to the weak coupling limit $\lambda \to 0$, which is the interesting one, because it corresponds to the continuum limit for the lattice. Of course the continuum coupling constant can have non-zero values, when the lattice coupling constant g tends to zero. All this is well known for finite N [3]. We assume that these features persist in the large N limit. The large N theory on an infinite, four-dimensional lattice has not been solved. In the two-dimensional theory, however, a third order phase transition for some $\lambda = \lambda_c$ has been found [4]. A similar transition has been reported [5] in a one-plaquette system in 2+1 dimensions with the Kogut-Susskind [6] continuous time Hamiltonian, and in some more complicated systems [7, 8]. This phase transition is an

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obstacle to the continuation of the theory from the easy strong coupling to the interesting weak coupling region. Therefore, it is of great interest that using instead of Wilson's action another formula proposed by Manton and corresponding to the same continuum limit [9], one finds no phase transition on the two-dimensional lattice [10]. Also convergence to the large N limit in two dimensions is faster and more smooth, when Manton's action is used instead of Wilson's [11]. In four dimensions a phase transition seems to persist, even when Manton's action is used. This, however, is a Monte Carlo calculation [12] and the conclusions are of a more qualitative character than in two dimensions.

In this paper we show that a simple extension of the model of Brézin, Itzykson, Parisi and Zuber (BIPZ) [13] exhibits the third order phase transition similar to those found in lattice gauge theories. Actually, the lattice gauge model considered by Wadia [5] is equivalent to a special case of the model considered here. Since the present model is both rather simple and rather general, it easily explains some features observed in more complicated calculations. In particular, one sees why the transition occurs, why it can be neither first nor second order, and why Manton's action gives more regular results than Wilson's.

2. *The model*

The model proposed here, which is a simple extension of BIPZ, is a quantum mechanical model. The wave function ψ depends on a hermitian $N\times N$ matrix M. Interpreting the two indices of M as colours, we can consider the elements of M as gluon fields in some U(N) theory. Since we intend to do the $N\to\infty$ transition, the distinction between U(N) and SU(N) is irrelevant. The Hamiltonian is

$$H = -\frac{1}{2m}\nabla^2 + W(M), \tag{2.1}$$

where

$$\nabla^2 = \sum_i \frac{\partial^2}{\partial M_{ii}^2} + \tfrac{1}{2}\sum_{i<j}\left(\frac{\partial^2}{\partial \operatorname{Re} M_{ij}^2} + \frac{\partial^2}{\partial \operatorname{Im} M_{ij}^2}\right), \tag{2.2}$$

and

$$W = \operatorname{Tr} f(M). \tag{2.3}$$

Function f is an $N\times N$ matrix-function of M, g and N — satisfying the relation

$$\operatorname{Tr} f(M) = \sum_i f(\lambda_i), \tag{2.4}$$

where λ_i are the eigenvalues of matrix M. Relation (2.4) holds whenever function $f(M)$ can be expanded in powers of M, for instance BIPZ have

$$f(M) = \tfrac{1}{2}M^2 + \frac{g}{N}M^4.$$

It is true more generally however: in particular f can have different power series expansions in different ranges of λ. In the following we put for simplicity $2m = 1$.

We assume that the ground state wave function depends on M only through the eigenvalues $\lambda_1, \ldots, \lambda_N$ and is totally symmetric in these variables. Then in the standard variational formula

$$E = \min \frac{\int d^{N^2}M((\nabla\psi)^2 + W\psi^2)}{\int d^{N^2}M\psi^2}. \tag{2.5}$$

The integration over the "angular" variable may be performed [13] and one finds

$$E = \min \frac{\int d^N\lambda \prod_{i<j} (\lambda_i-\lambda_j)^2 \sum_k \left[\left(\frac{\partial\psi}{\partial\lambda_k}\right)^2 + f(\lambda_k)\psi^2\right]}{\int d^N\lambda \prod_{i<j} (\lambda_i-\lambda_j)^2\psi^2}. \tag{2.6}$$

