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THE RUNNING COUPLING CONSTANT IN YANG-MILLS THEORY*

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The infra-red behavior of the running coupling constant $g^2(q^2)$ in Yang–Mills theory is calculated using a non-perturbative procedure based on the Schwinger–Dyson equations and the Slavnov–Taylor identities.

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1. Introduction

In these talks I will describe a program for calculating the infra-red behaviour of the axial gauge running coupling constant in Yang–Mills theory using a non-perturbative procedure based on the Schwinger–Dyson equations and the Ward (Slavnov–Taylor) identities. This work has been carried out in collaboration with [1–6] J. S. Ball, F. Zachariasen, R. Anishetty, S. K. Kim and P. Lucht.

The axial gauge running coupling constant $g^2(q^2)$ determines the one-gluon exchange force between a pair of heavy quarks interacting with momentum transfer q^2. For large q^2 the dependence of $g^2(q^2)$ upon q^2 in QCD can be calculated perturbatively because of asymptotic freedom (i.e., the effective coupling decreases at short distances). As $q^2 \to \infty$, $g^2(q^2)$ behaves as

$$g^2(q^2) \to \frac{1}{b\log(q^2/\Lambda^2)},$$

where $b = (\frac{11}{3} N_c - \frac{2}{3} N_f)/16\pi^2$. In this equation, N_c is the number of colours, (i.e., $N_c = 3$), N_f is the number of flavours (types of quarks), and the mass Λ^2 as determined from short-distance experiments like deep inelastic electron scattering is of the order $(400\,\text{MeV})^2$. Perturbation theory indicates that as q^2 decreases, $g^2(q^2)/4\pi$ increases to a value of order unity at which point perturbation theory estimates are no longer applicable.

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The outline of our program to calculate $g^2(q^2)$ at small q^2 is the following.

In the axial gauge the running coupling constant is determined by the gluon propagator $D^{ab}_{\mu\nu}(q) = \delta^{ab}D_{\mu\nu}(q)$. (The Greek letters μ, ν,... always refer to Lorentz indices, while $a, b, \ldots$ refer to colour indices.) We denote the "inverse" of $D^{ab}_{\mu\nu}(q)$ by $\Pi^{ab}_{\mu\nu}(q) = \delta^{ab}\Pi_{\mu\nu}(q)$, and the vacuum polarization tensor by $\tilde{\Pi}^{ab}_{\mu\nu}(q) = \delta^{ab}\tilde{\Pi}_{\mu\nu}(q)$. Then

$$\Pi_{\mu\nu}(q) = i(q^2 g_{\mu\nu} - q_\mu q_\nu) - \tilde{\Pi}_{\mu\nu}(q). \tag{1}$$

The Schwinger–Dyson equation then expresses $\tilde{\Pi}_{\mu\nu}(q)$ in terms of the interacting triple-gluon vertex $\Gamma^{abc}_{3\lambda\mu\nu}(p, q, r) \equiv \Gamma_3$ and the interacting quadruple-gluon vertex $\Gamma^{abcd}_{4\lambda\mu\nu\sigma}(p, q, r, s) \equiv \Gamma_4$ as shown in Fig. 1. Analogous Dyson equations determine Γ_3 in terms of Γ_4 and Γ_5, Γ_4 in terms of Γ_5 and Γ_6, etc.

The Slavnov–Taylor identity for Γ_{n+1} determines the longitudinal part of Γ_{n+1} in terms of Γ_n ($\Gamma_2 \equiv \Pi_{\mu\nu}$). This identity along with the requirements of Bose symmetry and the absence of kinematic singularities then determines Γ_{n+1} in terms of Γ_n when any one

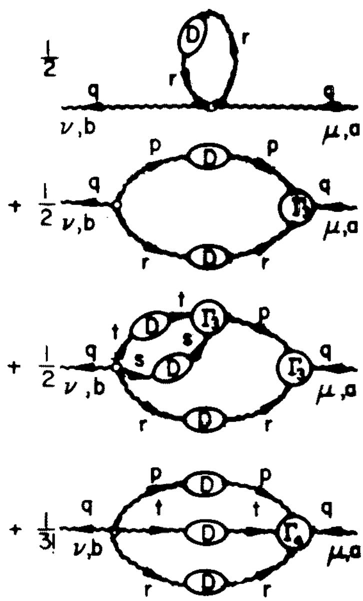

Fig. 1. Graphical representation of the Schwinger–Dyson expression for the vacuum polarization tensor $\tilde{\Pi}_{\mu\nu}(q)$

of the momenta entering into Γ_{n+1} is small [7]. This low momentum expression for Γ_{n+1} in terms of Γ_n can then be used in the Schwinger–Dyson equation for Γ_n. Furthermore Γ_{n+2} can be eliminated from this equation. There then results a closed set of equations for $D_{\mu\nu}$ $\Gamma_3, \Gamma_4 \ldots \Gamma_n$ which should be appropriate for studying the low momentum infra-red behaviour of the theory. The essential feature of this non-perturbative procedure for truncating the Schwinger–Dyson equations is that each stage yields vertices $\Pi_{\mu\nu}, \Gamma_3, \Gamma_4 \ldots \Gamma_n$

which are exact solutions of the Slavnov–Taylor identities. This is necessary for preserving the correct low momentum properties of the theory.

In the first stage of this procedure we use the Slavnov–Taylor identity to express Γ_3 in terms of $\Pi_{\mu\nu}$. The first two terms in the expression for $\tilde{\Pi}_{\mu\nu}$ (Fig. 1) are then re-expressed in terms of $\Pi_{\mu\nu}$ or equivalently $D_{\mu\nu}$ and, as we shall show, the last two terms do not contribute to the quantity of interest. Equation (1) then becomes a closed equation for $D_{\mu\nu}$. We look for solutions $D_{\mu\nu}(q)$, which for small q^2 have the same spin structure as the bare propagator, i.e.,

$$D_{\mu\nu}(q) = -\frac{iZ(q^2)}{q^2}\left[g_{\mu\nu}-\frac{(n_\mu q_\nu+q_\mu n_\nu)}{n\cdot q}+\frac{n^2 q_\mu q_\nu}{(n\cdot q)^2}\right]$$
$$\equiv Z(q^2)\Delta^{(0)}_{\mu\nu}(q). \tag{2}$$

In Eq. (2) the vector n^μ is a fixed vector which defines the gauge via the condition $n^\mu A^a_\mu(x) = 0$, or equivalently,

$$n^\mu D_{\mu\nu}(q) = 0. \tag{3}$$

There is a second possible spinor structure which satisfies Eq. (3). The basic reason for selecting the structure (2) is that the angular average of $\Delta^{(0)}_{\mu\nu}(q) = 0$, i.e.,

$$\int d\Omega_p \Delta^{(0)}_{\mu\nu}(p) = 0. \tag{4}$$

Because of Eq. (4), $Z(p^2)$ could behave like $1/p^2$ as $p^2 \to 0$ without introducing any infra--red singularities in the integrals determining $\tilde{\Pi}_{\mu\nu}(q)$ in Fig. 1. The second spinor structure satisfying Eq. (3) does not have a vanishing angular average. It can thus not have a coefficient which is as singular as $1/q^2$, because it would introduce an infra-red divergence of the type d^4p/p^4 into the integrals for $\tilde{\Pi}_{\mu\nu}(q)$. Since we will find that the solution for $Z(p^2)$ does behave like $1/p^2$ as $p^2 \to 0$, we conclude that $D_{\mu\nu}(q)$ must have the structure of Eq. (2) in the infra-red region.

The propagator $D_{\mu\nu}(p)$ and its "inverse" $\Pi_{\mu\nu}(p)$ are related by the equation [8],

$$\Pi_{\lambda\nu}(q)D_{\nu\mu}(q) = g_{\lambda\mu}-\frac{n_\lambda q_\mu}{n\cdot q}. \tag{5}$$

The additional term on the right-hand side of Eq. (5) makes it consistent with Eq. (3) and with the "Ward identity" for $\Pi_{\eta\nu}(q)$:

$$q^\mu \Pi_{\mu\nu}(q) = 0, \tag{6}$$

which states that the longitudinal part of $\Pi_{\mu\nu}(q)$ vanishes. Equations (2) and (5) then yield

$$\Pi_{\mu\nu}(q) = iZ^{-1}(q^2)\,(q^2 g_{\mu\nu}-q_\mu q_\nu). \tag{7}$$

The integral equation for Z is then obtained by contracting Eq. (1) with $n^\mu n^\nu/n^2$, which gives

$$q^2(Z^{-1}(q^2)-1)\left(1-\frac{(n\cdot q)^2}{n^2q^2}\right) = -i\,\frac{n^\mu\tilde{\Pi}_{\mu\nu}n^\nu}{n^2}. \tag{8}$$

The third and fourth terms in Fig. 1 are orthogonal to n_ν and thus do not contribute to Eq. (8). This is important because the Slavnov–Taylor identity determines the low momentum behaviour of Γ_4 in terms of Γ_3 and the equation for $Z^{-1}(q^2)$ would not be closed if it depended upon the Γ_4 contribution to $\tilde{\Pi}_{\mu\nu}(q)$.

