ON THE MAXWELLIAN TENSORS*

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Following the procedure first presented by Öktem we show that for every tensor field H, which is antisymmetric in the first two indices, we can attribute the so-called Maxwellian tensor M. Physical interpretation of tensors of that kind is given in the case of the electromagnetic field F and in the case of the curvature tensor field R in the framework of standard General Relativity and in the framework of the Einstein-Cartan theory of gravitation.

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1. The Maxwellian tensors

Suppose that the tensor field H is given with the components

$$H_{abM} = -H_{baM}.$$

Here index M denotes any set of indices.

Independently of the field equations satisfied by the field H, we can always consider for this field the following equalities

$$\nabla_a H_{bcM} + \nabla_b H_{caM} + \nabla_c H_{abM} = J_{abcM}, \tag{1a}$$

$$\nabla_a H^{ab}_{\ \ ..M} = \frac{4\pi}{c} J^{b}_{\ .M} \tag{1b}$$

with the corresponding tensors J_{abcM} and $J^{b}_{,M}$[1]. ∇ means the covariant derivative with respect to the given, linear and metric connection of space-time.

The equalities (1a) and (1b), possessing in the indices (a, b, c), the structure of the Maxwell equations are, in general, geometric identities.

Transvecting (1a) with the tensor $H^{bc}_{\ \ ..N}$ we get [1]

$$H^{bc}_{\ \ ..N}\nabla_a H_{bcM} + H^{bc}_{\ \ ..N}\nabla_b H_{caM} + H^{bc}_{\ \ ..N}\nabla_c H_{abM} = H^{bc}_{\ \ ..N} J_{abcM} \tag{2}$$

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[1] The equalities (1a) and (1b) are the definitions of the tensors J_{abcM} and $J^{b}_{\ .M}$.

or, if we use the antisymmetry of the field H in the first two indices

$$H^{bc}_{..N}\nabla_a H_{bcM} - 2H^{bc}_{..N}\nabla_b H_{acM} = H^{bc}_{..N}J_{abcM}. \tag{3}$$

From (3) we get

$$H^{bc}_{..N}\nabla_b H_{acM} - \tfrac{1}{2}H^{bc}_{..N}\nabla_a H_{bcM} = \tfrac{1}{2}H^{cb}_{..N}J_{abcM}. \tag{4}$$

Now, let us consider the relations obtained from (4) by transposition of the "indices" M and N

$$H^{bc}_{..M}\nabla_b H_{acN} - \tfrac{1}{2}H^{bc}_{..M}\nabla_a H_{bcN} = \tfrac{1}{2}H^{cb}_{..M}J_{abcN}. \tag{5}$$

Adding, side by side, to the equalities (5) the equalities (4), we obtain

$$H^{bc}_{..N}\nabla_b H_{acM} + H^{bc}_{..M}\nabla_b H_{acN} - \tfrac{1}{2}(H^{bc}_{..N}\nabla_a H_{bcM} + H^{bc}_{..M}\nabla_a H_{bcN})$$
$$= \tfrac{1}{2}(H^{cb}_{..N}J_{abcM} + H^{cb}_{..M}J_{abcN}). \tag{6}$$

With the help of (1b) we can easily bring the equalities (6) to the form

$$\nabla_b(H^{bc}_{..N}H_{acM} + H^{bc}_{..M}H_{acN} - \tfrac{1}{2}\delta^b_a H^{dc}_{..N}H_{dcM})$$
$$= \tfrac{1}{2}(H^{cb}_{..N}J_{abcM} + H^{cb}_{..M}J_{abcN}) + \frac{4\pi}{c}(J^c_{.M}H_{acN} + J^c_{.N}H_{acM}) \tag{7}$$

or, to the form

$$\nabla_b M^{b.}_{.aMN} = \tfrac{1}{2}(H^{cb}_{..N}J_{abcM} + H^{cb}_{..M}J_{abcN}) + \frac{4\pi}{c}(J^c_{.M}H_{acN} + J^c_{.N}H_{acM}), \tag{8}$$

where

$$M^{b.}_{.aMN} := H^{bc}_{..N}H_{acM} + H^{bc}_{..M}H_{acN} - \tfrac{1}{2}\delta^b_a H^{dc}_{..N}H_{dcM}. \tag{9}$$

We will call the tensor M, with the components given by (9), the Maxwellian tensor of the field H.

