THE NON-RELATIVISTIC SPACE-TIME MANIFOLDS

By E. Kapuścik

Institute of Nuclear Physics, Cracow*

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A new notion of non-relativistic space-time is proposed. The dimension of the space-time manifold depends on the order of approximation.

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The geometry of space-time in special relativity is determined by the group of Poincaré transformations:

$$\vec{x}' = \mathcal{R}\vec{x}+\gamma\vec{v}t+\vec{a}, \tag{1}$$

$$t' = \gamma t+\frac{(\vec{v}\cdot\mathcal{R}\vec{x})}{c^2}+b, \tag{2}$$

where

$$\gamma = \left(1-\frac{v^2}{c^2}\right)^{-1/2}, \tag{3}$$

and the matrix elements $\mathcal{R}_{ik}$ of the 3×3 matrix $\mathcal{R}$ satisfy the condition

$$\sum_{k=1}^{3}\mathcal{R}_{kl}\mathcal{R}_{kn} = \delta_{ln}+\sum_{k,s=1}^{3}\frac{v_k v_s\mathcal{R}_{kl}\mathcal{R}_{sn}}{c^2}. \tag{4}$$

In (1) and (2) $\vec{v}$ denotes the relative velocity of two inertial frames of reference in which the space-time coordinates are given by $(\vec{x}, t)$ and $(\vec{x}', t')$, respectively, while $\vec{a}$ and b describe the corresponding translations. The matrix $\mathcal{R}$ may be written in terms of the orthogonal matrix R which describes the pure rotation:

$$\mathcal{R}_{ik} = R_{ik}+\frac{\gamma^2}{\gamma+1}\cdot\frac{v_i}{c^2}\cdot\sum_{l=1}^{3}v_l R_{lk}. \tag{5}$$

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* Address: Instytut Fizyki Jądrowej, Radzikowskiego 152, 31-342 Kraków, Poland.

The standard way of getting the non-relativistic approximation to the above formulae is to take in them the formal limit $c^2 \to \infty$. In this way from (1) and (2) we get the Galileian transformations:

$$\vec{x}' = R\vec{x} + \vec{v}t + \vec{a}, \tag{6}$$

$$t' = t + b. \tag{7}$$

For the goals of non-relativistic physics it is necessary, however, to use the one-parameter extension of the Galileian group. This extension is obtained by adding to the four space-time coordinates $(\vec{x}, t)$ a new quantity θ with following transformation property

$$\theta' = \theta + \vec{v} \cdot R\vec{x} + \tfrac{1}{2} v^2 t + \varphi. \tag{8}$$

The appearance of this fifth "coordinate" looks mysteriously from the point of view of our "usual" non-relativistic intuition. Below, we shall find a satisfactory resolution of this puzzle by a suitable definition of the notion of non-relativistic limit of the Poincaré transformations.

The second question which we are going to explain is connected with the impossibility of taking in (1) and (2) the whole expansions of the right-hand sides into power series in c^{-2}. It is expected that if we could use such expansions we would obtain better "non--relativistic" approximations to (1) and (2) which presumably could describe some relativistic effects. It is well-known that such way is forbidden because it leads immediately to the violation of the group property of space-time transformations. Our method provides a new notion of non-relativistic space-time in which the group property of the relativity transformations is maintained.

The method which we propose to use in the investigation of the non-relativistic limit of the transformation properties (1) and (2) rests on the following observation. Assuming that the space-time coordinates $(\vec{x}, t)$ in a given inertial frame of reference are quantities of some fixed order of magnitude, we obtain from (1) and (2) that, in the non-relativistic domain, the transformed coordinates $(\vec{x}', t')$ are in general sums of quantities of different order of magnitude. Since the space-time coordinates in each inertial frame may be treated as those obtained from the space-time coordinates in another inertial frame, we shall assume that, for the purpose of non-relativistic physics, the space-time coordinates are represented as sums:

$$\vec{x} = \sum_{n=0}^{\infty} \vec{x}_n, \tag{9}$$

$$t = \sum_{n=0}^{\infty} t_n, \tag{10}$$

where on the right-hand sides the symbols with subscripts n denote quantities of n-th order of smallness. Obviously, in order to make this statement precise we should define what the order of smallness means. We shall not do that, however, because it is not necessary for our purpose. For us it is necessary only to rememeber that the product of the quantity of the n-th order of smallness by the quantity of the m-th order of smallness is a quantity of the order $(n+m)$.