Equation (8) then becomes an integral equation for $Z(p^2)$ which we will study in detail. The solution of this equation yields a running coupling constant which, for small q^2, behaves like

$$\frac{3}{4\pi^2}\, g^2(q^2) \xrightarrow[q^2\to 0]{} 0.215\left(\frac{M^2}{q^2}\right), \tag{9}$$

where M^2 is the subtraction point of the ultra-violet renormalization procedure and is defined by the normalization condition

$$\frac{3g^2(M^2)}{4\pi} = 1.$$

Thus the first stage of our program leads to a confining heavy quark potential. Furthermore, the strength given in Eq. (9) turns out to be of the expected order of magnitude. We argue that it is plausible that the $1/q^2$ small q^2 form for $g^2(q^2)$ is maintained beyond the first stage of our scheme, but that higher orders may influence its strength.

The content of these lectures is the following. In Section 2 we write the explicit form of the Schwinger–Dyson equations and the Slavnov–Taylor identities and indicate the relation between them. In Section 3 we write the low momentum form of Γ_3 in terms of Z and obtain the explicit form of the integral equation for $Z(p^2)$. In Section 4 and in the Appendix we show that gauge invariance requires subtraction terms in the equation for $Z(p^2)$ to remove ambiguities which are present if a gauge invariant regularization procedure is not used. In Section 5 we study the low momentum structure of the integral equation and indicate the reasons why $Z(q^2)$ has a $1/q^2$ low momentum behaviour. In Section 6 we renormalize the integral equation and determine the ultra-violet behaviour of the solution. In Section 7 we discuss briefly a numerical solution giving $Z(q^2)$ for all q^2. In Section 8 we consider the possible comparison of our solution with experiment.

2. The Schwinger–Dyson equations and the Slavnov–Taylor identities

We first write the explicit form of the first two graphs of Fig. 1 for $\tilde{\Pi}^{ab}_{\mu\nu}(q)$:

$$\tilde{\Pi}^{ab}_{\mu\nu}(q) = g_0^2/2 \int \frac{d^4p}{(2\pi)^4}\, \Gamma^{(0)bcca}_{4\nu\lambda\sigma\mu} D_{\lambda\sigma}(p)$$

$$+ g_0^2/2 \int \frac{d^4p}{(2\pi)^4}\, \Gamma^{(0)bcd}_{3\nu\lambda\sigma}(-q, -p, -r) D_{\lambda\lambda'}(p) D_{\sigma\sigma'}(r) \Gamma^{acd}_{3\mu\lambda'\sigma'}(q, p, r). \tag{10}$$

The bare triple gluon vertex $\Gamma_3^{(0)}$ and the bare quadrupole gluon vertex $\Gamma_4^{(0)}$ are given by

$$\Gamma^{(0)abc}_{3\lambda\mu\nu}(p, q, r) = f^{abc}[(p-q)_\nu g_{\lambda\mu} + (q-r)_\lambda g_{\mu\nu} + (r-p)_\mu g_{\lambda\nu}],$$

$$\Gamma^{(0)abcd}_{4\lambda\mu\nu\sigma} = -i[f^{abc}f^{cde}(g_{\mu\sigma}g_{\nu\lambda}-g_{\lambda\sigma}g_{\mu\nu}) + f^{bce}f^{dae}(g_{\nu\lambda}g_{\sigma\mu}-g_{\mu\lambda}g_{\nu\sigma}) + f^{ace}f^{bde}(g_{\mu\lambda}g_{\sigma\nu}-g_{\lambda\sigma}g_{\mu\nu})]. \tag{11}$$

The f^{abc} are the structure constants of the SU_3 colour group. The vertex $\Gamma^{(0)}_4$ appears on the left-hand side of the third and fourth graphs of Fig. 1. Since $n_\nu\Gamma^{(0)abcd}_{4\lambda\mu\nu\sigma}$ is a linear combination of n_λ, n_μ and n_σ and since $n_\lambda D_{\lambda\lambda'} = 0$, these terms do not contribute to $n_\nu\tilde{\Pi}_{\mu\nu}(q)$, as stated in the Introduction.

The Slavnov–Taylor identities for Γ_3 and Γ_4 are the following:

$$iq_\mu\Gamma^{bac}_{3\mu\lambda\sigma}(q, p, r) = f^{bac}[\Pi_{\lambda\sigma}(r)-\Pi_{\lambda\sigma}(p)], \tag{12}$$

$$iq_\mu\Gamma^{bade}_{4\mu\lambda\sigma\varrho}(q, p, t, r) = -f^{bac}\Gamma^{cde}_{3\lambda\sigma\varrho}(-(t+r), t, r)-f^{bdc}\Gamma^{cae}_{3\sigma\lambda\varrho}(-(p+r), p, r) -f^{bec}\Gamma^{cad}_{3\varrho\lambda\sigma}(-(p+t), p, t). \tag{13}$$

The relation between the Slavnov–Taylor identities and the Schwinger–Dyson equations is the following:

A) any vertices Γ_3 and Γ_4 satisfying Eqs. (12) and (13) yield a vacuum polarization tensor $\tilde{\Pi}_{\mu\nu}$ via the graphs of Fig. 1 which is transverse, i.e.

$$q^\mu\tilde{\Pi}_{\mu\nu} = 0, \tag{14}$$

or, equivalently, from Eq. (1), $\Pi_{\mu\nu}$ satisfies its Ward identity, Eq. (6);

B) neglect the last two terms in Fig. 1 for $\tilde{\Pi}_{\mu\nu}$. Then $\tilde{\Pi}_{\mu\nu}$ satisfies Eq. (14) provided Γ_3 satisfies Eq. (12).

The above statements are special cases of the following:

A*) use in the Schwinger–Dyson equation for Γ_n any vertices Γ_{n+1} and Γ_{n+2} which satisfy their "Ward" identities; then this expression for Γ_n automatically satisfies its Ward identity;

B*) in the Dyson equation for Γ_n, neglect Γ_{n+2} and use any Γ_{n+1} which satisfies its Ward identity; then this expression for Γ_n satisfies the Ward identity for Γ_n.

The above statements, which are straightforward to prove, tell us what approximations are consistent with gauge invariance.

3. *The integral equation for $Z(p^2)$*

As explained in the Introduction we expect the gluon propagator in the infra-red region to be proportional to the bare propagator, and hence $\Pi_{\mu\nu}(q)$ has the form of Eq. (7). We now use the Ward identity, Eq. (12), to determine the low momentum behaviour of Γ_3 in terms of Z^{-1}. Inserting the expression for $\Pi_{\mu\nu}(q)$ given by Eq. (7) into the right-hand side of Eq. (12), we can show that the most general solution of this equation for Γ_3 which satisfies Bose symmetry and is free of kinematic singularities has the following form [7]:

$$\Gamma^{abc}_{3\lambda\mu\nu}(p, q, r) = \Gamma^{(L)abc}_{3\lambda\mu\nu}(p, q, r)+\Gamma^{(T)abc}_{3\lambda\mu\nu}(p, q, r),$$

where

$$\Gamma^{(L)abc}_{3\lambda\mu\nu}(p,q,r) = f^{abc}\Bigg\{g_{\lambda\mu}[Z^{-1}(p)p_\nu - Z^{-1}(q^2)q_\nu]$$

$$-\frac{[Z^{-1}(p^2)-Z^{-1}(q^2)]}{p^2-q^2}[p\cdot q g_{\lambda\mu} - q_\lambda p_\mu](p-q)_\nu\Bigg\}$$

$$+\text{cyclic permutations of } p, q \text{ and } r, \tag{15}$$

and

$$\Gamma^{(T)abc}_{3\lambda\mu\nu}(p,q,r) = f^{abc}\{F(p^2,q^2,r^2)[p\cdot q g_{\lambda\mu} - p_\mu q_\lambda][p_\nu r\cdot q - q_\nu r\cdot p]$$

$$+G(p^2,q^2,r^2)[g_{\lambda\mu}(p_\nu q\cdot r - q_\nu p\cdot r) + (r_\lambda p_\mu q_\nu - q_\lambda p_\nu r_\mu)]/3\}$$

$$+\text{cyclic permutations of } p,\ q \text{ and } r, \tag{16}$$

and where $F(p^2, q^2, r^2) = F(q^2, p^2, r^2)$, $G(p^2, q^2, r^2) = G(q^2, p^2, r^2) = G(q^2, r^2, p^2)$. The transverse vertex $\Gamma_3^{(T)}$ expressed in terms of the undetermined functions F and G satisfies

$$p^\lambda \Gamma^{(T)abc}_{3\lambda\mu\nu}(p,q,r) = 0,$$

and vanishes linearly when any one of the three external momenta p, q, or r approaches zero. This is because each term in Eq. (16) contains an explicit factor of $p\times q\times r$ multiplying functions F or G which have no kinematic singularities. The low momentum structure of Γ_3 is then determined explicitly in terms of Z^{-1} according to Eq. (15).

