THE TOTAL FOUR MOMENTUM — ITS RELATION TO TRANSFORMATIONS OF COORDINATES AND FRAMES OF REFERENCE

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In this paper, the point of view is stressed that the total four momentum must primarily be defined with respect to a certain set of spacelike hypersurfaces which do not intersect. Each total four momentum defined with respect to a single hypersurface can then be handled like a free vector under affine transformations of coordinates. It may be possible that the total four momentum does not depend on the choice of a certain hypersurface from this set, which means a conservation law. This set of hypersurfaces is the main ingredient of the frame of reference. Replacing such a set of spacelike hypersurfaces by another, whose elements intersect those of the former one, is in principle equivalent to changing the frame of reference. It is also, on the other hand, possible that certain sets of spacelike hypersurfaces be equivalent, which means the principle of relativity. Then the problem is whether the total four momentum can be defined in the same way with respect to different frames of reference and whether it behaves like a four vector under the invariance group of mappings between the equivalent frames of reference. Not for all energy-momentum tensors the answer to this is affirmative. Several examples are discussed. They demonstrate the difference between such concepts as a transformation of coordinates and changing the frame of reference.

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1. Introduction

It is well-known that the total energy and momentum of a physical system can be defined without any difficulty if the system does not disturb certain symmetry properties of space-time in which the system is situated. This is realized for physical systems in Minkowski space-time. However, if such symmetry properties are not present, the total energy and momentum can only be defined with respect to certain classes of coordinate systems. But the total energy and momentum will not form components of a four vector if we make a transformation from one such class to another. Confusion arises if there is no clear understanding of what is the frame of reference (a certain set of observers) and how it is

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related to coordinate systems. It seems to be F. Klein's merit [1] to have given a clear distinction between these two different concepts. Certainly, F. Klein did not mention in his paper explicitly that his intension was to explain what was a transformation of coordinate systems and what, on the other hand, changing the frame of reference. The covariance of certain integral quantities defined with respect to the given frame of reference is only one part of the problem of "the total energy and momentum". The other one is how to define integral quantities with respect to different frames of reference. Here, the requirements of a certain principle of relativity can enter the problem. In frames of reference which are equivalent in the sense of this principle, integral quantities should be formed in the same way. And, moreover, these quantities should be the elements of a representation space of the group which interrelates such equivalent frames of reference.

Some time ago transformation properties of the total four momentum were related to the structure of physical systems. Some authors claimed that the total energy and momentum behaved like a four vector if the physical system under consideration was stable. In the context "stable" meant that certain integrals of the components of energy-momentum tensor taken over space-like hypersurfaces had to vanish [2]. We will see that the v. Laue theorem [3] quoted in this connection has, perhaps, quite another meaning, too. v. Laue considered physical systems which allow one to compute the total energy and momentum in the same way with respect to frames of reference which are equivalent under the principle of special relativity. For being able to do so, derivatives of fields the physical system is built from must tend to zero in a certain way if far away from the "centre". Components of the energy-momentum tensor enter the integral quantities in which we are interested. Originally, only energy-momentum tensors were considered which were bilinear expressions in the first derivatives of the fields. In the case of static fields, these derivatives behave like r^{-2} (r — radial coordinate) for $r \to \infty$. And then the components of the energy-momentum tensor behave like r^{-4}. But in the theory of gravitation proposed by Jordan and Brans and Dicke [11] an energy-momentum tensor was introduced which contained second derivatives of the field variables, too. This tensor behaves like r^{-3} for $r \to \infty$. In consequence, the total four momentum cannot be computed in the same way in all frames of reference which are equivalent under a principle of relativity. Perhaps, we can say that the particle concept has a more restricted meaning in this theory than in the theories mentioned above.

2. *v. Laue theorem*

Here, I would like to gather some ideas known as v. Laue theorem. Originally, this theorem was formulated by v. Laue in the framework of the theory of special relativity (SRT). In this paper, however, the theorem will be written in a form more suitable to such theories like Einstein's GRT.

The starting point is the assumption that there is a coordinate system in which time derivatives of the components of the energy-momentum tensor density are zero[1]:

$$\mathfrak{T}_{\mu}{}^{0}{}_{,0} = 0. \tag{1}$$

[1] Greek indices are ranging from 0 to 3 and Latin indices from 1 to 3.

If a local conservation law holds

$$\mathfrak{T}_{\mu}{}^{\nu}{}_{,\nu} = 0, \tag{2}$$

a consequence of (1) is

$$\mathfrak{T}_{\mu}{}^{l} = (x^{l}\mathfrak{T}_{\mu}{}^{k})_{,k}. \tag{3}$$

In (3) physical systems are considered whose energy-momentum tensor differs from zero only in a space-bounded region. Assuming the regularity of $\mathfrak{T}_{\mu}{}^{\nu}$, we obtain from (3) as a result of integration over a certain spacelike hypersurface x^0 = const that

$$\int_{x^0=\mathrm{const}} \mathfrak{T}_{\mu}{}^{l} d^3x = 0. \tag{4}$$

So, v. Laue theorem can be formulated in the following way: *There exist a Minkowski coordinate system such that under the assumptions* (1) *and* (2) *the net sum of tensions, as well as of the energy current density, taken at a certain time is zero.*

But, in case the system considered is not space-bounded, at first we have

$$\int_{x^0=\mathrm{const}} \mathfrak{T}_{\mu}{}^{l} d^3x = \int_{\partial V_3} x^{l}\mathfrak{T}_{\mu}{}^{k} dS_k, \tag{5}$$

and in order to obtain relation (4), $\mathfrak{T}_{\mu}{}^{k}$ must again behave like $r^{-(3+\alpha)}$ $(\alpha > 0)$ for $r \to \infty$.

