EFFECTIVE QUARK EQUATIONS

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Equations describing effective quarks in a meson are derived. The nature of quark confinement is discussed. The Bethe–Salpeter type equation for meson field is found and the form of binding potential is given.

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1. Introduction

It is generally assumed that mesons are composed of quarks. Free quarks are fermions and thus obey the Dirac equation. Description of a meson as a quark-antiquark bound pair leads to various quasi-independent quark models [1].

It is shown in Section 2 that the approach of quasi-independent quarks leads to the Duffin–Kemmer equations for a meson field if degrees of freedom, corresponding to the relative motion of quarks inside a meson, are removed. The aim of the present study was to find equations, describing effective (dressed) quarks inside a meson, from which the Duffin–Kemmer equations would follow. The effective quark equations are derived in Section 3 and discussed in the second quantization formalism in Section 4.

The notation of Bjorken and Drell [2] is used troughout. Indices $\alpha, \beta = 1, 2, m, n = 0, 1, 2, 3$, $P, Q, R = 1, 2, \ldots, 8$ indicate spinors, four-vectors, and SU(3) vectors, respectively. The structural constants of SU(3), f_{PQR}, are consistent with Gell-Mann convention [3].

2. Quark models and the Duffin–Kemmer field

A. The Bogolubov model

It is assumed in this model that free quarks obey the Dirac equation

$$(\gamma p - M)\psi_q = 0, \tag{1}$$

where $p^m = -i\partial/\partial x^m$ and M is the free quark mass. A meson is composed of quark and

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antiquark, held by potential V, and obeys the Bethe–Salpeter type equation [1b, 4]

$$((\gamma^{(1)}p^{(1)}-M)(\gamma^{(2)}p^{(2)}+M)+V)\psi_{q\bar{q}} = 0, \tag{2a}$$

$$\gamma^{(1)}\gamma^{(2)}\psi_{q\bar{q}} = \gamma^{(1)}\psi_{q\bar{q}}\gamma^{(2)}. \tag{2b}$$

This model can be considered as a special case of the more general MIT Bag theory [1c].

The potential V can be expanded in a series with respect to M

$$V = M^2 - MU + \ldots \tag{3}$$

and in the infinite mass limit, $M \to \infty$, Eq. (2) yields

$$(\gamma^{(1)}p^{(1)}-\gamma^{(2)}p^{(2)}-U)\psi_{q\bar{q}} = 0. \tag{4}$$

B. The Duffin–Kemmer equations

The plane-wave solutions of Eq. (4) are of the form

$$\psi_{k^{(1)}k^{(2)}} = |\varphi_{k^{(1)}}\rangle \langle\chi_{k^{(2)}}| \exp(-ik^{(1)}x^{(1)}-ik^{(2)}x^{(2)}). \tag{5}$$

Thus, the density matrix

$$\varrho_{k^{(1)}k^{(2)}} = |\varphi_{k^{(1)}}\rangle\,|\chi_{k^{(2)}}\rangle\,\langle\bar{\varphi}_{k^{(1)}}|\,\langle\bar{\chi}_{k^{(2)}}|, \tag{6}$$

fulfills the following equation

$$(\gamma^{(1)}k^{(1)}-\gamma^{(2)}k^{(2)}-U)\varrho_{k^{(1)}k^{(2)}} = 0, \tag{7a}$$

$$\gamma^{(i)}k^{(i)}\varrho = \varrho\gamma^{(i)}k^{(i)}, \quad i = 1, 2. \tag{7b}$$

If the Markov–Yukawa type conditions [5], to remove degrees of freedom of relative motion of quarks inside a meson, are imposed

$$k^{(1)} = k = -k^{(2)}, \tag{8}$$

then

$$(\tfrac{1}{2}(\gamma^{(1)}+\gamma^{(2)})k-(\tfrac{1}{2})U)\varrho_k = 0. \tag{9}$$

The representation of the Dirac matrices $\frac{1}{2}(\gamma_\mu^{(1)}+\gamma^{(2)})^m = \beta^m$ provides the reducible representation of the Duffin–Kemmer algebra and can be decomposed into 0, 5, 10 dimensional representations, corresponding to trivial, spin 0, spin 1 meson fields, respectively [5, 6]. Thus Eq. (9) reduces to the Duffin–Kemmer equation for the density matrix in momentum representation

$$(\beta k-m)\varrho_k = 0, \tag{10}$$

where $m = (\frac{1}{2})U$ represents the effective meson mass.