We now replace Γ_3 by $\Gamma_3^{(L)}$, Eq. (15), in Eq. (10) for $\tilde{\Pi}_{\mu\nu}(q)$. According to statement B in Section 2, this replacement preserves Eq. (14). Furthermore, it is exact when any one of the momenta p, q, or r is much smaller than the other two. Our assumption is that the remaining kinematic region does not dominate the infra-red behaviour of $\tilde{\Pi}_{\mu\nu}(q)$. The vacuum polarization $\tilde{\Pi}_{\mu\nu}(q)$ is then a function of Z alone, i.e., $\tilde{\Pi}_{\mu\nu} = \tilde{\Pi}_{\mu\nu}(Z)$. Equation (8) then becomes the following non-linear integral equation for $Z(q^2)$:

$$q^2\left(1-\frac{(n\cdot q)^2}{n^2q^2}\right)(Z^{-1}(q^2)-1) = -ig_0^2\int dp\,\frac{n\cdot r n\cdot(r-p)}{n^2}\,\Sigma_{\sigma\sigma'}$$

$$\left\{\frac{Z(p^2)}{Z(q^2)}\left[\frac{Z(r^2)-Z(q^2)}{r^2-q^2}\right](q-r)_\sigma q_{\sigma'} - \left[\frac{Z(p^2)-Z(r^2)}{p^2-r^2}\right](p\cdot r g_{\sigma\sigma'} - r_\sigma p_{\sigma'}) - Z(p^2)g_{\sigma\sigma'}\right\}, \tag{17}$$

where $r = -(p+q)$ and

$$\Sigma_{\sigma\sigma'} = \Delta^{(0)}_{\lambda\sigma}(p)\Delta^{(0)}_{\lambda\sigma'}(r).$$

In Eq. (17) $\int dp$ is a shorthand notation for

$$N_c\int\frac{d^4p}{(2\pi)^4}\,.$$

The right-hand side of Eq. (17) is in general a function of q^2 and

$$\gamma = \frac{n^2 q^2}{(n \cdot q)^2}. \tag{18}$$

We have assumed that $Z(q^2)$, which in general could depend upon q^2 and γ, is a function of q^2 alone. Any angular dependence in Z would produce infra-red divergences in Eq. (17), if $Z(p^2) \to 1/p^2$ as $p^2 \to 0$. Equation (4) would not be applicable because of the extra angular dependence introduced by the γ dependence of Z. The variable γ thus appears as a parameter in Eq. (17), and the same function $Z(q^2)$ must satisfy the equation for all values of γ.

In principle we can estimate the accuracy of Eq. (17) by proceeding to the second stage of our approximation scheme, in which we retain the full Γ_3 in Eq. (10) for $\tilde{\Pi}_{\mu\nu}$. In this second stage we then approximate the Schwinger–Dyson equation for Γ_3 by replacing the quadrupole gluon vertex Γ_4 appearing in this equation by its longitudinal part $\Gamma_4^{(L)}$. The vertex $\Gamma_4^{(L)}$ is given in terms of Γ_3 by an expression analogous to the right-hand side of Eq. (15), with Z^{-1} replaced by the invariant functions defining Γ_3. This approximation then yields coupled equations for Γ_3 and D. Although it is very difficult to carry out this next stage in practice, it may be possible to use it qualitatively to check the consistency of the $1/q^2$ infra-red behaviour of $Z(q^2)$.

4. Gauge invariant removal of ambiguities in the integral equation for $Z(p^2)$

We have pointed out that $\tilde{\Pi}_{\mu\nu}(q)$ is infra-red finite even if $Z(p^2)$ behaves like $1/p^2$ for small p^2. However, the integral, Eq. (10), defining $\tilde{\Pi}_{\mu\nu}(q)$ is not absolutely convergent; i.e., the region of integration in Eq. (10), where p approaches zero for fixed q, gives a contribution proportional to

$$\int d^4p D_{\lambda\lambda'}(p) = \int p^2 dp^2 \int d\Omega_p Z(p^2) \Delta^{(0)}_{\mu\nu}(p). \tag{19}$$

This region gives a vanishing contribution if the angular integration $d\Omega_p$ is carried out first, because of Eq. (4). However, if the p^2 integration is carried out first we obtain infinity. Thus, naively speaking, the value of the integral, Eq. (19), is zero times infinity which leads to a finite but ambiguous result. This means that if $Z(p^2) \to 1/p^2$ as $p^2 \to \infty$, the value of the vacuum polarization is sensitive to the procedure used to calculate the infra-red contribution to the integral.

This ambiguity is present only in $\tilde{\Pi}_{\mu\nu}(Z)$ (the contribution to $\tilde{\Pi}_{\mu\nu}$ from $\Gamma_3^{(L)}$). $\Gamma^{(T)abc}_{3\lambda\mu\nu}(p, q, r)$ behaves like p as $p \to 0$ for fixed q and this gives an absolutely convergent infra-red contribution to $\tilde{\Pi}_{\mu\nu}$ even if $Z(p^2)$ behaves like $1/p^2$ for small p^2. To understand this ambiguity, we can then consider $\tilde{\Pi}_{\mu\nu}(Z)$. We suppose that

$$Z(p^2) \xrightarrow[p^2 \to 0]{} \frac{M^2}{p^2} + Z_1(p^2),$$

where $p^2 Z_1(p^2) \to 0$ as $p^2 \to 0$. Write

$$\tilde{\Pi}_{\mu\nu}(Z) = \tilde{\Pi}_{\mu\nu}\left(\frac{M^2}{p^2}\right) + \tilde{\Pi}_{\mu\nu}(Z) - \tilde{\Pi}_{\mu\nu}\left(\frac{M^2}{p^2}\right).$$

The difference $\tilde{\Pi}_{\mu\nu}(Z) - \tilde{\Pi}_{\mu\nu}(M^2/p^2)$ contains integrals which are absolutely convergent in the infra-red region. The entire ambiguous contribution to $\tilde{\Pi}_{\mu\nu}$ then comes from $\tilde{\Pi}_{\mu\nu}(M^2/p^2)$.

By dimensional arguments we can write

$$\tilde{\Pi}_{\mu\nu}\left(\frac{M^2}{p^2}\right) = C_1 M^2 g_{\mu\nu} + C_2 M^2 \frac{n_\mu n_\nu}{n^2}, \tag{20}$$

where C_1 and C_2 are finite functions of γ whose values are sensitive to the infra-red regularization procedure. On the other hand, we know that $\tilde{\Pi}_{\mu\nu}(Z)$ satisfies Eq. (14), which is usually sufficient to demonstrate that

$$\lim_{q \to 0} \tilde{\Pi}_{\mu\nu}(q) = 0. \tag{21}$$

In the case where $Z(p^2)$ has a $1/p^2$ singularity and the integrals are not uniquely defined we must impose Eq. (21) to remove the ambiguity. We then regulate the integrals for $\tilde{\Pi}_{\mu\nu}$ so that $C_1 = C_2 = 0$. However, instead of regulating the integrals we can equivalently replace $\tilde{\Pi}_{\mu\nu}(Z)$ by

$$\tilde{\Pi}_{\mu\nu}(Z) - \tilde{\Pi}_{\mu\nu}\left(\frac{M^2}{p^2}\right). \tag{22}$$

The difference Eq. (22) is well behaved in the infra-red region and the usual argument leading to Eq. (21) applies to this difference. This infra-red subtraction is somewhat analogous to the ultra-violet subtraction which removes the quadratically divergent contribution to the gluon mass if dimensional regularization is not used. Since we will not use dimensional regularization, we must first perform this ultra-violet subtraction.