Substituting

$$\varphi(\lambda_1, \ldots, \lambda_N) = \prod_{i<j} (\lambda_i-\lambda_j)\psi(\lambda_1, \ldots, \lambda_N), \tag{2.7}$$

one finds further

$$E = \min \frac{\int d^N\lambda \sum_k \left[\left(\frac{\partial\varphi}{\partial\lambda_k}\right)^2 + f(\lambda_k)\varphi^2\right]}{\int d^N\lambda\varphi^2}. \tag{2.8}$$

This corresponds to the ground state energy of a system of N non-interacting particles, with the single particle energy levels defined by the single particle Schrödinger equation

$$\left[-\frac{\partial^2}{\partial\lambda^2} + f(\lambda)\right]\varphi_i(\lambda) = e_i\varphi_i(\lambda). \tag{2.9}$$

According to (2.7), and keeping in mind that function ψ is totally symmetric in the eigenvalues $\lambda_1, \ldots, \lambda_N$: $\varphi(\lambda_1, \ldots, \lambda_N)$ is totally antisymmetric in these arguments. Thus the N particles are fermions and the ground state energy according to the Pauli principle is

$$E = \sum_{i=1}^{N} e_i, \tag{2.10}$$

where the summation extends over the N lowest eigenvalues of (2.9). These results are valid for any N, but the problem of finding E further simplifies in the large N limit.

3. *The large N limit*

In the large N limit almost all the states contributing to the sum (2.10) are highly excited and the typical wave length in the wave function is small. Under such conditions it is legitimate to use the WKB or the Thomas-Fermi approximation. Then [13]

$$E = \int \frac{d\lambda dp}{2\pi} [p^2 + f(\lambda)]\theta(\mu - p^2 - f(\lambda)), \tag{3.1}$$

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4. *The phase transition*

A discontinuity in the derivative of integral $I(\beta)$ occurs, when at some β_0 the evolution law for the classical turning points changes suddenly. We present two typical cases. In each the transition occurs when $\beta V(\lambda_0) = \mu$ and the character of the transition depends on the behaviour of function $V(\lambda)$ for λ close to λ_0. Let us choose $\lambda_0 > 0$ and assume

$$V_a(\lambda) = V_0 + V_1'|\lambda-\lambda_0| + V_1(\lambda-\lambda_0) + O(|\lambda-\lambda_0|^2) \tag{4.1}$$

and

$$V_b(\lambda) = \begin{cases} V_0 - V_2(\lambda-\lambda_0)^2 + O(|\lambda-\lambda_0|^3) & \text{for } \lambda \leqslant \lambda_0 \\ \infty & \text{for } \lambda > \lambda_0, \end{cases} \tag{4.2}$$

where V_0, V_1, V_1' and V_2 are constants. In both cases we assume that $V(-\lambda) = V(\lambda)$ and that for $\beta V(\lambda)$ different from μ, but sufficiently close to it, the inverse of $dV(\lambda)/d\lambda$ is bounded. Examples are shown in Figs 1 and 2. In either case, when the chemical potential changing steadily crosses βV_0, the velocity of the classical turning points changes suddenly.

In order to study qualitatively the discontinuity of the derivative (3.9), it is enough to replace the potential $V(\lambda)$ by its approximation (4.1) or (4.2) for positive λ and use the corresponding formula for λ negative. For the two cases one obtains after elementary integrations

$$I_a(\beta) = C_a(\beta) - \frac{8V_1'}{V_1^2-(V_1')^2}\sqrt{|\mu-\beta V_0|}\,\theta(\beta V_0-\mu) \tag{4.3}$$

$$I_b(\beta) = C_b(\beta) + \frac{\ln|\mu-\beta V_0|}{\sqrt{\beta V^2}} \tag{4.4}$$

where $C_a(\beta)$ and $C_b(\beta)$ are regular in the vicinity of $\beta = \mu/V_0$. In either case the inverse of $I(\beta)$ and consequently the second derivative (3.7) is continuous, but the third derivative (3.9) has a discontinuity at $\beta = \mu/V_0$. An exception is the trivial case $V_1' = 0$, when the potential $V_a(\lambda)$ has no kink at $\lambda = \lambda_0$ — then there is no third order transition there.