3. *Transformation properties of integral quantities — their independence of the choice of the hypersurface*

Let us suppose that there is a certain local conservation law

$$\mathfrak{A}_{\mu}{}^{\nu}{}_{,\nu} = 0. \tag{6}$$

In the case of SRT, $\mathfrak{A}_{\mu}{}^{\nu}$ is the energy-momentum tensor density $\mathfrak{T}_{\mu}{}^{\nu}$ of the physical system under consideration. For theories of gravitation, $\mathfrak{A}_{\mu}{}^{\nu}$ is the sum of $\mathfrak{T}_{\mu}{}^{\nu}$ and of a term $\mathfrak{F}_{\mu}{}^{\nu}$ which is a contribution from the gravitational field. For metric theories of gravitation, this part is a tensor density with respect to affine coordinate transformation only.

In the framework of GRT, Einstein has considered the question of the existence of integral quantities which are conserved. The assertion that the total energy and momentum are time-independent means that some of the integrals of $\mathfrak{T}_{\mu}{}^{\nu}$ do not depend on the choice of certain spacelike hypersurfaces. Einstein made the choice of coordinate systems which was most suitable for those spacelike hypersurfaces. As a result of this, a clear-cut distinction between coordinate systems and frames of reference became obliterated. It is said that this was the starting point for F. Klein's papers.

A similar problem arises when you try to define the total energy and momentum of the electromagnetic field of a charge distribution. The discussion of this question was continued untill 1960, and even later [5], although Klein's paper was published in 1918.

F. Klein considered only the mathematical aspect of the problem. It was the discussion which went on for many years that made a little clearer what was the content of the principles of general relativity and of general covariance. In consequence, it has been understood that the choice of a family of spacelike hypersurfaces means essentially the choice of the frame of reference. According to the point of view taken here, the frame of reference (a family of observers) is the congruence of timelike curves which covers the whole space-time. The spacelike sections of this congruence by a hypersurface, which always exist, define the moment at which observers do their measurement [6]. Then, it is clear that by giving such a congruence of timelike curves and a family of spacelike hypersurfaces nothing is said about the coordinate system which can be used for describing these two geometrical objects. Surely, there will be coordinate systems which are more suitable for describing them than some others [7].

We are mainly interested in transformation properties of integral quantities under the global Lorentz transformation. Therefore we restrict the class of coordinate systems to those we can consider as rectilinear in an affine space. Since we do not exclude the case that space-time can be curved, perhaps the following remark is not needless here. We suppose that space-time is a differentiable manifold which can be endowed with a metric whose curvature is not zero and which is, besides that, endowed with an affine structure. To simplify matters, we restrict the family of spacelike hypersurfaces in such a way that it only contains hypersurfaces which are flat with respect to the second structure. With the affine structure, there are given four constant vector fields the components of which are $K_A{}^\mu$ ($A, B, \ldots$ enumerate the vector fields and range from 0 to 3). Then a special coordinate system exists in which

$$K_A{}^\mu = \delta_A{}^\mu. \tag{7}$$

Together with (6), we also have

$$(K_A{}^\mu \mathfrak{A}_\mu{}^\nu)_{,\nu} = 0. \tag{8}$$

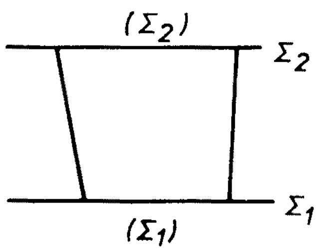

Fig. 1

We integrate (8) over a four volume the boundary of which consists of three parts (Σ_1), (Σ_2) and Z (see Fig. 1). (Σ_1) and (Σ_2) are subsets of two spacelike hypersurfaces Σ_1 and Σ_2, respectively, and are connected by a subset Z of a timelike hypersurface. Then we have

$$\int_{(\Sigma_2)} K_A{}^\mu \mathfrak{A}_\mu{}^\nu d\Sigma_\nu - \int_{(\Sigma_1)} K_A{}^\mu \mathfrak{A}_\mu{}^\nu d\Sigma_\nu + \int_Z K_A{}^\mu \mathfrak{A}_\mu{}^\nu d\Sigma_\nu = 0. \tag{9}$$

Let us now suppose that $\mathfrak{A}_\mu{}^\nu$ does not vanish only in a space-bounded region. Then, $\mathfrak{A}_\mu{}^\nu$ is equal to zero on Z. In this case, (9) means that the quantities

$$P_\mu(\Sigma) = \int_{(\Sigma)} \mathfrak{A}_\mu{}^\nu d\Sigma_\nu \tag{10}$$

do not depend on the choice of the hypersurface. This remains also true when Σ_1 and Σ_2 intersect. To prove this, we have to take a third hyperplane Σ_3, such that (Σ_3) does not intersect both (Σ_1) and (Σ_2) and apply (9) to the two combinations (Σ_3), (Σ_1) and (Σ_3), (Σ_2); see Fig. 2.