We show in the Appendix, using Ward's identity for Γ_3, that the general expression for the ultra-violet subtraction is

$$ig_0^2 \int dp \left\{ \frac{\partial}{\partial p_\mu} [\Gamma^{(0)}_{3\nu\lambda\sigma}(-p, -q, -r) D_{\lambda\sigma}(r)] \right.$$
$$\left. + [D_{\lambda\sigma}(r) - D_{\lambda\sigma}(p)] \left(\frac{\partial}{\partial q_\mu} - \frac{\partial}{\partial p_\mu}\right) \Gamma^{(0)}_{3\nu\lambda\sigma}(-q, -p, -r) \right\} \equiv \tilde{\Pi}^{\mathrm{UV}}_{\mu\nu}. \tag{23}$$

Using dimensional regularization we could translate variables in Eq. (23) and obtain

$$\tilde{\Pi}^{\mathrm{UV}}_{\mu\nu} = 0.$$

We choose not to do this and instead replace $\tilde{\Pi}_{\mu\nu}$ by

$$\tilde{\Pi}_{\mu\nu} - \tilde{\Pi}^{\mathrm{UV}}_{\mu\nu} \equiv \tilde{\Pi}^{\mathrm{Q}}_{\mu\nu}. \tag{24}$$

Our final expression for the vacuum polarization which is equivalent to performing both infra-red and ultra-violet regularization procedures is then obtained by replacing $\tilde{\Pi}_{\mu\nu}$ by $\tilde{\Pi}^{Q}_{\mu\nu}$ in Eq. (22), i.e.,

$$\tilde{\Pi}_{\mu\nu}(Z) \rightarrow \tilde{\Pi}^{Q}_{\mu\nu}(Z) - \tilde{\Pi}^{Q}_{\mu\nu}\left(\frac{M^2}{p^2}\right). \tag{25}$$

This replacement generates additional subtraction terms in Eq. (17) which remove from the equation potential quadratically divergent ultra-violet contributions as well as finite infra-red contributions to the gluon mass. In the Appendix we use the Schwinger–Dyson equation for $\tilde{\Pi}_{\mu\nu}$ along with the Slavnov–Taylor identity for Γ_3 to obtain Eq. (25).

5. *The low momentum structure of the integral equation*

Equation (17) can be written in the form

$$Z^{-1}(q^2) = 1 + g_0^2 \int dpK(p, q)Z(p^2) + \frac{g_0^2}{Z(q^2)} \int dpL(p, q)Z(p^2)Z(r^2), \tag{26}$$

where

$$K(p, q) = \frac{n \cdot rn \cdot (r-p)}{in^2 q^2 p^2 r^2} \frac{\Sigma_{\sigma\sigma'}}{(1-1/\gamma)} \left[g_{\sigma\sigma'} + \frac{p \cdot r g_{\sigma\sigma'} - r_\sigma p_{\sigma'}}{p^2 - r^2} + \frac{(q-r)_\sigma q_{\sigma'}}{r^2 - p^2} \right],$$

$$L(p, q) = \frac{n \cdot rn \cdot (r-p)}{in^2 q^2 p^2 r^2 (1-1/\gamma)} \Sigma_{\sigma\sigma'} \frac{(q-r)_\sigma q_{\sigma'}}{r^2 - q^2},$$

and

$$\Sigma_{\sigma\sigma'} = \Delta^{(0)}_{\lambda\sigma}(p)\Delta^{(0)}_{\lambda\sigma'}(r).$$

The subtractions in Eq. (25) generate additional terms in K and L which make all integrals in Eq. (26) infra-red unambiguous and guarantees the absence of a gluon mass.

We will now try to determine what low momentum behaviour of $Z(p^2)$ is compatible with Eq. (26) [9, 10]. This discussion will make use of the fact that K and L both contain a factor

$$\frac{1}{p^2 r^2} = \frac{1}{p^2 (p+q)^2}$$

and that the angular average of the bare propagator vanishes, Eq. (4). We will make various ansätz about the low momentum behaviour of $Z(p^2)$ to insert in the right-hand side of Eq. (26). For each input $Z_{\text{in}}(p^2)$, we will estimate the low q^2 behaviour of $Z^{-1}_{\text{out}}(q^2)$, the output obtained from the integration on the right-hand side, and will then compare for consistency. In most cases the arguments will depend mostly on the dimensions and convergence properties of the resulting integrals.

Assume

(a) $$Z_{\text{in}}(p) \xrightarrow[p^2 \to 0]{} \text{const}.$$

Then the logarithmic singularity at $q^2 = 0$ generated by the factor $1/p^2(p+q)^2$ in K and L yields an output [11]

$$Z_{\text{out}}^{-1}(q^2) \xrightarrow[q^2\to 0]{} 1+(\text{const}) \log q^2,$$

where 1 is the Born term on the right-hand side of Eq. (26).

Similarly,

(b)
$$Z_{\text{in}}(p^2) \xrightarrow[p^2\to 0]{} \log p^2$$

gives

$$Z_{\text{out}}^{-1}(q^2) \xrightarrow[q^2\to 0]{} 1+(\text{const}) (\log q^2)^2.$$

(c)
$$Z_{\text{in}}(p^2) \xrightarrow[p^2\to 0]{} \frac{1}{\log p^2}$$

gives

$$Z_{\text{out}}^{-1}(q^2) \xrightarrow[q^2\to 0]{} 1+(\text{const}) \log(\log q^2).$$

Perhaps these infra-red logarithms sum to a power. We try

(d)
$$Z_{\text{in}}(p^2) \xrightarrow[p^2\to 0]{} \left(\frac{M^2}{p^2}\right)^{\alpha}, \quad 0<\alpha<1.$$

The integrals in Eq. (26) are all infra-red finite and their low q^2 behaviour is determined by their dimension, i.e.,

$$Z_{\text{out}}^{-1}(q^2) \xrightarrow[q^2\to 0]{} 1+(\text{const}) \left(\frac{M^2}{q^2}\right)^{\alpha}.$$

Suppose

(e)
$$Z_{\text{in}}(p^2) \xrightarrow[p^2\to 0]{} \frac{M^2}{p^2}.$$

Then because of Eq. (4), the integrals in Eq. (26) are still infra-red finite and hence

$$Z_{\text{out}}^{-1}(q^2) \xrightarrow[q^2\to 0]{} 1+C\left(\frac{M^2}{q^2}\right),$$

where the constant C is a linear combination of the constants C_1 and C_2 appearing in Eq. (20). We have seen that gauge invariance implies $C_1 = C_2 = 0$. Hence $C = 0$ and consistent infra-red behaviour between Z_{in} and Z_{out}^{-1} is possible.

The two ingredients making possible a $1/q^2$ infra-red behaviour are Eq. (4), which makes C finite, and the gauge invariance of our approximation procedure which makes $C = 0$. If the axial gauge propagator did not satisfy Eq. (4), an arbitrary introduction of an infra-red cut-off would have generated an inconsistent Z_{out}^{-1} containing a factor $\log q^2$. Note that in example (d) a non-vanishing coefficient of the $(M^2/q^2)^{\alpha}$ output is to be expected since it does not violate gauge invariance. This output then dominates the Born term and is inconsistent with the input, just as in the first three examples.

We have seen that an M^2/p^2 input generates an output which is simply the Born term 1. To obtain consistency the output Z^{-1} must behave like q^2 for small q^2, i.e., the Born term 1 must be cancelled by small p^2 corrections to the leading M^2/p^2 behaviour of $Z_{in}(p^2)$. We now consider various possibilities for these corrections.

Suppose

(f) $$Z_{in}(p^2) \xrightarrow[p^2\to 0]{} \frac{M^2}{p^2} + \text{const.}$$

Then

$$Z_{out}^{-1}(q^2) \xrightarrow[q^2\to 0]{} 1 + (\text{const}) \log q^2.$$

The $\log q^2$ output is generated by the constant part of the input just as in Case (a). Thus there can be no constant term in $Z_{in}(p^2)$ which means there is no $1/p^2$ pole in $D_{\mu\nu}(p)$.

Suppose

(g) $$Z_{in}(p^2) \xrightarrow[p^2\to 0]{} \left(\frac{M^2}{p^2}\right) + \frac{1}{\log p^2}.$$

Then

$$Z_{out}^{-1}(q^2) \xrightarrow[q^2\to 0]{} 1 + \log(\log q^2),$$

which is also inconsistent.

We now try an input of the form

(h) $$Z_{in}(p^2) \xrightarrow[p^2\to 0]{} A\frac{M^2}{p^2} + B\frac{p^2}{M^2}.$$

Then

$$Z_{out}^{-1}(q^2) \xrightarrow[q^2\to 0]{} 1 + D_0 + D_1\frac{q^2}{M^2} + \dots .$$

Consistency requires that A and B be chosen so that D_0 cancels the Born term, i.e., $1+D_0 = 0$, and so that $D_1 = 1/A$, for all values of γ. The numerical study [6] of Eq. (26) discussed in Section 7 provides support for the idea that this can be done for the values of γ between 2 and 10 for which calculations were carried out.