The Duffin–Kemmer equations in the spin 0 case have the following properties [7]

$$(\beta p-m)\phi = 0, \tag{11a}$$

$$(\beta k-m)v_k = 0 = \bar{v}_k(\beta k-m), \tag{11b}$$

$$v_k = \begin{bmatrix} k^0 \\ k^1 \\ k^2 \\ k^3 \\ m \end{bmatrix} m^{-1/2}, \quad \bar{v}_k = v_k^+ \beta', \beta' = \begin{bmatrix} +1 & & & & \\ & -1 & & & \\ & & -1 & & \\ & & & -1 & \\ & & & & +1 \end{bmatrix}, \tag{11c}$$

$$\varrho_k = |v_k\rangle \langle \bar{v}_k| \tfrac{1}{2} m. \tag{11d}$$

The field $\phi(x)$ has the form

$$\phi(x) = \int (dk)\,(a(k)v_k \exp(-ikx) + a^+(k)v_{-k} \exp(+ikx)), \tag{11e}$$

and the operators $a(k)$, $a^+(k)$ obey the Bose commutation relations.

3. *Equations for effective quarks*

A. Derivation of the effective quark equations

The system of equations

$$p^{1\dot{1}}\chi = m\zeta^{1\dot{1}}, \quad p^{2\dot{1}}\chi = m\zeta^{2\dot{1}}, \quad p^{2\dot{2}}\zeta^{1\dot{1}} - p^{1\dot{2}}\zeta^{2\dot{1}} = m\chi, \tag{12a}$$

where $p^{\alpha\dot{\beta}} = (\sigma_m p^m)^{\alpha\dot{\beta}}$, due to the identity $p^{1\dot{1}}p^{2\dot{2}} - p^{1\dot{2}}p^{2\dot{1}} = p_m p^m$, implies for $m \neq 0$ that χ obeys the Klein–Gordon equation. Eq. (12a) can be written in matrix form

$$(\not{p} - m)\psi_q(x) = 0, \tag{12b}$$

where

$$\not{p} = \varrho^n p_n, \quad \psi_q(x) = \begin{pmatrix} \zeta^{1\dot{1}} \\ \zeta^{2\dot{1}} \\ \chi \end{pmatrix}, \tag{12c}$$

and the matrices ϱ^m are of the form

$$\varrho^0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \varrho^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \varrho^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{pmatrix},$$

$$\varrho^3 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{13}$$

In momentum representation Eq. (12) reads

$$(\not{K}-m)u_k = 0, \tag{14a}$$

$$\bar{u}_k(\not{K}-m) = 0. \tag{14b}$$

The solutions of Eq. (14)

$$u_k = \begin{pmatrix} k^0+k^3 \\ k^1+ik^2 \\ m \end{pmatrix} m^{-1/2}, \quad \bar{u}_k = (k^0-k^3, -k^1+ik^2, m)m^{-1/2} \tag{15}$$

are independent in contradistinction to the Duffin–Kemmer case, Eq. (11b). The pair $u_k, \bar{u}_k$ is equivalent to $v_k(\bar{v}_k)$ because the four-vector k^m is equivalent to the spinor $k^{\alpha\dot{\beta}} = (\sigma_m k^m)^{\alpha\dot{\beta}}$. Algebraic properties of the representation (13) are collected in Appendix.