As a result of integration over a hypersurface Σ, in a certain system of affine coordinates, we obtain four numbers $P_\mu(\Sigma)$. We cannot ask what the transformation law of $P_\mu(\Sigma)$

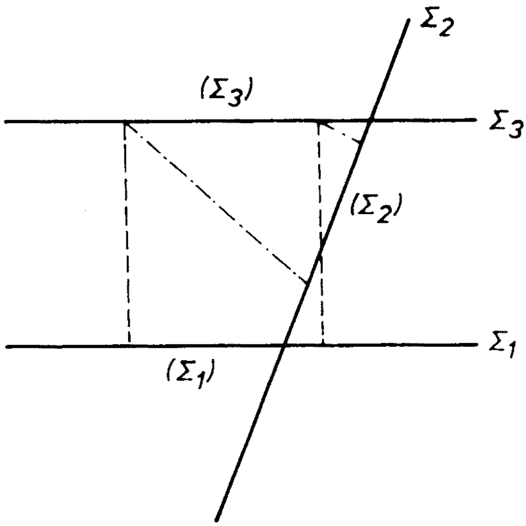

Fig. 2

is, but must instead define how $P_\mu(\Sigma)$ behaves when the affine coordinate system is changed. A reasonable definition seems to be that $P_\mu(\Sigma)$ are components of a free four vector with respect to affine coordinate transformations. This means

$$P_{\mu'}(\Sigma) = \frac{\partial x^\mu}{\partial x^{\mu'}} P_\mu(\Sigma). \tag{11}$$

Because of the local conservation law (6), it does not matter which hypersurface Σ is chosen for computing $P_\mu(\Sigma)$ with respect to a certain coordinate system. If there were no local conservation law for $\mathfrak{A}_\mu{}^\nu$, we could yet determine $P_\mu(\Sigma)$. But now, these quantities would depend on the chosen hypersurface Σ. And we could define the quantities $P_\mu(\Sigma)$ to be vectors with respect to affine coordinate transformations in this case, too.

As an explanation why we should define $P_\mu(\Sigma)$ to be the components of a free four vector, one can point at the transformation properties of $\mathfrak{A}_\mu{}^\nu$ with respect to affine coordinate transformations. No matter whether there is a local conservation law (6) or not, we have

$$\int_{(\Sigma)} K_A{}^\mu \mathfrak{A}_\mu{}^\nu d\Sigma_\nu = \int_{(\Sigma)} K_A{}^{\mu'} \mathfrak{A}_{\mu'}{}^{\nu'} d\Sigma_{\nu'} \tag{12}$$

when replacing one affine coordinate system by another, because $K_A{}^\mu \mathfrak{A}_\mu{}^\nu d\Sigma_\nu$ are four scalars. From (12) we obtain

$$\int_{(\Sigma)} \mathfrak{A}_\mu{}^\nu d\Sigma_\nu = \frac{\partial x^{\mu'}}{\partial x^\mu} \int_{(\Sigma)} \mathfrak{A}_{\mu'}{}^{\nu'} d\Sigma_{\nu'} \tag{13}$$

or

$$P_\mu(\Sigma) = \frac{\partial x^{\mu'}}{\partial x^\mu} P_{\mu'}(\Sigma). \tag{14}$$

So we get the transformed $P_{\mu'}(\Sigma)$ by integrating the transformed $\mathfrak{A}_{\mu'}{}^{\nu'}$ over the same hypersurface Σ and transforming the result like a vector. This is rather trivial in the case when the physical system is space-bounded. It is not so trivial in field theory where the fields are spread over the whole space-time.

Let us now come back to the case when a local conservation law (6) holds and $\mathfrak{A}_\mu{}^\nu$ differs from zero on a space-bounded region only. Then we have

$$P_\mu(\Sigma_2) = P_\mu(\Sigma_1). \tag{15}$$

Furthermore, let us suppose that Σ_2 is given by $x^{0'} = \text{const}'$ and Σ_1 by $x^0 = \text{const}$, where $x^{0'}$ is the time coordinate in an affine coordinate system $\{x^{\mu'}\}$ and x^0 is the time coordinate in another affine coordinate system $\{x^\mu\}$. Instead of (15), we can write then

$$\int_{x^0=\text{const}} \mathfrak{A}_\mu{}^0 d^3x = \frac{\partial x^{\mu'}}{\partial x^\mu} \int_{x^{0'}=\text{const}'} \mathfrak{A}_{\mu'}{}^{0'} d^3x' \tag{16}$$

or

$$P_\mu(x^0 = \text{const}) = \frac{\partial x^{\mu'}}{\partial x^\mu} P_{\mu'}(x^{0'} = \text{const}'). \tag{17}$$

Eqs. (16) and (17) correspond to the requirements of the special relativity principle. The two hypersurfaces $x^{0'} = \text{const}'$ and $x^0 = \text{const}$ belong to frames of reference which are equivalent under this principle of relativity. Then, this principle seems to require that physical quantities must be measured in the same way (here: projection of $\mathfrak{A}_\mu{}^\nu$ on $d\Sigma_\nu = \delta_\nu{}^0 d^3x$) with respect to equivalent frames of reference. Quantities obtained in this way in equivalent frames of reference should be elements of a representation space of the group relating equivalent systems of reference (here: vectors).