The above discussion is, of course, very crude. We have presented it here to indicate how the direct coupling of Z to Z^{-1} in a gauge invariant manner which is characteristic of the Schwinger–Dyson equations of a non-Abelian gauge theory leads naturally to a gluon propagator behaving like $1/q^2$ for small q^2. This coupling distinguishes non-Abelian gauge theory from other self-coupled field theories like ϕ^3 or ϕ^4.

Finally, we ask how the above arguments change when we use the full vertex Γ_3 in the Schwinger–Dyson equation? To answer this question definitively we would have to carry out the next stage of our approximation scheme. However, qualitatively we note that the difference between the full vertex Γ_3 and our approximation $\Gamma_3^{(L)}$ is a transverse vertex, $\Gamma_3^{(T)}$ which contains an explicit factor $p\times q\times r$ [see Eq. (16)]. Thus the integrals generated by the contribution of $\Gamma_3^{(T)}$ to $\tilde{\Pi}_{\mu\nu}$ will be (because of $p \leftrightarrow -p$ symmetry) more convergent in the infra-red by a factor of p^2. They should therefore not give the dominant contributions to the infra-red singularity structure of $\tilde{\Pi}_{\mu\nu}$. Consequently the contribution of $\Gamma_3^{(T)}$ to $\tilde{\Pi}_{\mu\nu}$ should not change essentially the previous arguments suggesting a $(1/q^2)^2$ behaviour of the gluon propagator. However, we expect that $\Gamma_3^{(T)}$ will influence the value of the coefficient of the $1/q^2$ singularity in Z since it is determined by a cancellation between the Born term and the infra-red constant part of $\tilde{\Pi}_{\mu\nu}$. This constant part of $\tilde{\Pi}_{\mu\nu}$ should receive contributions from $\Gamma_3^{(T)}$. It is then plausible that the $1/q^2$ infra-red behaviour of Z may also be a feature of the full theory but that the coefficient of $1/q^2$ should differ from the value obtained from Eq. (26).

6. Renormalization of the integral equation for $Z(q^2)$ and the ultra-violet behaviour of the solution

Iteration of Eq. (27) generates the perturbation theory solution, possessing logarithmic ultra-violet divergences. We now renormalize these equations and show that the solutions, when expressed in terms of a suitably defined renormalized coupling constant, are ultra-violet finite. We will then use renormalization group to determine the large q^2 behaviour of the running coupling constant and the renormalized propagator $\tilde{Z}(q^2)$.

We define the renormalized propagator $\tilde{Z}(q^2)$ by the equation

$$Z(q^2) = Z(M^2)\tilde{Z}(q^2), \tag{27}$$

so that it is normalized at an arbitrary mass scale M^2, i.e.,

$$\tilde{Z}(q^2)|_{q^2=M^2} = 1. \tag{28}$$

Substituting Eq. (27) into Eq. (26) yields

$$\tilde{Z}^{-1}(q^2) = Z(M^2)+g_0^2Z^2(M^2)\int dpK(p,q)\tilde{Z}(p^2) + \frac{g_0^2Z^2(M^2)}{\tilde{Z}(q^2)}\int dpL(p,q)\tilde{Z}(p^2)\tilde{Z}(r^2). \tag{29}$$

The normalization condition Eq. (28) then becomes

$$1 = Z(M^2)+g_0^2Z^2(M^2)\int dpK(p,q)\tilde{Z}(p^2)|_{q=M} + g_0^2Z^2(M^2)\int dpL(p,q)\tilde{Z}(p^2)\tilde{Z}(r^2)|_{q=M}. \tag{30}$$

Subtracting Eq. (30) from Eq. (29) yields

$$\tilde{Z}^{-1}(q^2)-1 = g_0^2Z^2(M^2)\int dpK(p,q)\tilde{Z}(p^2)|_{\mathrm{R}} + g_0^2\frac{Z^2(M^2)}{\tilde{Z}(q^2)}\int dpL(p,q)\tilde{Z}(p^2)\tilde{Z}(r^2)|_{\mathrm{R}} - \left(1-\frac{1}{\tilde{Z}(q^2)}\right)g_0^2Z^2(M^2)\int dpL(p,q)\tilde{Z}(p^2)\tilde{Z}(r^2)|_{q=M}. \tag{31}$$

In Eq. (31) we have used the notation $|_R$ to indicate the difference between the integral evaluated at q and the integral evaluated at M, e.g.,

$$\int dpK(p,q)\tilde{Z}(p^2)|_R \equiv \int dpK(p,q)\tilde{Z}(p^2) - \int dpK(p,q)\tilde{Z}(p^2)|_{q=M}. \tag{32}$$

We now define a renormalized coupling constant $g^2(M^2)$ as

$$g^2(M^2) = \frac{g_0^2 Z^2(M^2)}{1 - g_0^2 Z^2(M^2) \int dpL(p,q)\tilde{Z}(p^2)\tilde{Z}(r^2)|_{q=M}}. \tag{33}$$

Then Eq. (31) can be written as

$$\tilde{Z}^{-1}(q^2) - 1 = g^2(M^2) \int dpK(p,q)\tilde{Z}(p^2)|_R + \frac{g^2(M^2)}{\tilde{Z}(q^2)} \int dpL(p,q)\tilde{Z}(p^2)\tilde{Z}(r^2)|_R. \tag{34}$$

Perturbation theory evaluation of Eq. (33) for $g^2(M^2)$ yields logarithmically divergent integrals, which arise both from the expansion of Eq. (30) for $Z(M^2)$ and from the explicit integral in the denominator of Eq. (33). We choose g_0^2 to depend upon the cut-off so that $g^2(M^2)$ is finite. Then Eq. (34), determining the renormalized propagator $\tilde{Z}(q^2)$ in terms of the renormalized coupling constant $g^2(M^2)$, has a solution which is finite to every order in perturbation theory because only subtracted integrals appear in the equation. We then write

$$\tilde{Z}(q^2) = \tilde{Z}\left(\frac{q^2}{M^2}, g^2(M^2)\right),$$

with

$$\tilde{Z}(1, g^2(M^2)) = 1. \tag{28'}$$

We want to solve Eq. (34) without recourse to perturbation theory. We have given arguments in the previous section that $Z(q^2)$ and hence $\tilde{Z}(q^2)$ should behave like $1/q^2$ as $q^2 \to 0$. We will now use the renormalization group to determine $\tilde{Z}(q^2)$ as $q^2 \to \infty$. These two limits will then provide a guide for seeking an approximate numerical solution for all q^2.

We first use Eq. (30) to rewrite Eq. (33) for $g^2(M^2)$ in the following way:

$$g^2(M^2) = \frac{g_0^2 Z(M^2)}{1 + g_0^2 Z(M^2) \int dpK(p,q)\tilde{Z}(p^2)|_M}. \tag{35}$$

Writing Eq. (35) for a different value, M'^2, and the eliminating g_0^2 gives $g^2(M'^2)$ in terms of $g^2(M^2)$ as

$$g^2(M'^2) = \frac{g^2(M'^2)\dfrac{Z(M'^2)}{Z(M^2)}}{1 + g^2(M^2)\left[\int dpK(p,q)\tilde{Z}(p^2)|_{q=M'} - \int dpK(p,q)\tilde{Z}(p^2)|_{q=M}\right]}. \tag{36}$$

From Eqs. (27) and (28′) we have

$$\frac{Z(M'^2)}{Z(M^2)} = \frac{\tilde{Z}\left(\frac{q^2}{M^2},\, g^2(M^2)\right)}{\tilde{Z}\left(\frac{q^2}{M'^2},\, g^2(M'^2)\right)},$$

since $Z(q^2)$ does not depend upon M. Setting $q^2 = M'^2$ then yields

$$\frac{Z(M'^2)}{Z(M^2)} = \tilde{Z}\left(\frac{M'^2}{M^2},\, g^2(M^2)\right).$$

Using this result in Eq. (36) and calling $M'^2 = q^2$ gives us

$$g^2(q^2) = \frac{g^2(M^2)\tilde{Z}\left(\frac{q^2}{M^2},\, g^2(M^2)\right)}{1 + g^2(M^2)\int dpK(p, q)\tilde{Z}(p^2)|_R}. \tag{37}$$

Equation (37) yields a finite expression for $g^2(q^2)$ in terms of $g^2(M^2)$, because $\tilde{Z}$ is finite and the denominator in the equation is also finite.