5. *Discussion*

The model presented here has been derived from quantum mechanical considerations. It seems, however, to be more general than its derivation suggests. In particular, studying the gauge theory of one plaquette in 2+1 dimensions in the framework of the Kogut--Susskind formalism, Wadia [5] reduced his problem to what differs only by irrelevant rescaling and change of boundary conditions from the model shown in Fig. 2. In the case shown in the figure all the integrations may be performed explicitly. In particular one finds

$$I(\beta) = \begin{cases} \dfrac{4}{\sqrt{\beta}}K\left(\dfrac{\mu}{\beta}\right) & \text{for } \beta \geqslant \mu \\[2ex] \dfrac{4}{\sqrt{\mu}}K\left(\dfrac{\beta}{\mu}\right) & \text{for } \beta \leqslant \mu, \end{cases} \tag{5.1}$$

where

$$K(m) = \int_0^{\pi/2} \frac{dx}{\sqrt{1-m\sin^2 x}} \tag{5.2}$$

is the standard elliptic functions of the first kind. The strong coupling expansion is [14]

$$I(\beta) = \frac{2\pi}{\sqrt{\mu}}\left(1+\sum_{n=1}^{\infty}\frac{(2n-1)!!}{(2n)!!}\left(\frac{\beta}{\mu}\right)^n\right) \tag{5.3}$$

and diverges for $\beta \geqslant \mu$. One could, however, try methods of Padé, or others, to improve the convergence. In order to see better the analytic structure of the model, one may rewrite relations (3.4) and (3.5) in the form

$$E = \mu N - 4\,\mathrm{Re}\int_0^{\pi/2} \frac{d\lambda}{3\pi}\sqrt{\mu-\beta\sin^2\lambda}^{\,3} \tag{5.4}$$

$$N = 2\,\mathrm{Re}\int_0^{\pi/2} \frac{d\lambda}{\pi}\sqrt{\mu-\beta\sin^2\lambda}\,. \tag{5.5}$$

The integrals are analytic in the β plane cut from $\beta = \mu$ to $\beta = \infty$. This situation should be contrasted with that for the potential shown in Fig. 1. Here the two branches of the potential are a priori completely unrelated and there may be no analytic continuation from the large β to the small β region.

The phase transition occurs, when the velocity of the classical turning points corresponding to a steady rise of the chemical potential (or β), changes suddenly. This generalizes the remark of Gross and Witten [4], who noticed that in their model the transition occurs, when the eigenvalues λ_i fill all the range $[-\pi, \pi]$, and the allowed range cannot expand any more, i.e. the turning points stop.

Accepting that all the physics is in the continuum, i.e. in the weak coupling limit, while the lattice is a purely mathematical device, one finds that only the behaviour of $V(\lambda)$ near its minimum is important. This remark can be used to change the singularity structure of the model. Adding kinks at finite distances from the minimum, one can introduce an arbitrary number of phase transitions without affecting the continuum limit. On the other hand, replacing in the potential shown in Fig. 2 $\sin^2\lambda$ by λ^2, one reduces the singularity in the derivative of (4.4) to the milder singularity in the derivative of (4.3). In Wadia's model [5] this would correspond to a replacement of Wilson's action by Manton's action. Thus in this model, Manton's action is less effective in eliminating the phase transition than in the two-dimensional lattice QCD, where the singularity has been reported to disappear [10]. In the present model the phase transition may be completely eliminated by introducing $V(\lambda) = \lambda^2$ in the whole range $(-\infty, \infty)$. This again leaves the continuum limit unaffected and from this point of view is acceptable.