Let us stress once again: transformation properties of $P_\mu(\Sigma)$ are coupled with changing from one affine coordinate system to another one. But primarily they have nothing to do with passing over from one spacelike hypersurface to another. If we choose a new hypersurface which intersects the former one, this means essentially that we changed the frame of reference. In this latter case we have to ask whether there is a tensor relationship between quantities which are measured in equivalent frames of reference in the same way.

I have already pointed out that we have started from an $\mathfrak{A}_\mu{}^\nu$ which vanishes outside a space-bounded region. For field theories, this is in general not fulfilled. In this case one has to be cautious. Although Einstein says in [4] that the components $g_{\mu\nu}$ of the metric tensor shall be constant outside a certain space-bounded region and that the energy-momentum tensor is zero there, it should be understood, of course, that he has in mind a system which is put into the Minkowski space, that means $g_{\mu\nu} \neq \eta_{\mu\nu}$ is spread over the whole space-time, but $g_{\mu\nu} \to \eta_{\mu\nu}$ far away from the "centre" of the physical system. It seems that F. Klein interpreted this statement of Einstein too literally and did not take into account that $\mathfrak{A}_\mu{}^\nu$ is not identically zero outside a space-bounded region.

Our starting point is again that a local conservation law (6) holds. If $\mathfrak{A}_\mu{}^\nu$ is not identically zero outside a space-bounded region then (depending on how $\mathfrak{A}_\mu{}^\nu$ and/or its derivatives tend to zero at large distances from the "centre") the quantities

$$\int_\Sigma \mathfrak{A}_\mu{}^\nu d\Sigma_\nu, \tag{18}$$

where we have to integrate over the whole hypersurface Σ, can depend on Σ. If the two hypersurfaces Σ_1 and Σ_2 do not intersect, the volume of the timelike hypersurface Z behaves like r^2 for $r \to \infty$. In this case, $\mathfrak{A}_\mu{}^\nu$ has to approach zero like $r^{-(2+\alpha)}$ ($\alpha > 0$) to assure that the integrals (18) do not depend on the choice of non-intersecting hypersurfaces. If we choose such a coördinate system that the family of hypersurfaces $x^0 = \text{const}$ coincides with the family of spacelike hypersurfaces Σ of the chosen system of reference, then this independence of the choice of a hypersurface in the integrals (18) expresses the time independence of

$$P_\mu(x^0 = \text{const}) = \int_{x^0=\text{const}} \mathfrak{A}_\mu{}^0 d^3x. \tag{19}$$

Now, we can proceed as described above. We attach the quantities $P_\mu(x^0 = \text{const})$ to each point of the affine coordinate system and transform them like a four vector when passing over to another affine coordinate system:

$$P_{\mu'}(x^0 = \text{const}) = \frac{\partial x^\mu}{\partial x^{\mu'}} P_\mu(x^0 = \text{const}). \tag{20}$$

Let us stress once again, this vector field can depend on the choice of the family $x^0 = \text{const}$. As above, also in the case when $\mathfrak{A}_\mu{}^\nu$ does not identically vanish outside a space-bounded region, we have

$$P_{\mu'}(x^0 = \text{const}) = \int_{x^0=\text{const}} \mathfrak{A}_{\mu'}{}^{\nu'} d\Sigma_{\nu'}. \tag{21}$$

Now, the question is again what is the relation between the quantities

$$P_\mu(x^0 = \text{const}) = \int_{x^0=\text{const}} \mathfrak{A}_\mu{}^0 d^3x \tag{22}$$

and

$$P_{\mu'}(x^{0'} = \text{const}') = \int_{x^{0'}=\text{const}'} \mathfrak{A}_{\mu'}{}^{0'} d^3x', \tag{23}$$

which are measured in the same way with respect to equivalent frames of reference. In order to obtain the relations (16), (17) also in the case when $\mathfrak{A}_\mu{}^\nu$ does not identically vanish outside a space-bounded region we must require that the integrals (18) do not depend on the choice of a hypersurface, no matter to which family it belongs. To deal with this situation we take again a hypersurface Σ_3 which does not intersect both Σ_1 and Σ_2 (see Fig. 3). If we want the relation

$$\int_{\Sigma_3} \mathfrak{A}_\mu{}^\nu d\Sigma_\nu = \int_{\Sigma_1} \mathfrak{A}_\mu{}^\nu d\Sigma_\nu \tag{24}$$

to be valid, we must require that (18) taken over the timelike hypersurface which connects (Σ_3) and (Σ_1) approaches zero for $r \to \infty$. Now, however, $d\Sigma_\nu$ behaves like r^3 for $r \to \infty$

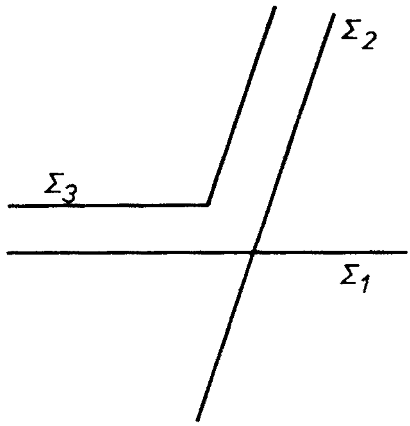

Fig. 3

and, therefore, $\mathfrak{A}_\mu{}^\nu$ must behave like $r^{-(3+\alpha)}$ ($\alpha > 0$) for $r \to \infty$. If this is assured, relation (24) holds also for the combination of Σ_3 and Σ_2, and, in consequence, also for Σ_1 and Σ_2.