The relation, Eq. (37), between $g^2(q^2)$ and $\tilde{Z}(q^2)$ for our approximate equation differs from the relation

$$g^2(q^2) = q^2(M^2)\tilde{Z}\left(\frac{q^2}{M^2},\, g^2(M^2)\right) \tag{38}$$

valid in exact axial gauge Yang–Mills theory. However, they are compatible in the infra-red region for which our approximation procedure is appropriate for the following reason. We have argued that the $1/q^2$ infra-red behaviour of $\tilde{Z}$ may be independent of our approximation. If this is true, then Eq. (38) yields a $g^2(q^2)$ with this same infra-red behaviour. On the other hand, Eq. (37), valid in our approximation, also yields a $g^2(q^2)$ which behaves like $1/q^2$ for small q^2 if our solution for $\tilde{Z}(q^2)$ has this behaviour. In this case the effect of the denominator in Eq. (37) will be to modify the coefficient of $1/q^2$. This is compatible with the fact that we expect the coefficient of $1/q^2$ in $\tilde{Z}$ to differ from its value in the full theory.

In the intermediate and high q^2 region, Eq. (37) can give a significant difference between $g^2(q^2)$ and $\tilde{Z}(q^2)$. In such cases we must identify $g^2(q^2)$ rather than $\tilde{Z}(q^2)$ as the quantity of physical interest. This is because our approximate non-perturbative wave function renormalization constant correctly renormalizes the two point function but not the higher point functions since they have not been included in the first stage of our approximation. (In the next stage when we include Γ_3 it will also renormalize the three-point function, etc.)

When comparing to experiment we must first carry out the exact renormalization of all multiparticle amplitudes, re-expressing them in terms of the exact physical coupling constant. We then can use our approximate $g^2(q^2)$ in these formulae to estimate the infra-red behaviour of the process. For example, the general renormalization argument relates the one-gluon exchange potential between a pair of heavy quarks to the axial gauge running coupling constant. We can then use our approximation to $g^2(q^2)$ to calculate the long-range part of the potential.

We now calculate $g^2(q^2)$ and $\tilde{Z}(q^2)$ for large q^2. We first note that by the usual argument $g^2(q^2)$ satisfies the renormalization group equation:

$$q\,\frac{dg}{dq} = \beta(g). \tag{39}$$

Expanding Eqs. (37) and (34) in a power series in $g^2(M^2)$ we obtain

$$g^2(q^2) = g^2(M^2)\,\{1 - g^2(M^2)\int dp(K(p,q)+L(p,q))|_{\mathrm{R}} - g^2(M^2)\int dpK(p,q)|_{\mathrm{R}}\} + \dots . \tag{40}$$

The $-(K+L)$ term in Eq. (40) arises from the $\tilde{Z}(q^2)$ term in the numerator of Eq. (37) and gives the usual perturbation theory result for $g^2(q^2)$. The $-K$ term comes from the denominator of Eq. (37) and reflects the particular form of the renormalization dictated by our integral equation. Equations (39) and (40) then give the following g^3 contribution to $\beta(g)$

$$\beta(g) = -\frac{g^3}{16\pi^2}\left[\tfrac{11}{3}N_c + \tfrac{5}{3}N_c\right] = -\frac{g^3}{16\pi^2}\left[\tfrac{16}{3}N_c\right]$$

$$= \tfrac{16}{11}\left[-\frac{g^3}{16\pi^2}\cdot\tfrac{11}{3}N_c\right]. \tag{41}$$

The usual $\frac{11}{3}N_c$ contribution comes from the $K+L$ term in Eq. (40), while the additional $\frac{5}{3}N_c$ arises from the K term. Because of the negative sign of β we can insert Eq. (41) into the right-hand side of Eq. (39) to obtain the large q^2 behaviour of $g^2(q^2)$. The result is clearly

$$g^2(q^2) \xrightarrow[q^2\to\infty]{} \frac{1}{\frac{16}{11}\,b_{\mathrm{YM}}\log\left(\dfrac{q^2}{M^2}\right)}, \tag{42}$$

where

$$b_{\mathrm{YM}} = \tfrac{11}{3}\,\frac{N_c}{16\pi^2}.$$

Thus our approximation, appropriate to the small q^2 region, yields a result for large q^2 which differs from the exact asymptotic formula by a factor $\frac{11}{16}$. It is then not unreasonable qualitatively even in the large q^2 domain.

Finally, from Eqs. (34) and (42) the standard renormalization group argument yields

$$\tilde{Z}(q^2) \xrightarrow[q^2\to\infty]{} \frac{\text{const.}}{\left(\log\left(\dfrac{q^2}{M^2}\right)\right)^{16/11}}. \tag{43}$$

In this case the factor $\frac{16}{11}$ appears in the power of the logarithm. This vividly reflects the fact that our approximate $\tilde{Z}(q^2)$ cannot be identified with the renormalized coupling constant.

7. *Approximate numerical solution [6]*

We have seen that $\tilde{Z}(q^2)$ has the large q^2 behaviour of Eq. (43) and for small q^2 we expect that it behaves like

$$Z(q^2) \xrightarrow[q^2\to 0]{} A\left(\frac{M^2}{q^2}\right) + B\left(\frac{q^2}{M^2}\right). \tag{44}$$

Guided by these limits, Ball and Zachariasen [6] used the following three-parameter trial function to seek a numerical solution of Eq. (34):

$$\tilde{Z}(q^2) = \frac{AM^2}{q^2} + \frac{(1-A)q^2(M^2+\mu_2^2)}{M^2(q^2+\mu_2^2)}\left(\frac{1}{1+\frac{16}{11}\,g^2(M^2)\log\left(\frac{q^2+\mu_3^2}{M^2+\mu_3^2}\right)}\right)^{11/16}. \tag{45}$$

The function, Eq. (45), is normalized [Eq. (28)] and has the asymptotic behaviours given by Eqs. (43) and (44) for all values of the parameters A, μ_2^2 and μ_3^2. This trial function was then inserted into the right-hand side of Eq. (34) and the parameters were varied so that the output matched the input for as large a range of q^2 as was compatible with the accuracy of the numerical integrations. In Fig. 2, the solid curve is a plot of $\tilde{Z}^{-1}(q^2)$, Eq. (45), obtained from using the best values of the parameters in the case $g^2(M^2) = 2\pi$. For this curve $A = 0.375$, $\mu_2 = 0.5M^2$ and $\mu_3 = 0.6M^2$. The dots on the curve represent

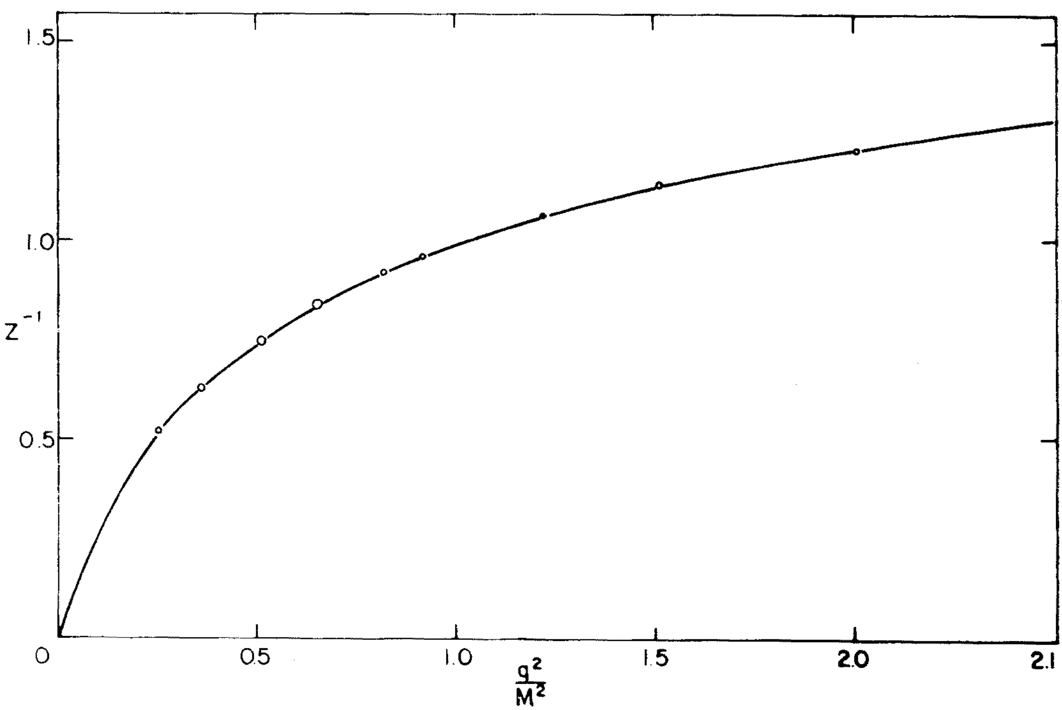

Fig. 2. $\tilde{Z}^{-1}(q^2)$ as given by Eq. (45) with $A = 0.375$, $\mu_2^2 = 0.5\,M^2$, $\mu_3^2 = 0.6\,M^2$, for $g^2(M^2) = 2\pi$

the output values of $\tilde{Z}^{-1}(q^2)$ calculated from the integrals on the right-hand side of Eq. (34). The thickness of the dots represents the difference between the input and output values of $\tilde{Z}^{-1}$ at various values of q^2. The agreement between input and output is within two per cent for all values of q^2 where calculations were made. Thus we feel that it is likely that our equation has a solution which varies continuously from its large q^2 behaviour, Eq. (43), to the singular low q^2 behaviour given by Eq. (44).