Let us now decide how to treat the other quantities, like $\vec{v}$, $\mathscr{R}$, $\vec{a}$ and b, in (1) and (2). Looking at the composition laws for these quantities:

$$\vec{v}_{12} = \frac{\gamma_1\vec{v}_1+\mathscr{R}_1\vec{v}_2}{\gamma_1+\dfrac{(\vec{v}_1\cdot\mathscr{R}_1\vec{v}_2)}{c^2}}, \tag{11}$$

$$(\mathscr{R}_{12})_{ik} = (\mathscr{R}_1\mathscr{R}_2)_{ik}+\gamma_1\frac{v_{1i}}{c^2}\sum_{l=1}^{3} v_{2l}\mathscr{R}_{2lk}, \tag{12}$$

$$\vec{a}_{12} = \vec{a}_1+\mathscr{R}_1\vec{a}_2+\gamma_1\vec{v}_1 b_2, \tag{13}$$

$$b_{12} = b_1+\gamma_1 b_2+\frac{(\vec{v}_1\cdot\mathscr{R}_1\vec{a}_2)}{c^2}, \tag{14}$$

it is easily seen that the assumption that they are quantities of fixed order of smallness immediately leads to a contradiction. In order to see this let us look, for example, at the formula (11). Assuming that $\vec{v}_1$ and $\vec{v}_2$ are quantities of the first order of smallnes, we get from (11) that $\vec{v}_{12}$ is a sum of quantities of all odd orders of smallness. Since every velocity may be regarded as composed of two or more smaller velocities, the only assumption consistent with the composition law (11) for velocities is to represent all $\vec{v}$'s as sums:

$$\vec{v} = \sum_{n=0}^{\infty} \vec{v}_{2n+1}, \tag{15}$$

where $\vec{v}_n$ are quantities of the n-th order of smallness.

A similar argument, applied to (12), shows that the matrices $\mathscr{R}$ must be represented as

$$\mathscr{R} = \sum_{n=0}^{\infty} \mathscr{R}_{2n}, \tag{16}$$

where, from (5), it follows that

$$\mathscr{R}_0 = R. \tag{17}$$

Finally, from (13) and (14) we get the representations for $\vec{a}$ and b:

$$\vec{a} = \sum_{n=0}^{\infty} \vec{a}_n, \tag{18}$$

$$b = \sum_{n=0}^{\infty} b_n. \tag{19}$$

Substituting into (1) and (2) all these representations and comparing the terms of the same order of smallness on both sides, we obtain an infinite set of transformation properties for $\vec{x}_n$ and t_n. Explicitly, up to quantities of the third order of smallness, we have:

$$\vec{x}'_0 = R\vec{x}_0+\vec{a}, \tag{20}$$

$$t'_0 = t_0+b, \tag{21}$$

$$\vec{x}'_1 = R\vec{x}_1+\vec{v}t_0+\vec{d}, \tag{22}$$

$$t'_1 = t_1+\frac{(\vec{v}\cdot R\vec{x}_0)}{c^2}+e, \tag{23}$$

$$\vec{x}'_2 = R\vec{x}_2+Q\vec{x}_0+\frac{(\vec{v}\cdot R\vec{x}_0)}{2c^2}\vec{v}+\vec{v}t_1+\vec{f}, \tag{24}$$

$$t'_2 = t_2+\frac{v^2}{2c^2}t_0+\frac{(\vec{v}\cdot R\vec{x}_1)}{c^2}+g, \tag{25}$$

$$\vec{x}'_3 = R\vec{x}_3+Q\vec{x}_1+\frac{(\vec{v}\cdot R\vec{x}_1)}{2c^2}\vec{v}+\vec{v}t_2+\vec{u}t_0+\vec{h}, \tag{26}$$

$$t'_3 = t_3+\frac{v^2}{2c^2}t_1+\frac{(\vec{v}\cdot R\vec{x}_2)+(\vec{v}\cdot Q\vec{x}_0)+(\vec{u}\cdot R\vec{x}_0)}{c^2}+k, \tag{27}$$

where for latter convenience we change shlighty the notation:

$$\vec{a}_0 = \vec{a}, \quad \vec{a}_1 = \vec{d}, \quad \vec{a}_2 = \vec{f}, \quad \vec{a}_3 = \vec{h},$$

$$b_0 = b, \quad b_1 = e, \quad b_2 = g, \quad b_3 = k,$$

$$\vec{v}_1 = \vec{v}, \quad \vec{u} = \frac{v^2}{2c^2}\vec{v}+\vec{v}_3,$$

$$(\mathscr{R}_2)_{ik} = \frac{v_i}{2c^2}\sum_{l=1}^{3} v_l R_{lk}+Q_{ik}. \tag{28}$$

From the condition (4) we get the following condition for the matrix Q:

$$R^TQ+QR^T = 0. \tag{29}$$

Under this condition, the transformation laws (20)–(27) possess the group property with the following composition laws:

$$R_{12} = R_1R_2, \tag{30}$$

$$(Q_{12})_{ik} = (R_1Q_2+Q_1R_2)_{ik}+\frac{v_{1i}\sum_{i=1}^{3} v_{2l}R_{2lk}-(R_1\vec{v}_2)_i\sum_{l=1}^{3} v_{1l}(R_1R_2)_{lk}}{2c^2}, \tag{31}$$

$$\vec{v}_{12} = v_1 + R_1\overleftarrow{v}_2, \tag{32}$$

$$\vec{u}_{12} = \vec{u}_1 + R_1\vec{u}_2 + Q_1\vec{v}_2 + \frac{(\vec{v}_1 \cdot R_1\vec{v}_2) + v_2^2}{2c^2}\vec{v}_1, \tag{33}$$

