SCALAR TENSOR THEORIES IN *P*-SPACE TIMES

BY S. N. PANDEY

Department of Applied Sciences, M.M.M. Engineering College, Gorakhpur*

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The vacuum field equations of scalar meson, Sen–Dunn scalar tensor theory and that of Brans–Dicke scalar tensor theory are studied in *P*-space times in a unified way.

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1. Introduction

The plane wave solutions of Bondi et al. [1] of Einstein equations are not really plane as a plane electromagnetic wave in Maxwell's theory. But, there do exist homogeneous plane gravitational waves, according to Ebner [2], for which g_{ij} can be put in the form

$$g_{ij} = g_{ij}(Z), \quad Z = \sum a_i x^i, \quad g^{ij} Z_{,i} Z_{,j} = 0. \tag{1.1}$$

Using (1.1), Takeno [3] defined the plane gravitational wave, and studied the wave solutions of various field equations in general relativity and those of non-symmetric unified field theories. In his studies, the space times $\boldsymbol{P}$ and $\boldsymbol{H}$ play an important role, and they are interpreted as "plane wave like" and "plane wave" solutions, respectively. The former is characterized by the following two tensor conditions:

(*i*) It admits a parallel null vector field.

(*ii*) It is a gravitational null field.

On the other hand, in studying the wave solutions of various field equations in general relativity we obtained a space time represented by the line element [4–6]

$$ds^2 = -Adx^2 - Bdy^2 - (1-E)dz^2 - 2Edzdt + (1+E)dt^2,$$
$$A = A(Z), \quad B = B(Z), \quad E = E(x, y, Z), \quad (Z \equiv z - t), \tag{1.2}$$

in cartesian like coordinate system $x^i = (x, y, z, t)$. This space time is a $\boldsymbol{P}$ as it satisfies the conditions (*i*) and (*ii*). Further, the simplest form of the line element of $\boldsymbol{P}$ is obtained if we put $A = B = 1$. We call them $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ respectively. An important difference between

* Address: Department of Applied Sciences, M. M. M. Engineering College, Gorakhpur-273010, India.

them lies in the energy content of the gravitational wave. In P_1 gravitational wave does carry some energy and momentum in the direction of propagation whereas in P_2 it does not do so [6].

Therefore we consider here in the coordinate system of P_1, the plane wave like solutions of scalar tensor theories, namely, scalar meson field equation [7], scalar tensor theory of Sen–Dunn [8] and Brans–Dicke scalar tensor theory [9]. These theories are coupled with the gravitational field via Einstein's field equations.

2. *Field equation*

In vacuum the field equations for a scalar meson field are [7]

$$R_{ij}-\tfrac{1}{2}g_{ij}R = -\varphi_{,i}\varphi_{,j}+\tfrac{1}{2}g_{ij}(\varphi_{,l}\varphi^{,l}-m^2\varphi^2) \tag{2.1}$$

where we have set $K = 1$, and where φ is the meson field and m is the mass of the meson. Equation (2.1) can easily be rewritten as

$$R_{ij} = -\varphi_{,i}\varphi_{,j}+\tfrac{1}{2}g_{ij}m^2\varphi^2. \tag{2.2}$$

Similarly the vacuum field equations of Sen–Dunn scalar tensor theory [8]

$$R_{ij}-\tfrac{1}{2}g_{ij}R = \sigma_{,i}\sigma_{,j}-\tfrac{1}{2}g_{ij}\sigma_{,l}\sigma^{,l}, \tag{2.3}$$

where $e^{\sigma\sqrt{2/3}}$ is the scalar field in the theory, can be rewritten in the form

$$R_{ij} = \sigma_{,i}\sigma_{,j}. \tag{2.4}$$

The vacuum field equations of Brans–Dicke scalar tensor theory [9]

$$R_{ij}-\tfrac{1}{2}g_{ij}R = -(\omega/\psi^2)[\psi_{,i}\psi_{,j}-\tfrac{1}{2}g_{ij}\psi_{,l}\psi^{,l}]-(1/\psi)(\psi_{,i;j}-g_{ij}\Box\psi) \tag{2.5}$$

can accordingly be put in the form

$$R_{ij} = -(\omega/\psi^2)\psi_{,i}\psi_{,j}-(1/\psi)(\psi_{,i;j}+\tfrac{1}{2}g_{ij}\Box\psi), \tag{2.6}$$

where ψ is the scalar field, ω the dimensionless constant and $\Box\psi \equiv \psi_{;i}{}^{;i}$. Here a coma and semicolon denote partial and covariant differentiations respectively.