The output points on Fig. 2 were calculated for a value of $\gamma = 4$. The output points obtained by using $\gamma = 2$ in the integral equation are essentially indistinguishable from these. For $\gamma \approx 10$ the output deviates from the input by about 5% at intermediate values of q^2. Since we only expect that $\tilde{Z}(q^2)$ should be independent of γ in the infra-red and ultra--violet regions some γ dependence might be expected. It is encouraging that it is not too drastic. This dependence is now being further investigated by Ball.

Knowing the solution of Eq. (34) for one value of $g^2(M_1^2)$, one can obtain it for any other value $g^2(M_2^2)$ from the equation

$$\tilde{Z}\left(\frac{q^2}{M_2^2},\ g(M_2^2)\right) = \frac{\tilde{Z}\left(\frac{q^2}{M_1^2},\ g(M_1^2)\right)}{\tilde{Z}\left(\frac{M_2^2}{M_1^2},\ g(M_1^2)\right)}. \tag{46}$$

Equation (46) was actually used to extend the range of q^2 for which the input and the output could be compared.

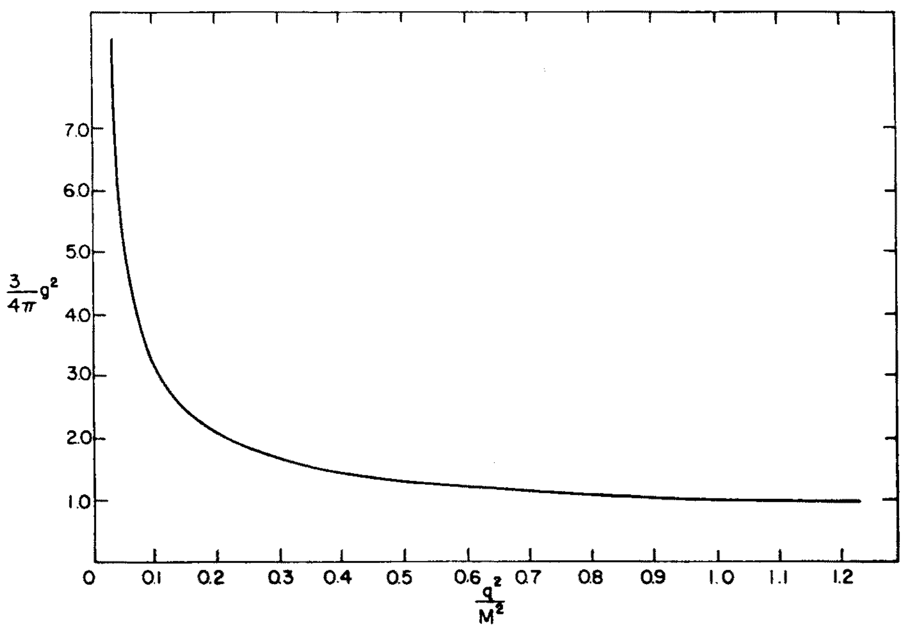

Fig. 3. $\frac{3}{4\pi} g^2(q^2)$ calculated from the solution for $\tilde{Z}^{-1}(q^2)$ of Fig. 2, using Eq. (37) for $g^2(q^2)$

Using the solution, Eq. (45), for $\tilde{Z}(q^2)$ we can calculate the running coupling constant from Eq. (37) as a function of q^2/M^2. The value of the scale M^2 was arbitrary. We define M^2 by the normalization condition

$$\frac{3}{4\pi} g^2(M^2) = 1. \tag{47}$$

With this definition of M^2, the function $\frac{3}{4\pi} g^2(q^2)$, obtained by substituting Eq. (45) into Eq. (37), is plotted in Fig. 3 as a function of q^2/M^2. In the limit as $q^2 \to 0$, $\frac{3}{4\pi} g^2(q^2)$ has the $0.215(M^2/q^2)$ behaviour given by Eq. (9).

In Fig. 4, we plot $-\beta(g)/g$ as a function of $\sqrt{\frac{3}{4\pi}}\, g$. This curve starts from the weak coupling quadratic behaviour and makes a transition in the vicinity of $\frac{3}{4\pi} g^2 \approx 1$ to the constant behaviour for large g^2.

Finally we remark that Eq. (34) for $\tilde{Z}$ has been obtained by contracting the vacuum polarization tensor $\tilde{\Pi}_{\mu\nu}$ with $n^\mu n^\nu$. A second integral equation can be obtained by contracting

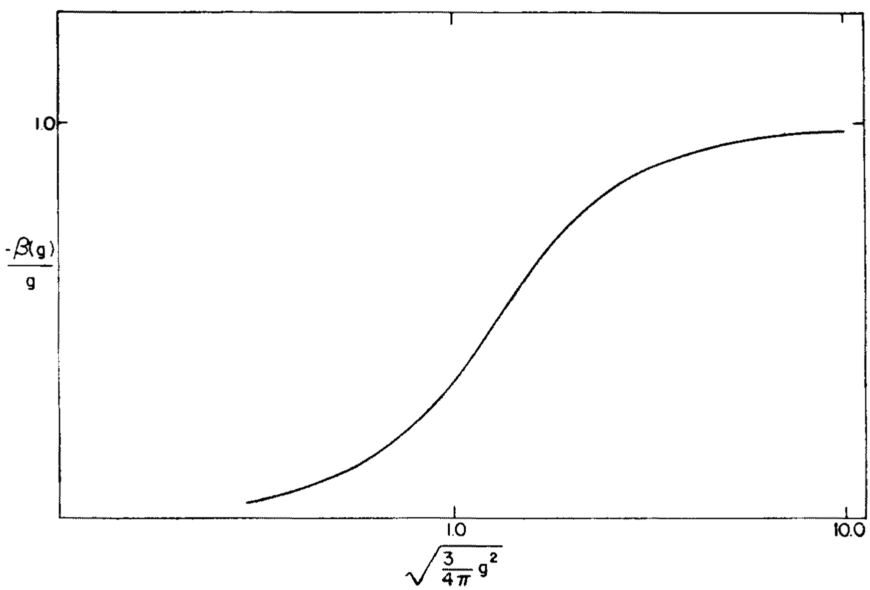

Fig. 4. $-\beta(g)/g$ vs $\sqrt{\frac{3}{4\pi}}\, g$

$\tilde{\Pi}_{\mu\nu}$ with $g^{\mu\nu}$. The third and the fourth terms in Fig. 1 (one of which involves Γ_4) contribute to this contraction. However, we have seen that it is consistent with Eq. (14) to omit these terms and it would be very interesting to check the extent to which our solution satisfies the equation for $g^{\mu\nu}\tilde{\Pi}_{\mu\nu}$ if we continue to neglect Γ_4.

8. *Qualitative comparison with data*

We now give a rough comparison of our calculated running coupling constant to a phenomenologically determined heavy quark potential used to fit spectroscopic data by Richardson [12]. His potential corresponds to a running coupling constant,

$$g_R(q^2) = \frac{1}{b_R \log\left(1 + \frac{q^2}{\Lambda^2}\right)}, \tag{48}$$

where

$$b_R = \frac{1}{16\pi^2}(11 - \tfrac{2}{3} N_f).$$

He takes $N_f = 3$ and finds a good fit with $\Lambda^2 = (400\text{ MeV})^2$, a charmed quark mass of 1490 MeV and a b quark mass of 4180 MeV.

In the limit of large q^2, Eq. (48) takes on the asymptotic freedom result, while for small q^2

$$\frac{3}{4\pi} g_R^2(q^2) \xrightarrow[q^2 \to 0]{} \frac{3}{4\pi} \frac{\Lambda^2}{b_R q^2} \approx \frac{4\Lambda^2}{q^2}. \tag{49}$$

The next term in the low q^2 expansion is a constant so his potential differs significantly from the solution of our equations. However, it is interesting to compare the coefficient of the $1/q^2$ term in Richardson's phenomenological potential with the low q^2 behaviour,

$$\frac{3}{4\pi} g^2(q^2) \xrightarrow[q^2 \to 0]{} 0.215 \frac{M^2}{q^2}$$

of the solution of our equation.