So we see that for proving the time independence of $P_\mu(x^0 = \text{const})$ all we need is that $\mathfrak{A}_\mu{}^\nu$ behaves like $r^{-(2+\alpha)}$ ($\alpha > 0$) for $r \to \infty$. To prove, however, that $P_\mu(\Sigma)$ can be determined in the same way with respect to equivalent frames of reference, we need more: $\mathfrak{A}_\mu{}^\nu$ must approach zero like $r^{-(3+\alpha)}$ ($\alpha > 0$) for $r \to \infty$.

Now, once again we start with (23) and express the integrand in terms of such $\mathfrak{A}_\mu{}^\nu$ which follow from $\mathfrak{A}_{\mu'}{}^{\nu'}$ as a result of a special global Lorentz transformation (a special affine transformation). We have

$$x^1 = \frac{x^{1'} - \frac{v}{c} x^{0'}}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad x^2 = x^{2'}, \quad x^3 = x^{3'}, \quad x^0 = \frac{x^{0'} - \frac{v}{c} x^{1'}}{\sqrt{1 - \frac{v^2}{c^2}}}, \tag{25}$$

and

$$\mathfrak{A}_{\mu'}{}^{0'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\mathfrak{A}_\mu{}^0 + \frac{v}{c} \mathfrak{A}_\mu{}^1}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{26}$$

This gives

$$P_\mu(x^{0'} = \text{const}') = \int\limits_{x^{0'}=\text{const}'} \mathfrak{A}_{\mu'}{}^{0'} \sqrt{1-\frac{v^2}{c^2}}\, d^3x$$

$$= \frac{\partial x^\mu}{\partial x^{\mu'}} \Bigg[\int\limits_{x^{0'}=\text{const}'} \mathfrak{A}_\mu{}^0 d^3x + \frac{v}{c} \int\limits_{x^{0'}=\text{const}'} \mathfrak{A}_\mu{}^1 d^3x \Bigg]. \tag{27}$$

Here we have to take into account that integration is taken over the hypersurface $x^{0'} = \text{const}'$. This is not influenced by changing the integration variables. If $\mathfrak{A}_\mu{}^\nu$ depends on x^0, we have to choose $x^0(x^k)$ in such a way that $(x^0(x^k), x^k)$ is a point on the hypersurface $x^{0'} = \text{const}'$. Only in the case when $\mathfrak{A}_\mu{}^\nu (x^k, x^0)$ does not depend on x^0, we can pass from $x^{0'} = \text{const}'$ to $x^0 = \text{const}$ in the second line of (27). Then, in order that $P\,(x^{0'} = \text{const}')$ could be measured, with respect to the frame of reference to which the coordinate system $\{x^{\mu'}\}$ is adapted, in the same way as $P_\mu(x^0 = \text{const})$ with respect to the frame of reference to which the coordinate system $\{x^\mu\}$ is adapted, we must have

$$\int\limits_{x^0=\text{const}} \mathfrak{A}_\mu{}^1 d^3x = 0. \tag{28}$$

Sometimes ([2], [8]) one can read that if (28) is fulfilled, then $P_\mu(\Sigma)$ is a four vector. We have seen that (28) is not related to the transformation properties of $P_\mu(\Sigma)$ provided we restrict ourselves to transformations of affine coordinates. Eq. (28) is rather a requirement of the special relativity principle.

Of course, (28) reminds of v. Laue's theorem. Therefore, let me examine the presumptions which lead to

$$P_\mu(x^{0'} = \text{const}') = \int\limits_{x^{0'}=\text{const}'} \mathfrak{A}_{\mu'}{}^{0'} d^3x' = \frac{\partial x^\mu}{\partial x^{\mu'}} \int\limits_{x^0=\text{const}} \mathfrak{A}_\mu{}^0 d^3x$$

$$= \frac{\partial x^\mu}{\partial x^{\mu'}}\, P_\mu(x^0 = \text{const}). \tag{29}$$

In (27) we have used only the definition of $P_\mu(x^0 = \text{const})$ and the transformation properties of $\mathfrak{A}_{\mu'}{}^{\nu'}(x^{\varrho'})$. Especially, we did not ask whether there is a conservation law. We were led to (28) by supposing that $\mathfrak{A}_\mu{}^\nu$ does not depend on x^0. On the other hand, if a conservation law holds, then the assumption that $\mathfrak{A}_\mu{}^0{}_{,0} = 0$ and $\mathfrak{A}_\mu{}^k$ behaves like $r^{-(3+\alpha)} (\alpha > 0)$ for $r \to \infty$ implies

$$\int\limits_{x^0=\text{const}} \mathfrak{A}_\mu{}^l d^3x = 0. \tag{30}$$

In this case nothing is said about the time dependence of $\mathfrak{A}_\mu{}^k$. Here, I do not intend to answer the question whether these requirements are all independent and whether they mean a further restriction on $\mathfrak{A}_\mu{}^\nu$. I remark only that (28) is fulfilled as a consequence of *v.* Laue's theorem.