B. Equations for the density matrix

If the density matrix

$$\varrho_k = |u_k\rangle\langle\bar{u}_k|1/2m, \tag{16}$$

is introduced, then it follows from Eq. (14) that

$$(\not{K}-m)\varrho_k = 0 = \varrho_k(\not{K}-m), \tag{17}$$

and the equality

$$k^n = m\,\mathrm{Tr}\,(\varrho_k\varrho^n), \quad n = 0, 1, 2, 3 \tag{18}$$

demonstrates the equivalence of ϱ_k and v_k.

The quark–antiquark field

$$\psi_{q\bar{q}}(x, x) = \int (dk)\,(a(k)\varrho_k \exp(-ik(x-x'))+a^+(k)\varrho_{-k}\exp(ik(x-x'))),$$
$$(dk) = d^3\mathbf{k}/((2\pi)^3\omega_k)^{1/2}, \quad \omega_k = +(\mathbf{k}^2+m^2)^{1/2}, \tag{19}$$

obeys the following equations

$$(\not{p}-m)\psi_{q\bar{q}} = 0 = \psi_{q\bar{q}}(\not{p}-m), \tag{20a}$$

$$(\not{p}+m)\psi_{q\bar{q}} = 0 = \psi_{q\bar{q}}(\not{p}+m), \tag{20b}$$

and describes neutral mesons of the Duffin–Kemmer theory, Eq. (11e), due to Eq. (18).

C. Lorentz transformations of quark equations

The matrices λ^{mn}

$$2\lambda^{mn} = \begin{bmatrix} 0 & \lambda_1 & \lambda_2 & \lambda_3 \\ -\lambda_1 & 0 & -i\lambda_3 & i\lambda_2 \\ -\lambda_2 & i\lambda_3 & 0 & -i\lambda_1 \\ -\lambda_3 & -i\lambda_2 & i\lambda_1 & 0 \end{bmatrix}, \tag{21}$$

where $\lambda_1, \lambda_2, \lambda_3$ are the generators of SU(3) algebra [3], and satisfy the commutation relations for the angular momentum operators of $D^{1/2} \oplus D^0$ representation, have the following commutation properties

$$[\varrho^l, 2\lambda^{mn}] = g^{lm}\varrho^n - g^{ln}\varrho^m + ie^{lmnp}\varrho^p, \tag{22}$$

$l, m, n, p = 0, 1, 2, 3$
where e^{lmnp} is the Ricci antisymmetric tensor:
Thus

$$\Lambda^{-1}\varrho^m\Lambda = \varrho^m + \varrho^n\Omega^m_n = L^m_n\varrho^n, \tag{23a}$$

$$\Lambda = 1 + \lambda^{mn}\omega_{mn}, \quad \Lambda^{-1} = 1 - \lambda^{mn}\omega_{mn}, \tag{23b}$$

$$\Omega^{mn} = \omega^{mn} + i/2e^{mnrs}\omega_{rs}, \tag{23c}$$

and Eqs. (14) are covariant under Lorentz complex transformations L^m_n of the coordinate four-vector $x^m \to x^{m\prime} = L^m_n\, x^n, L^m_n = \delta^m_n + \Omega^m_n$

$$(\Lambda^{-1}(\not{k} - m)\Lambda)\Lambda^{-1}u_k = 0, \tag{24a}$$

$$\bar{u}_k\Lambda(\Lambda^{-1}(\not{k} - m)\Lambda) = 0. \tag{24b}$$

According to Eqs. (24), u_k and $\bar{u}_k$ are transformed by Λ^{-1} and Λ respectively, i.e. have quantum numbers of opposite sign (cf. Eq. (23b)). The meaning of complex Lorentz transformations (23) is clear. If $\omega^{01} \neq 0$ then $\Omega^{01} = \omega^{01}$, $\Omega^{23} = -i\omega^{01}$, what corresponds to boost in x^0x^1 plane and rotation about an immaginary angle in x^2x^3 plane.

