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ASYMPTOTICAL INTEGRATION OF CURRIE-HILL EQUATIONS

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The Currie-Hill conditions for the relativistic world-line invariance of a Newtonian-like dynamical system of interacting particles are asymptotically integrated for $x \to \infty$.

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1. *Introduction*

In 1960 Havas and Plebański [1] showed that, for a system of interacting particles, Newtonian causality, also called predictivity [2, 3], is not incompatible with special relativity, as had been generally thought before. Subsequently, there has been a gradual revival of interest in the study of Newtonian-like dynamical systems. As a result, Currie [4] and Hill [5] found the necessary conditions that the accelerations of the particles must satisfy. Bel [6] proved that such conditions are also sufficient. These conditions, that constitute a first order system of nonlinear partial differential equations must be satisfied by the accelerations in order to have relativistic world-line invariance. More precisely, the Currie-Hill equations guarantee the relativistic form invariance of Newtonian equations of motions i.e. in each inertial frame the accelerations expressed as functions of positions and velocities have the same functional form.

Currie-Hill equations constitute a system of partial quasilinear equations, which are very difficult to handle. In the present note we study the behaviour of the solutions of Currie-Hill equations for large interparticle separations $x = x_2 - x_1$ in the case of the straight line motions of two particles. We require that the forces have the Coulomb-like behaviour for $x \to \infty$ and express the solutions as power series in $1/x$. The coefficients of such a series satisfy the linear partial differential equations, which one is able to integrate one after another. We find the first two coefficients of this series.

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2. *Integration*

One-dimensional Currie-Hill equations take the following form[1]:

$$(1-v_1^2)\frac{\partial F_1}{\partial v_1}+(1-v_2^2+xF_2)\frac{\partial F_1}{\partial v_2}-xv_2\frac{\partial F_1}{\partial x}+3v_1F_1=0, \tag{1a}$$

$$(1-v_2^2)\frac{\partial F_2}{\partial v_2}+(1-v_1^2-xF_1)\frac{\partial F_2}{\partial v_1}-xv_1\frac{\partial F_2}{\partial x}+3v_2F_2=0, \tag{1b}$$

where $x \equiv x_2-x_1$ is the relative position, v_1 and v_2 are the particle velocities, $F_1 = dv_1/dt$, $F_2 = dv_2/dt$ are the particle accelerations.

In our system of units the accelerations $F_n(x, v_1, v_2)$ ("forces") have the same dimension as $1/x$ so if we want to obtain the "forces" which asymptotically behave as $1/x^\alpha$, $\alpha > 1$ we must suppose that the solutions contain some constant l_0 of length dimension, so it will be convenient to write the forces F_n in the following form:

$$F_n = f_n(y, v_1, v_2)/x \quad \text{where} \quad n = 1, 2, \quad y = x/l_0$$

and f_n is a dimensionless function to be determined. Since we are trying to find the solutions in the limit $x \to \infty$, it is convenient to introduce the variable $z = 1/y$. After these changes the equations take the form:

$$(1-v_1^2)\frac{\partial f_1}{\partial v_1}+(1-v_2^2+f_2)\frac{\partial f_1}{\partial v_2}+v_2z\frac{\partial f_1}{\partial z}+(v_2+3v_1)f_1=0, \tag{2a}$$

$$(1-v_2^2)\frac{\partial f_2}{\partial v_2}+(1-v_1^2-f_1)\frac{\partial f_2}{\partial v_1}+v_1z\frac{\partial f_2}{\partial z}+(v_1+3v_2)f_2=0. \tag{2b}$$

One can easily see that if one wants to obtain the forces having Coulomb-like asymptotic behaviour and which are analytic functions at infinity (for $z = 0$) one is obliged to write the functions f_n in the following form:

$$f_1 = z\sum_{i=0}^{\infty} a_i z^i, \quad f_2 = z\sum_{i=0}^{\infty} b_i z^i, \quad a_0 \neq 0,\ b_0 \neq 0. \tag{3}$$

We insert the expressions (3) into equations (2) and thus receive an infinite system of linear partial differential equations, which has the form:

$$(1-v_1^2)\frac{\partial}{\partial v_1}a_0+(1-v_2^2)\frac{\partial}{\partial v_2}a_0+(2v_2+3v_1)a_0=0, \tag{4a}$$

$$(1-v_1^2)\frac{\partial}{\partial v_1}a_1+(1-v_2^2)\frac{\partial}{\partial v_2}a_1+(3v_2+3v_1)a_1+b_0\frac{\partial}{\partial v_2}a_0=0, \ldots, \tag{4b}$$