This is justified by the fact that the tensor M has, in the indices (a, b, c, d), the same structure and properties as the symmetric energy-momentum tensor T of the electromagnetic field F [2].

2. *On the physical meaning of the Maxwellian tensors*

There exist the three following, well known, classical fields having physical meaning whose components are antisymmetric in the first two indices: the electromagnetic field F, the torsion field S and the curvature tensor field R[2].

In the case of the electromagnetic field F, we have

$$H_{ab} = F_{ab} = -F_{ba},$$

and the equalities (1a) and (1b) are the Maxwell equations in vacuum [2].

[2] We are using here Schouten's notation [3] for the curvature and torsion tensors.

If the connection of the space-time is symmetric than

$$J_{abc} = 0.$$

The Maxwellian tensor (9) possesses, in this case, the following components

$$M^{b\cdot}_{\cdot a} = F^{bc}F_{ac} + F^{bc}F_{ac} - \tfrac{1}{2}\,\delta^b_a F^{dc}F_{dc} = 2(F^{bc}F_{ac} - \tfrac{1}{4}\,\delta^b_a F^{dc}F_{dc}) = -8\pi T^{b\cdot}_{\cdot a}, \tag{10}$$

where

$$T^{b\cdot}_{\cdot a} := -\frac{1}{4\pi}\,(F^{bc}_{\cdot\cdot}F^{\cdot\cdot}_{ac} - \tfrac{1}{4}\,\delta^b_a F^{dc}F_{dc}) \tag{11}$$

are the components of the metric energy-momentum tensor T of the electromagnetic field [2].

In the space-time with torsion, we have for the torsion tensor S

$$H_{abM} = S_{abc} = -S_{bac}.$$

In this case the equalities (1a) and (1b) are the geometric identities and the Maxwellian tensor (9) has the components

$$M^{b\cdots}_{\cdot apq} = 2S^{bc}_{\cdot\cdot(p|}S_{ac|q)} - \tfrac{1}{2}\,\delta^b_a S^{dc}_{\cdot\cdot p}S^{\cdot\cdot\cdot}_{dcq}. \tag{12}$$

The possible physical meaning of the Maxwellian tensor with the components (12) will be examined, especially in the framework of the relativistic theories of the gravitational field with the quadratic Lagrangian.

In the case of the curvature tensor field R the situation is the following

$$H^{\cdot\cdot}_{abM} = R^{\cdot\cdot\cdot\cdot}_{abcd} = -R^{\cdot\cdot\cdot\cdot}_{bacd};$$

(1a) are the Bianchi identities and (1b) are some consequences of these identities [3]. Moreover, in the space-time with a symmetric linear connection we have

$$J_{abcM} = 0.$$

Maxwellian tensor (9) has the following components

$$M^{b\cdots\cdot}_{\cdot aklpq} = R^{bc\cdot\cdot}_{\cdot\cdot kl}R^{\cdot\cdot\cdot\cdot}_{acpq} + R^{bc\cdot\cdot}_{\cdot\cdot pq}R^{\cdot\cdot\cdot\cdot}_{ackl} - \tfrac{1}{2}\,\delta^b_a R^{bc\cdot\cdot}_{\cdot\cdot kl}R^{\cdot\cdot\cdot\cdot}_{bcpq}. \tag{13}$$

The contraction of the tensor in the indices (k, p) gives the Bel-Robinson tensor [4].

In the framework of standard General Relativity the Maxwellian tensor with the components given by (13) does not immediately possess physical meaning. However, its contraction in the indices (k, p), i.e., the Bel-Robinson tensor has physical meaning as the Maxwellian part of the superenergy tensor of the gravitational field which we identify with the Riemannian connection of the space-time [5].

We have a similar situation in the framework of the Einstein-Cartan theory of gravitation [5]. The physical meaning of the Maxwellian tensor determined by (13) in the framework of the relativistic theories of gravitation with quadratic Lagrangian is examined.

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