4. *Examples*

a. Electromagnetic field in Minkowski space-time

The question whether the total energy and momentum of an electromagnetic field form a four vector, was discussed for a long time [5]. In this case, we have

$$ {}_{\mathrm{E}}P_{\mu'}(x^{0'} = \mathrm{const}') = \int\limits_{x^{0'}=\mathrm{const}'} {}_{\mathrm{E}}\mathfrak{T}_{\mu'}{}^{0'} d^3x', \tag{31} $$

where ${}_{\mathrm{E}}\mathfrak{T}_{\mu'}{}^{\nu'}$ are the components of the energy-momentum tensor density expressed in terms of the field strength tensor $F_{\mu'\nu'}$ in the well-known way. In [2] you can find considerations similar to those you have read here in connection with (25), ..., (28). There you can read the conclusion that ${}_{\mathrm{E}}P_{\mu}(x^{0'} = \mathrm{const}')$ is a four vector, if

$$ \int\limits_{x^0=\mathrm{const}} {}_{\mathrm{E}}\mathfrak{T}_{\mu}{}^{k} d^3x = 0 \tag{32} $$

holds. The question of time dependence of ${}_{\mathrm{E}}\mathfrak{T}_{\mu}{}^{\nu}$ was not discussed there, but it seems to have been implicitly supposed that ${}_{\mathrm{E}}\mathfrak{T}_{\mu}{}^{\nu}{}_{,0} = 0$, because the authors had in mind a particle model and thought of the Coulomb field. According to our discussion here, it is clear that (32) is not needed for the proof that ${}_{\mathrm{E}}P_{\mu}(x^{0'} = \mathrm{const}')$ are the components of a four vector with respect to affine coordinate transformations. If we choose a family of spacelike hypersurfaces to which the coordinate system $\{x^{\mu}\}$ is adapted, which means that the family of hypersurfaces is given by $x^{0'} = \mathrm{const}'$, then ${}_{\mathrm{E}}P_{\mu'}(x^{0'} = \mathrm{const}')$ are from their definition components of a free four vector with respect to changes of the affine coordinate system. On the other hand, (32) makes sure that ${}_{\mathrm{E}}P_{\mu}(x^0 = \mathrm{const})$ can be calculated in the same way for all frames of reference which are equivalent under the special relativity principle. If there is only the electromagnetic field present, then we will have ${}_{\mathrm{E}}\mathfrak{T}_{\mu}{}^{\nu}{}_{,\nu} = 0$ and if the Maxwell equations had had a static regular solution (which, of course, is not the case) which had behaved like the Coulomb field for $r \to \infty$, then (32) would have been automatically fulfilled. In the case when there is another field present besides the electromagnetic one, we have, in general,

$$ {}_{\mathrm{E}}\mathfrak{T}_{\mu}{}^{\nu}{}_{,\nu} \neq 0 \tag{33} $$

and ${}_{\mathrm{E}}P_{\mu}(\Sigma)$ is a free four vector which depends on the choice of the system of reference [5]. In this case, it is not possible to change the affine coordinate system and calculate ${}_{\mathrm{E}}P_{\mu}(\Sigma)$ in the same way as it has been done in the former case using the frame of reference to which the coordinate system $\{x^{\mu'}\}$ is adapted now. From the point of view suggested by the special relativity principle, the meaning of this result seems to be that it is senseless to split the total energy and momentum of a more complex physical system into various parts.

b. Energy-momentum complex of GRT

In the framework of GRT C. Møller [9] has discussed the energy-momentum complex ${}_{\mathrm{L}}\mathscr{T}_{\mu}{}^{\nu}$ defined by

$$ {}_{\mathrm{L}}\mathscr{T}_{\mu}{}^{\nu} = \chi_{\mu}{}^{\nu\varrho}{}_{,\varrho}, \tag{34} $$

$$ \chi_{\mu}{}^{\nu\varrho} = -\chi_{\mu}{}^{\varrho\nu} = \frac{\sqrt{-g}}{\kappa}(g_{\mu\alpha,\beta} - g_{\mu\beta,\alpha})g^{\nu\alpha}g^{\varrho\beta}. \tag{35} $$

It is worth mentioning that Lorentz took into consideration this energy-momentum complex a long time ago [8] and that his paper was known to Klein [1].

Because of the antisymmetry of $\chi_\mu{}^{\nu\varrho}$, the relation

$$ {}_{\mathrm{L}}\mathcal{T}_\mu{}^\nu{}_{,\nu} = 0 \tag{36}$$

is fulfilled identically.

The total energy and momentum with respect to the frame of reference to which the coordinate system $\{x^{\mu'}\}$ is adapted is defined by

$$ {}_{\mathrm{L}}P_{\mu'}(x^{0'} = \mathrm{const}') = \int\limits_{x^{0'}=\mathrm{const}'} {}_{\mathrm{L}}\mathcal{T}_{\mu'}{}^{0'} d^3x'. \tag{37}$$

I will consider only such physical systems for which a coordinate system exists that $\chi_\mu{}^{\nu\varrho}{}_{,0} = 0$ assuming that this coordinate system is an affine one. At large distances from the matter which creates the gravitational field the line element can be written in the form

$$ ds^2 = \left(1-\frac{2M}{r}\right)dt^2 - \left(1+\frac{2M}{r}\right)\delta_{kl}dx^k dx^l. \tag{38}$$