Boosts in x^0x^3 plane and rotations in x^1x^2 plane are distinguished. If the new generators

$$\lambda^{03\prime} = \lambda_3 + (\tfrac{1}{3})^{1/2}\lambda_8, \quad \lambda^{12\prime} = \lambda_3 - (\tfrac{1}{3})^{1/2}\lambda_8, \tag{25}$$

are introduced, then the following equations

$$[\varrho^l, 2\lambda^{mn\prime}] = g^{lm}\varrho^n - g^{ln}\varrho^m, \quad mn = 03, 12, \quad l = 0, 1, 2, 3 \tag{26}$$

are fulfilled and thus $\lambda^{mn\prime}$ correspond to real Lorentz transformations $L^m_n = \delta^m_n + \omega^m_n$.

It is possible to transform the momentum four-vector from the rest-mass frame to a moving frame applying real Lorentz transformations only (cf. Eqs. (23), (26)):

$$\begin{aligned} k^{m(0)} = (m, 0, 0, 0) \xrightarrow{\lambda^{01}} (k^{0\prime}, k^{1\prime}, 0, 0) \xrightarrow{\lambda^{12\prime}} & (k^{0\prime}, k^1, k^2, 0) \\ & \downarrow \lambda^{03\prime} \\ & k^m = (k^0, k^1, k^2, k^3). \end{aligned} \tag{27}$$

It follows from Eqs. (23, (26) that the little group of the four-vector $k^{m(2)}$ is 0(2).

The meson equation, however, can be written in explicitly covariant form. The solution of Eq. (17)

$$\varrho_k = (\not{k}(+m)m - (k^2 - \not{k}^2))/2m^2, \tag{28}$$

has the following properties (cf. Appendix):

$$0 = (\not{k} - m)\varrho_k = (k^2 - m^2), \qquad \varrho_k^2 = \varrho_k, \qquad \mathrm{Tr}\, \varrho_k = 1, \tag{29}$$

and hence the covariance of Eq. (17) follows.

The quark equation (12) is also invariant under CPT transformation $x \to x' = -x$

$$(\not{p} - m)CPT\psi_q(x^0, \boldsymbol{x}) = 0, \tag{30}$$

$$CPT\psi_q(x^0, \boldsymbol{x}) = \eta\varrho^5\psi_q(-x^0, -\boldsymbol{x}), \qquad \varrho^5 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & +1 \end{pmatrix}, \qquad |\eta| = 1. \tag{31}$$

It is interesting, that Eq. (12) is not invariant under any other discrete transformations.

4. The Lagrange formalism

It is easily checked that fields

$$\psi_q(x) = \int (dk)\,(a(k)u_k \exp(-ikx) + b^+(k)u_{-k} \exp(ikx)), \tag{32}$$

$$\overline{\psi}_q(x) = \int (dk)\,(a^+(k)\overline{u}_k \exp(ikx) + b(k)\overline{u}_{-k} \exp(-ikx)), \tag{33}$$

fulfill Eq. (12) and

$$\overline{\psi}_q(\not{p} + m) = 0, \tag{34}$$

respectively. The four-vector field

$$j^m(x) = \overline{\psi}_q(x)\varrho^m\psi_q(x), \tag{35}$$

obeys the continuity equation and leads to conserved charge Q:

$$Q = \int d^3\boldsymbol{x}\overline{\psi}_q(x)\varrho^0\psi_q(x) = \int d^3\boldsymbol{k}(a^+(k)a(k) + b(k)b^+(k)), \tag{36a}$$

where the following normalization was used

$$u_{-k} = \begin{pmatrix} -k^0 & -k^3 \\ -k^1 & -ik^2 \\ m & \end{pmatrix} m^{-1/2}, \qquad \overline{u}_{-k} = -(-k^0 + k^3, k^1 - ik^2, m)m^{-1/2}, \tag{37a}$$

$$\overline{u}_k u_k = 2m, \qquad \overline{u}_{-k}u_{-k} = -2m. \tag{37b}$$

Thus the Fermi anticommuting relations have to be obeyed by the operators a, a^+, b, b^+ to get the proper form of the charge operator [2]

$$Q = \int d^3\boldsymbol{x} : \overline{\psi}_q(x)\varrho^0\psi_q(x) := \int d^3\boldsymbol{k}(a^+(k)a(k) - b^+(k)b(k)), \tag{36b}$$

and the field ψ_q describes fermions, accordingly.