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[1] We use the system of units in which $c = 1$.

$$(1-v_1^2)\frac{\partial}{\partial v_1}a_n+(1-v_2^2)\frac{\partial}{\partial v_2}a_n+[(n+2)v_2+3v_1]a_n+\tfrac{1}{2}\left(b_0\frac{\partial}{\partial v_2}a_{n-1}\right.$$

$$\left.+b_1\frac{\partial}{\partial v_2}a_{n-2}+\ldots+n_{n-2}\frac{\partial}{\partial v_2}a_1+b_{n-1}\frac{\partial}{\partial v_2}a_0\right)=0,\ \ldots, \tag{4c}$$

$$(1-v_1^2)\frac{\partial}{\partial v_1}b_n+(1-v_2^2)\frac{\partial}{\partial v_2}b_n+[(n+2)v_1+3v_2]b_n-\tfrac{1}{2}\left(a_0\frac{\partial}{\partial v_1}b_{n-1}\right.$$

$$\left.+a_1\frac{\partial}{\partial v_1}b_{n-2}+\ldots+a_{n-2}\frac{\partial}{\partial v_1}b_1+a_{n-1}\frac{\partial}{\partial v_1}b_0\right)=0,\ \ldots\ . \tag{4d}$$

Each of the above equations can be integrated [7] if we know the solutions of all the previous ones. Integrating the first two equations from (4), we obtain

$$a_0=(1-v_1^2)^{3/2}(1-v_2^2)\phi(u), \tag{5}$$

$$a_1=(1-v_1^2)^{3/2}(1-v_2^2)^{3/2}\left\{-\phi\left(\frac{1}{u}\right)\frac{d\phi(u)}{du}v_1+2\phi\left(\frac{1}{u}\right)\phi(u)\right.$$

$$\left.\times\left[\frac{(1-v_1^2)(1-v_2^2)}{(v_1-v_2)^2}\ln\left(\frac{1-v_1^2}{1-v_1v_2}\right)+\frac{v_1(1-v_1v_2)}{v_1-v_2}\right]+\psi(u)\right\}, \tag{6}$$

where ϕ, ψ are arbitrary functions, $u=\frac{1+v_1}{1-v_1}\cdot\frac{1-v_2}{1+v_2}$. In these equations $a_i(v_1,v_2)=(-1)^{i+1}b_i(v_2,v_1)$ for $i=0,1,\ldots$ as a result of the requirement that equations (1) should be invariant with respect to the transposition of particles.

3. *Remarks*

Analysing the system of equations (4), it is easily seen that the power series expansion (3) must be infinite. Terminating the expansion at a_N or b_N leads to situation in which all a_i, b_i must be equal to zero.

It is difficult to discuss the convergence of the series in the whole range of the variable z. From the solutions obtained, coefficients devoid of singularities can be selected, so that a finite radius of series convergence can be expected to exist; the results of the papers [8, 9] confirm this.

It is interesting to compare the expressions (5), (6) with those obtained by Hill in his paper [5] where he expands the forces of particles interacting with the Lienard-Wiechert potentials into a power series of x. Taking the functions ϕ, ψ in the form; $\phi=1$,

$$\psi=\mathrm{cth}\,(h)-\mathrm{sh}\,(h)\ln[1-\mathrm{th}\,(h)],$$

where

$$h=\tfrac{1}{2}\ln\left(\frac{1-v_1}{1-v_1}\cdot\frac{1-v_2}{1+v_2}\right)$$

in our expressions, we find that the coefficients at the proper powers of $1/x$ will be the same as those of the paper cited above, if one puts $l_0 = e^2/m$.

If we perform the Taylor expansion of the expression (5) around the points $v_1 = 0$, $v_2 = 0$ (valid for small velocities) and keep the linear and quadratic terms only we get the expression

$$a_0 \cong (1-\tfrac{3}{2}v_1^2)(1-v_2^2).$$

Thus the force F_1 takes the form

$$F_1 = \frac{l_0}{x^2}(1-\tfrac{3}{2}v_1^2)(1-v_2^2)$$

which coincides with the force obtained from the Lagrangian of Darwin [10] if one puts $l_0 = \dfrac{e^2}{m}$.

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