In [8], an argument similar to that given here (see formulas (25) to (28)) is meant to be a proof that (28) is a necessary condition for ${}_{\mathrm{L}}P_\mu$ to be components of a four vector. As has been already frequently stated here (28) means really something different. Formula (28) must be fulfilled if one wants to be consistent with the requirements of the special relativity principle. The quantities ${}_{\mathrm{L}}P_\mu$ must be calculated in the same way with respect to all equivalent systems of reference. We can simply accept the definition that ${}_{\mathrm{L}}P_{\mu'}(x^{0'} = \mathrm{const}')$, being originally calculated by integration of ${}_{\mathrm{L}}\mathcal{T}_{\mu'}{}^{\nu'}$ in that system of reference to which the affine coordinate system $\{x^{\mu'}\}$ is adapted, is a four vector. This means that in another coordinate system $\{x^\mu\}$ ${}_{\mathrm{L}}P_\mu(x^{0'} = \mathrm{const}')$ is given by

$$ {}_{\mathrm{L}}P_\mu(x^{0'} = \mathrm{const}') = \frac{\partial x^{\mu'}}{\partial x^\mu}\, {}_{\mathrm{L}}P_{\mu'}(x^{0'} = \mathrm{const}'). \tag{39}$$

This statement is completely independent of (28). *But if* (28) *does not hold, then the total four momentum vector with the components* ${}_{\mathrm{L}}P_\mu(x^{0'} = \mathrm{const}')$ *will in general not be equal to the vector* ${}_{\mathrm{L}}P_\mu(x^0 = \mathrm{const})$ *that is the total four momentum vector with respect to the system of reference to which the coordinate system* $\{x^\mu\}$ *is adapted.*

Taking into account (35) and (38), one gets [8]

$$ \int\limits_{x^0=\mathrm{const}} {}_{\mathrm{L}}\mathcal{T}_\mu{}^1 d^3x = \int\limits_{x^0=\mathrm{const}} \chi_\mu{}^{1k}{}_{,k} d^3x = \oint \chi_\mu{}^{1k} dS_k \neq 0 \tag{40}$$

and so the energy-momentum complex (34), (35) really satisfies the statement which is printed above in italics. If we, however, start with v. Freud's superpotential $h_\mu{}^{\nu\varrho}$ [10] defined by

$$ h_\mu{}^{\nu\varrho} = -h_\mu{}^{\varrho\nu} = \frac{g_{\mu\alpha}}{2\kappa\sqrt{-g}}\left[(-g)\left(g^{\nu\alpha}g^{\beta\varrho} - g^{\varrho\alpha}g^{\nu\beta}\right)\right]_{,\beta} \tag{41}$$

and with the energy-momentum complex

$$\theta_\mu{}^\nu = h_\mu{}^{\nu\varrho}{}_{,\varrho}, \tag{42}$$

then (28) is satisfied. From the point of view dictated by the special principle of relativity $\theta_\mu{}^\nu$ is a "better" energy-momentum complex than ${}_{\mathrm{L}}\mathscr{T}_\mu{}^\nu$.

In [2] you can read that (28) means stability of the physical system considered. If this had been true in general, we would have concluded that, in GRT stability depends on which energy-momentum complex we choose. But this is nonsense, since physical properties like stability of a system cannot depend on the mathematical description used.

c. Scalar-tensor theory of gravitation

We start with the representation of the theory [11] in which the weak equivalence principle is satisfied. The "Einstein" equations then read

$$R_{\mu\nu}-\tfrac{1}{2}\, g_{\mu\nu}R = \frac{8\pi}{c^4}\frac{1}{\psi}\, T_\mu{}^\nu + \frac{\omega}{\psi^2}(\psi^\mu\psi^\nu-\tfrac{1}{2}\, g_{\mu\nu}\psi_\alpha\psi^\alpha)+\frac{1}{\psi}(\psi_{;\mu\nu}-g_{\mu\nu}\Box\psi). \tag{43}$$

The scalar field ψ, which can be interpreted as analogous to the gravitational coupling "constant", is an additional degree of freedom of the theory. It is well-known that the Einstein tensor density may be written as

$$\sqrt{-g}\,(R_\mu{}^\nu-\tfrac{1}{2}\,\delta_\mu{}^\nu R) = h_\mu{}^{\nu\varrho}{}_{,\varrho}-\mathfrak{f}_\mu{}^\nu. \tag{44}$$

Then, as it follows from (43),

$$\left[\frac{8\pi}{c^4}\,\mathfrak{T}_\mu{}^\nu+\psi\mathfrak{f}_\mu{}^\nu+\frac{\omega}{\psi}(\psi_\mu\psi^\nu-\tfrac{1}{2}\,\delta_\mu{}^\nu\psi_\alpha\psi^\alpha)+\psi_{;\mu}{}^\nu-\delta_\mu{}^\nu\Box\psi\right]_{,\nu} = h_\mu{}^{\nu\varrho}{}_{,\varrho}\psi_\nu, \tag{45}$$

which, in general, does *not* mean that the quantity

$$P_\mu^{(1)}(\Sigma) := \int\limits_\Sigma \left[\mathfrak{T}_\mu{}^\nu+\frac{\psi c^4}{8\pi}\,\mathfrak{f}_\mu{}^\nu+\frac{\omega c^4}{8\pi\psi}(\psi_\mu\psi^\nu-\tfrac{1}{2}\,\delta_\mu{}^\nu\psi_\alpha\psi^\alpha)+\psi_{;\mu}{}^\nu-\delta_\mu{}^\nu\Box\psi\right]d\Sigma_\nu \tag{46}$$