To make this comparison we must relate M^2 to Λ^2, which follows from the normalization condition, Eq. (47). Normalizing $g_R^2(q^2)$ in this way yields

$$M^2 = \Lambda^2(e^{3/4\pi b_R} - 1) \approx 65\Lambda^2 \approx 20(\text{GeV})^2. \tag{50}$$

Note that the -1 in Eq. (50) is negligible and the value of M^2 is essentially determined by the large q^2 behaviour of $g_R^2(q^2)$, which is just asymptotic freedom. Putting $M^2 \approx 65\,\Lambda^2$ in our solution we obtain

$$\frac{3}{4\pi} g^2(q^2) \xrightarrow[q^2 \to 0]{} 0.215\left(\frac{65\Lambda^2}{q^2}\right) \sim 14\Lambda^2/q^2 \sim \frac{2.3(\text{GeV})^2}{q^2}. \tag{51}$$

Comparing Eqs. (49) and (51) we see that the low momentum coupling of our solution is about three times stronger than that determined by Richardson.

This can be understood qualitatively by the fact that our solution has a suppressed high momentum behaviour because of the factor $\frac{16}{11}$ in Eq. (42) and because of the absence of fermions. Since, as seen from Fig. 3, the value of $\frac{3}{4\pi} g^2(q^2)$ near $q^2/M^2 \sim 1$ is essentially given by its asymptotic high momentum form, we must normalize it by an anomalously large factor to raise it to the value 1. This in turn increases the value of its low momentum behaviour.

We can estimate this effect by supposing that the asymptotic form of the exact theory agreed with our approximation, i.e., $b_R = \frac{11}{16} b_{YM}$. In this case $\frac{3}{4\pi b_R} = \frac{3\pi}{4}$, and thus

$$\frac{3}{4\pi} g_R^2(q^2) \to \frac{3\pi}{4} \Lambda^2/q^2 \sim 2.5\Lambda^2/q^2. \tag{49'}$$

From Eq. (50) we would obtain $M^2 \approx 11\Lambda^2$, and hence our solution would have the low momentum behaviour

$$\frac{3}{4\pi} g^2(q^2) \approx \frac{(0.21)}{q^2}(11\Lambda^2) \sim 2.3\Lambda^2/q^2, \tag{51'}$$

in close agreement with Eq. (49′).

The point of this discussion is that it is difficult to make a precise comparison of the low q^2 behaviour with experiment because the normalization condition involves the large q^2 behaviour which differs by a factor $\frac{11}{16}$ from the correct value. The comparison will then be somewhat sensitive to how one accounts for this factor as well as for fermions. E.g., we could either choose to compare Eq. (49) with Eq . (51) or Eq. (49′) with Eq. (51′). However, in either case our zero parameter calculation yields a long-range one-gluon force of qualitatively the right strength.

Altarelli [13] has suggested other ways to check the implications of our calculation. Recall that Eqs. (50) and (51) only make use of the asymptotic freedom result for $g^2(q^2)$ since the -1 in Eq. (50) is negligible. The estimate, Eq. (50), for M^2 then does not depend upon Richardson's fit. From Fig. 2, we note that $\frac{3}{4\pi} g^2(q^2)$ does not begin to deviate significantly from its asymptotic freedom expression unless q^2 is of order 0.1 to 0.2 M^2. Using $M^2 \approx 20\,(\text{GeV})^2$ from Eq. (50), we see that our solution predicts no substantial deviation from asymptotic freedom down to values of a few $(\text{GeV})^2$, which is consistent with experiment. It is also interesting to evaluate the value of our running coupling constant at a nucleon size. Putting $q^2 = m_\pi^2$ in Eq. (51) yields

$$g^2/4\pi \approx \tfrac{1}{3} \frac{(2.3)}{(0.02)} \approx 40,$$

which is of the order of magnitude of the expected size of the strong interaction coupling, at a pion Compton wavelength.

The discussion is clearly only semi-quantitative, but it indicates that our low q^2 running coupling constant is not unreasonable from an empirical point of view.

9. *Summary*

We have presented a systematic approach to calculating the infra-red behaviour of Yang–Mills theory using the Schwinger–Dyson equations and the Ward identities. It is argued that this theory leads to an axial gauge running coupling constant which behaves like $1/q^2$ as $q^2 \to 0$. In the first stage of our approximation scheme we have an explicit zero parameter expression for $g^2(q^2)$ given by the curve of Fig. 3. The strength of the long-range confining part of this curve is of the order of magnitude of the value obtained from spectroscopic data on heavy quark systems as well as from other empirical estimates.

Our program, when carried to the next stage, will hopefully make clearer the extent of the generality of the $1/q^2$ infra-red behaviour of $g^2(q^2)$. It also might give us some estimate of the two-gluon exchange force.

APPENDIX

Ward's identity and the vanishing of the gluon mass

Starting from Eq. (10) for $\tilde{\Pi}_{\mu\nu}$ and assuming that Γ_3 satisfies the Slavnov–Taylor identity, Eq. (12), it is easy to derive the following identity

$$\tilde{\Pi}^{Q}_{\mu\nu} \equiv \tilde{\Pi}_{\mu\nu} - \tilde{\Pi}^{UV}_{\mu\nu} = -\frac{ig_0^2}{2}\int dp q_\delta \frac{\partial}{\partial q_\mu}[\Gamma^{(0)}_{3\nu\lambda\sigma}(-q, -p, -r)D_{\lambda\lambda'}(p)D_{\sigma\sigma'}(r)\Gamma_{3\delta\lambda'\sigma'}(q, p, r)], \tag{A.1}$$

where $\tilde{\Pi}^{UV}_{\mu\nu}$ is given by Eq. (23). If we use dimensional regularization $\tilde{\Pi}^{UV}_{\mu\nu} = 0$, and $\tilde{\Pi}^{Q}_{\mu\nu} = \tilde{\Pi}_{\mu\nu}$. Equivalently, we can simply replace $\tilde{\Pi}_{\mu\nu}$ by $\tilde{\Pi}^{Q}_{\mu\nu}$ when we calculate the vacuum polarization. Because of the derivative on the right-hand side of Eq. (A.1), $\tilde{\Pi}^{Q}_{\mu\nu}$ contains only logarithmic ultra-violet divergences. One can show that $\tilde{\Pi}^{UV}_{\mu\nu}$ vanishes if we put $Z = 1/p^2$. Hence $\tilde{\Pi}^{Q}_{\mu\nu}$, like $\tilde{\Pi}_{\mu\nu}$, is infrared finite, if Z is set equal to $1/p^2$. Furthermore, Eq. (A.1) implies

$$\lim_{q\to 0} \tilde{\Pi}^{Q}_{\mu\nu}(q) = 0, \tag{A.2}$$

provided we can take the $\lim_{q\to 0}$ under the integral sign in the integral on the right-hand side of Eq. (A.1). Now if Z did not contain a strong infra-red singularity this integral would be sufficiently convergent to justify the interchange of the limit with the integral sign. Hence we would obtain a vanishing gluon mass by Eq. (A.2).

On the other hand, we have seen that $\tilde{\Pi}^{Q}_{\mu\nu}$, although infra-red finite, is not absolutely convergent and hence depends upon the infra-red regularization procedure. We can remove this ambiguity by regulating the integral in Eq. (A.1) so that the $\lim_{q\to 0}$ can be taken under the integral sign. This eliminates possible finite infra-red contributions to the gluon mass just as dimensional regularization eliminates potential quadratically divergent ultra--violet contributions.

Equivalently if we do not use an infra-red regularization procedure which implements (A.2), we then must subtract $\lim_{q\to 0} \tilde{\Pi}^{Q}_{\mu\nu}(q)$ from $\tilde{\Pi}^{Q}_{\mu\nu}(q)$, just as we must subtract $\tilde{\Pi}^{UV}_{\mu\nu}$

from $\tilde{\Pi}_{\mu\nu}$ if we do not use dimensional regularization. Thus we see that gauge invariance dictates that we must replace $\tilde{\Pi}_{\mu\nu}(q)$ by

$$\tilde{\Pi}^{Q}_{\mu\nu}(q) - \lim_{q\to 0} \tilde{\Pi}^{Q}_{\mu\nu}(q), \tag{A.3}$$

which gives us Eq. (25) when $\tilde{\Pi}_{\mu\nu}(q)$ depends only upon Z.

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[10] These arguments will be similar to those used by Mandelstam to study the low momentum behaviour of his equation and they lead to similar results. I am indebted to S. Mandelstam for a helpful discussion of this point.
[11] In the discussion that follows, γ is held fixed as $q^2 \to 0$. The constants entering into the output are functions of γ and consistency must be obtained for all values of γ.
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