Equations (12), (34) can be derived from the following Lagrangian

$$L_q = i/2(\bar{\psi}_q \varrho^m \partial\psi_q/\partial x^m - \partial\psi_q/\partial x^m \varrho^m \psi_q) - m\bar{\psi}_q\psi_q, \tag{37c}$$

which yields the four-momentum operator

$$P^m = \int d^3k k^m (a^+(k)a(k) + b^+(k)b(k)). \tag{38}$$

5. *Closing remarks*

We have shown that the Duffin–Kemmer equations for the meson field can be derived from the quasi-independent quark model of Bogolubov.

On the other hand, the Duffin–Kemmer equations have been shown in the spin 0 case to be equivalent to the pair of equations (14), describing quark and antiquark fields. Thus the quark fields, obeying Eqs. (14), should be interpreted as effective dressed fields inside the meson.

The results of Section 3C demonstrate the presence of a distinguished axis in the quark–antiquark system. Single quark field is covariant with respect to boosts in this direction and rotations around this axis. On the other hand, all other Lorentz transformations violate the covariance of quark equations (14). Accordingly, the little group of the momentum four-vector in the rest-mass frame $k^{m(0)}$ is O (2). A single quark is noncovariantly defined with respect to the distinguished axis — connecting quark and antiquark in a meson. Thus single quarks should be particles of only partially covariant character in contradistinction to a quark–antiquark pair a meson which obeys the covariant equation (17). The quark confinement in the present model is due to lack of the full Lorentz covariance of quark equations (14).

It should be remarked that the meson field (19) obeys due to (20) the following equation

$$(\not{p} - m)(\not{p}' + m)\psi_{q\bar{q}}(x, x') = 0 \tag{39}$$

which can be written in the Bethe–Salpeter form (cf. Eq. (2))

$$(\vec{\not{p}} - M)\psi_{q\bar{q}}(x, x')(\overleftarrow{\not{p}}' + M) + V\psi_{q\bar{q}}(x, x') = 0, \tag{40a}$$

$$V = (m - M)^2. \tag{40b}$$

Therefore, Equations (12), (34) describe meson constituents, i.e. quark and antiquark fields, respectively.

APPENDIX

Let us summarize basic properties of the algebra (13). Matrices ϱ^m fulfill the following equation

$$\varrho^{(l}\varrho^m\varrho^{n)} = g^{(lm}\varrho^{n)}, \quad l, m, n = 0, 1, 2, 3, \tag{A1}$$

where (lmn) is the symmetrizer. There are also other algebras obeying (A1) [7]. They are related to the Duffin–Kemmer equations spin 0 and spin 1 [6] and to the Maxwell equations for the vector $E+iB$ [7, 8], and fulfill a milder condition then (A1)

$$\varrho^l\varrho^m\varrho^n+\varrho^n\varrho^m\varrho^l = g^{lm}\varrho^n+g^{nm}\varrho^l. \tag{A2}$$

Due to (A1) the equality

$$(\not{K}-m)d(k) = d(k)(\not{K}-m) = k^2-m^2, \quad \not{K}^3 = \not{K}m^2, \tag{A3}$$

holds, with

$$d(k) = (\not{K}+m)-1/m(k^2-\not{K}^2). \tag{A4}$$

Matrices ϱ^m are connected with the generators of SU (3) algebra

$$(\varrho^0, \varrho^1, \varrho^2, \varrho^3) = (\lambda_4, -i\lambda_7, -i\lambda_6, -i\lambda_5), \tag{A5}$$

where Gell-Mann representation of λ matrices is used [3]. It should be also noted that the generators of Lorentz transformations, Eqs. (21), (25) are also composed of SU(3) generators, λ_1, λ_2, λ_3, λ_8.

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