in conserved. If the integrals in (46) were defined, that means were finite with respect to a certain affine coordinate system, we could define $P_\mu^{(1)}(\Sigma)$ to be the components of a free vector with respect to affine coordinate transformations. In general, this free vector would depend on the chosen hypersurface and there would be only one affine coordinate system in which $P_\mu^{(1)}(\Sigma)$ is calculated as described in (46). For each other affine coordinate system we would obtain

$$P_{\mu'}{}^{(1)}(\Sigma) = \frac{\partial x^\nu}{\partial x^{\mu'}}\, P_\nu{}^{(1)}(\Sigma). \tag{47}$$

There is, however, another possibility of defining the total energy and momentum in the scalar-tensor theory of gravitation. At first sight, this possibility seems to be quite satisfactory, but soon, it discloses some oddity of the scalar-tensor theory of gravitation. According to the idea of this theory, the scalar field ψ is created by local and cosmic matter distributions. A consequence of this is that ψ does not approach zero for $r \to \infty$, but it tends to ψ_0, the value of the cosmic scalar background field. We shall see below what it means.

From (44) we get the local conservation law

$$\left[\frac{8\pi}{c^4}\frac{1}{\psi}\mathfrak{T}_\mu{}^\nu + \mathfrak{t}_\mu{}^\nu + \frac{\omega}{\psi^2}(\psi_\mu\psi^\nu - \tfrac{1}{2}\delta_\mu{}^\nu\psi_\alpha\psi^\alpha) + \frac{1}{\psi}(\psi_{;\mu}{}^\nu - \delta_\mu{}^\nu\Box\psi)\right]_{,\nu} = 0, \qquad (48)$$

where $8\pi/(c^4\psi)$ is the effective coupling "constant". Now, we multiply (48) by $1/\kappa_0$, where κ_0 is the value of this coupling "constant" at a certain point of space-time inside the local physical system under consideration. Then, we define

$$P_\mu{}^{(2)}(\Sigma) := \int_\Sigma \left[\frac{8\pi}{c^4\psi}\frac{1}{\kappa_0}\mathfrak{T}_\mu{}^\nu + \frac{1}{\kappa_0}\mathfrak{t}_\mu{}^\nu + \frac{\omega}{\kappa_0\psi^2}(\psi_\mu\psi^\nu - \tfrac{1}{2}\delta_\mu{}^\nu\psi_\alpha\psi^\alpha) + \frac{1}{\kappa_0}(\psi_{;\mu}{}^\nu - \delta_\mu{}^\nu\Box\psi)\right] d\Sigma_\nu \qquad (49)$$

and look upon $P_\mu^{(2)}(\Sigma)$ as being the total energy and momentum. The procedure follows now the lines as described above. We define $P_\mu^{(2)}(\Sigma)$ to be the components of a free four vector with respect to affine coordinate transformations. From the linear approximation which holds at large distances from a local non-radiating physical system, neglecting additionally the variability of the cosmic scalar background field ψ_0, it follows that $P_\mu^{(2)}(\Sigma)$ do not depend on how Σ is chosen from a set of non-intersecting spacelike hypersurfaces. In this case, the first derivatives of all fields behave like r^{-1} and the second ones like r^{-2}, at least, for $r \to \infty$. But, even though we have the conservation law (48), we cannot conclude that $P_\mu^{(2)}(\Sigma)$ and $P_\mu^{(2)}(\Sigma')$, where Σ and Σ' belong to two different families of non-intersecting spacelike hypersurfaces, are numerically equal. To obtain such a result we need the expressions

$$\frac{8\pi}{c^4}\frac{1}{\psi}\mathfrak{T}_\mu{}^\nu + \mathfrak{t}_\mu{}^\nu + \frac{\omega}{\psi^2}(\psi_\mu\psi^\nu - \tfrac{1}{2}\delta_\mu{}^\nu\psi_\alpha\psi^\alpha) + \frac{1}{\psi}(\psi_{;\mu}{}^\nu - \delta_\mu{}^\nu\Box\psi) \qquad (50)$$

to behave like $r^{-(3+\alpha)}$ $(\alpha > 0)$ for $r \to \infty$. However, ψ tends to $\psi_0 \neq 0$ and the second derivatives behave like r^{-3} for $r \to \infty$ in the case of non-radiating systems. So, it is the last term in (50) which makes that $P_\mu^{(2)}(\Sigma)$ depends on the choice of the frame of reference and cannot be calculated in the same way (given by (49)) for any equivalent frame of reference. Of course, we can make a choice and calculate with respect to a certain family of hypersurfaces in the way described by (49). The value of $P_{\mu'}^{(2)}(\Sigma)$ is then a linear combination of the numerical values of $P_\mu^{(2)}(\Sigma)$. We could do the same with respect to a family of hypersurfaces which contains Σ'. But then $P_\mu^{(2)}(\Sigma')$ would, in general, not be equal to $P_{\mu'}^{(2)}(\Sigma)$ nor to the mentioned linear combination of $P_\mu^{(2)}(\Sigma)$. This seems to indicate that the concept of the total energy and momentum is an idea less meaningful in the scalar--tensor theory of gravitation than in GRT.

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