The equation (2.6) can further be reduced to the form

$$R^*_{ij} = -(\omega+\tfrac{3}{2})\psi^*_{,i}\psi^*_{,j} \tag{2.7}$$

if one sets

$$g^*_{ij} = \psi g_{ij},$$

$$R^*_{ij} \equiv \text{the Ricci tensor formed out of } g^*_{ij},$$

$$\psi^* = \log\psi. \tag{2.8}$$

Thus, it is evident that the vacuum field equations of these theories, that is, (2.2), (2.4) and (2.7) are quite similar in appearance although they are different as they describe different systems.

Therefore, we propose to consider the field equation

$$R_{ij} = \alpha\mu_{,i}\mu_{,j}, \quad (\alpha \neq 0) \tag{2.9}$$

in the coordinate system of $\boldsymbol{P}_1$, where μ can be assigned the role of scalar fields for the above theories. If $\alpha = 0$ or $\mu = \text{const}$ the field equation (2.9) reduces to the Einstein vacuum field equation.

3. *Solution*

In the coordinate system of $\boldsymbol{P}_1$, it can easily be seen that

$$\mu = \mu(Z). \tag{3.1}$$

Further, it should be mentioned that the Brans–Dicke metric g_{ij} also takes the form under consideration on performing the coordinate transformation

$$z^* = \tfrac{1}{2}(z+t)+\tfrac{1}{2}\int \psi dZ,$$

$$t^* = \tfrac{1}{2}(z+t)-\tfrac{1}{2}\int \psi dZ. \tag{3.2}$$

Therefore the solutions of field equation (2.9) are

(a) $\mu = \text{const}$, vacuum Ricci tensor,

(b) $$\alpha\mu'^2 - K = 0, \tag{3.3}$$

where

$$K = (A''/A+B''/B)/2-(A'^2/A^2+B'^2/B^2)/4-(E_{,11}/A+E_{,22}/B)/2,$$

$$(A' = \partial A/\partial Z,\ E_{,1} = \partial E/\partial x,\ \text{etc.}). \tag{3.4}$$

This last implies that K depends only on Z, and hence that the metric coefficient E satisfies

$$\text{a)} \quad M = 0 \quad \text{or} \quad \text{b)}\ M = f(Z), \tag{3.5}$$

where

$$M = E_{,11}/A+E_{,22}/B.$$

This is true directly of g_{ij} in both the meson and Sen–Dunn cases. In the former case this means that the solution of massive scalar meson field in vacuum does not exist in $\boldsymbol{P}_1$.

Further it is true of $E^* = \psi^*E$ in the Brans–Dicke case whence it is also true of E in this third case.

The solution

$$\mu = \int (K/\alpha)^{1/2} dz \tag{3.6}$$

gives ψ and σ in the first two cases and gives ψ^* in terms of A^*, B^* and E^* ($= \psi^*A$, ψ^*B and ψ^*E) in the third case using

$$Z^* = \int \psi dz = z^* - t^*.$$

Thus even in this third case all solutions can be generated by choosing A^*, B^* and E^*, though it is not as easy as in the first two cases to solve for ψ with a given A, B and E: for this the second order equation is still required.

4. Remarks

As is seen in the above, the scalar fields of these theories depend on $Z \equiv z-t$ in the coordinate system under consideration. Also there is no restriction imposed on the dependence of E on Z. However, (3.5a) in $\boldsymbol{P}_2$ means that E is a harmonic function of x and y. If so, then $\boldsymbol{P}_2$ identically satisfies the Einstein vacuum field equation. This implies that $\mu = \text{const}$. Hence $\boldsymbol{P}_2$ in which E is a harmonic function of x and y can not be the solution of these scalar tensor theories.

Further, a form of E that satisfies (3.5b) is

$$E = x^2F_1+2xyF_2+y^2F_3+xF_4+yF_5+F_6, \quad F_i = F_i(Z), \quad i = 1, \ldots, 6. \tag{4.1}$$

So, $\boldsymbol{P}_1$ (or $\boldsymbol{P}_2$) with (4.1) satisfies $E_{,abc} = 0$, $(a, b, c = 1,2)$, and hence it is an $\boldsymbol{H}$ [3], [6]. Hence the solutions of these scalar tensor theories in $\boldsymbol{P}$ where E is given by (4.1) are "plane wave solutions".

Furthermore, the value of dimensionless constant in Brans–Dicke theory in $\boldsymbol{P}_1$ is given by

$$\omega = (\psi/\psi')^2(M-L-\psi''/\psi') \tag{4.2}$$

where

$$L = (\sqrt{A})''/\sqrt{A}+(\sqrt{B})''/\sqrt{B}$$

and is positive or negative according as

$$M > \text{or} < L+\psi''/\psi'. \tag{4.3}$$

Obviously in $\boldsymbol{P}_2$, $L \equiv 0$, and then (4.2) becomes

$$\omega = (\psi/\psi')^2(M-\psi''/\psi'). \tag{4.4}$$

Editorial note. This article was proofread by the editors only, not by the author.

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