$$\vec{a}_{12} = \vec{a}_1 + R_1\vec{a}_2, \tag{34}$$

$$b_{12} = b_1 + b_2, \tag{35}$$

$$\vec{d}_{12} = \vec{d}_1 + R_1\vec{d}_2 + \vec{v}_1 b_2, \tag{36}$$

$$e_{12} = e_1 + e_2 + \frac{(\vec{v}_1 \cdot R_1\vec{a}_2)}{c^2}, \tag{37}$$

$$f_{12} = \vec{f}_1 + R_1\vec{f}_2 + Q_1\vec{a}_2 + \vec{v}_1 e_2 + \frac{(\vec{v}_1 \cdot R_1\vec{a}_2)}{2c^2}, \tag{38}$$

$$g_{12} = g_1 + g_2 + \frac{v_1^2 b_2 + 2(\vec{v}_1 \cdot R_1\vec{d}_2)}{2c^2}, \tag{39}$$

$$\vec{h}_{12} = \vec{h}_1 + R_1\vec{h}_2 + Q_1\vec{d}_2 + \frac{(\vec{v}_1 \cdot R_1 d_2)}{2c^2}\vec{v}_1 + \vec{v}_1 g_2 + \vec{u}_1 b_2, \tag{40}$$

$$k_{12} = k_1 + k_2 + \frac{v_1^2 e_2}{2c^2} + \frac{(\vec{v}_1 \cdot R_1\vec{f}_2) + (\vec{v}_1 \cdot Q_1\vec{a}_2) + (\vec{u}_1 \cdot R_1\vec{a}_2)}{c^2}. \tag{41}$$

The geometrical properties of the quantities $\vec{x}_n$ and t_n are specified by the transformation rules (20)–(27). Looking at these transformation rules we see that it is possible to reinterpret the meaning of $\vec{x}_n$ and t_n in the following way. Instead of treating $\vec{x}_n$ and t_n as components of $\vec{x}$ and t of the different order of magnitude, from now on we may treat them as coordinates of points of some new manifold which replaces the four-dimensional relativistic space-time. While the former procedure was in fact a numerical approximation to the relativistic transformation rules, this reinterpretation gives the real clue to what we call a non-relativistic approximation. The coordinates $\vec{x}_n$ and t_n do not need to be quantities of a given order of smallness since the transformation rules do not remember such statement and consequently we may "forget" the limitation of their meaning. In this way, for the N-th order of non-relativistic approximation, we obtain the $4(N+1)$-dimensional manifold which is a proper arena for the quasi-relativistic description of space-time events from the non-relativistic point of view.

Let us add to this conclusion a few remarks concerning the lowest order cases. In the zeroth order of non-relativistic approximation we obtain the four-dimensional manifold with geometry determined by the transformation rules (20) and (21). This shows that in this case we have the Aristotelian model of space-time. Our derivation, however, shows that this picture is applicable to space-time events in some neighbourhood of the light-cone in the non-relativistic limit.

In the first order of approximation we obtain the eight-dimensional manifold with the geometry determined by the transformation rules (20)–(23). This manifold contains the four-dimensional submanifold defined by the conditions

$$\vec{x}_0 = t_1 = 0, \tag{42}$$

which is invariant under the subgroup of transformations with

$$\vec{a} = e = 0. \tag{43}$$

It is easily seen that this submanifold realizes the usual Galileian space-time with the usual Galilei relativity group. In view of our original procedure, the conditions (42) mean nothing else than the restriction to the deep time like region of the Minkowski space-time.

In the second order of approximation we have the twelve-dimensional manifold with geometry specified by the transformation rules (20)–(25). This manifold contains the five-dimensional submanifold determined by the conditions

$$\vec{x}_0 = \vec{x}_2 = t_1 = 0, \tag{44}$$

which is invariant under the subgroup of transformations for which

$$\vec{a} = e = \vec{f} = 0. \tag{45}$$

Denoting

$$\theta = c^2 t_2,$$

$$\varphi = c^2 g, \tag{46}$$

we get just the five-dimensional Galileian space-time, where the one-parameter extension of the Galilei group operates. In this way we have found the answer to the question of the origin of the fifth dimension in non-relativistic physics.

In the third order of approximation we obtain the sixteen-dimensional manifold with geometry specified by the transformation rules (20)–(27). This manifold contains the eight-dimensional submanifold defined by the conditions

$$\vec{x}_0 = \vec{x}_2 = t_1 = t_3 = 0, \tag{47}$$

which is invariant under the subgroup of transformations for which

$$\vec{a} = e = \vec{f} = k = 0. \tag{48}$$

This subgroup is a new relativity group which, apart from non-relativistic physics, describes the first order corrections coming from relativistic physics. The precise meaning of this statement needs, however, further investigations which are beyond the scope of the present paper.