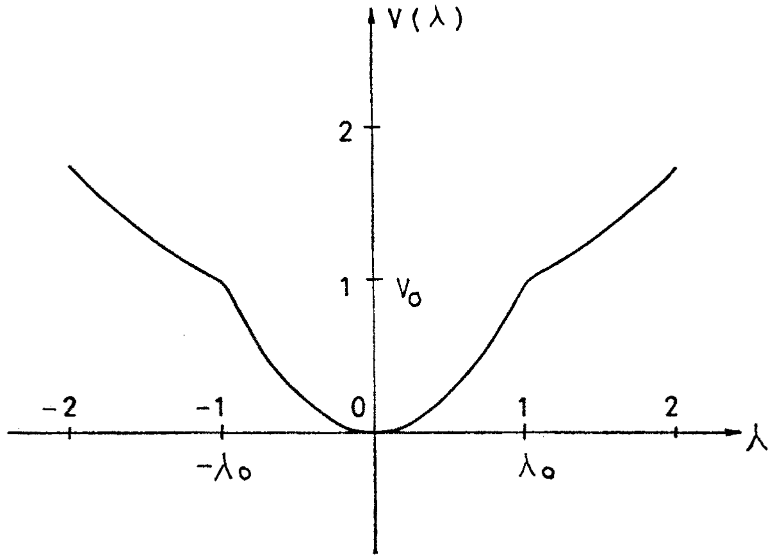

Fig. 1. Potential belonging to class (4.1): $V(\lambda) = \begin{cases} \lambda^2 \text{ for } |\lambda| \leqslant 1 \\ \frac{3}{4}+\frac{1}{4}\lambda^2 \text{ for } |\lambda| \geqslant 1. \end{cases}$ Here $\lambda_0 = 1$, $V_0 = 1$, $V_1 = 5/4$, $V_1' = -3/4$

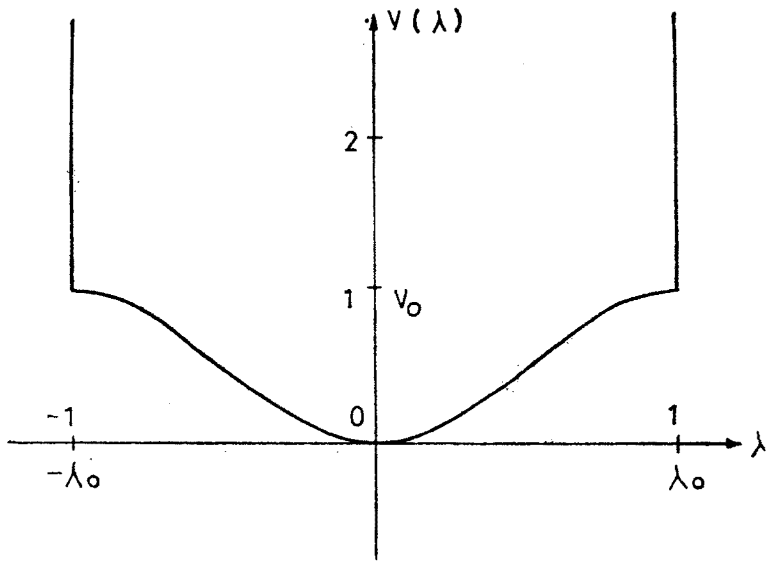

Fig. 2. Potential belonging to class (4.2:) $V(\lambda) = \begin{cases} \sin^2 \lambda \text{ for } |\lambda| \leqslant \pi/2 \\ \infty \quad \text{ for } |\lambda| > \pi/2. \end{cases}$ Here $\lambda_0 = \frac{\pi}{2}$, $V_0 = 1$, $V_2 = 2$

6. *Conclusions*

The model presented in this paper offers a simple interpretation of the third order transitions observed in some lattice models of the SU($N \to \infty$) gauge theory. In these models the eigenvalues of the matrix formed by the "gluon" fields cover a certain range. This range changes with changing temperature ($1/\beta$). According to the present model, when the rate of change of the size of this region as function of β suffers a discontinuity, a third order transition occurs. It is easily seen that neither a first nor a second order transition can be generated by this mechanism.

In gauge theories, a given continuum limit is consistent with an infinite variety of possible lattice actions. In our model this goes over into the statement that given the behaviour of the potential $V(\lambda)$ in the vicinity of its minimum, one can choose arbitrarily $V(\lambda)$ at finite distances from the minimum. Within this freedom it is possible to generate any prescribed number of phase transitions, or to eliminate them altogether. From this point of view, Manton's action is an improvement over Wilson's, but it is possible to do even better by rejecting the limitation that the eigenvalues must be all contained in the range $[-\pi, \pi]$.

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