THE UNIQUENESS OF THE REGULARIZATION PROCEDURE

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On the grounds of the BPHZ procedure, the criteria of correct regularization in perturbation calculations of QFT are given, together with the prescription for dividing the regularized formulas into the finite and infinite parts.

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1. *Introduction*

In order to avoid infinite results in perturbation calculations of QFT, we must regularize the distributions coming into indefinite products and after all calculations we have to "take off" the regularization.

Up to now many methods of regularization were worked out [1–6]. For each method we have to compare separately the finite results with the results given by the R-operation in the BPHZ procedure [7–9]. This procedure, in principle, may be used instead of any regularization [10], but in practice we often use regularization.

In this investigation we look for a simple, easy to check and universal criterion satisfied by any correct regularization. The second question we deal with, lying close to the former one is how to define the so-called infinite part to ensure consistency with the BPHZ procedure.

2. *The BPHZ procedure*

In order to fix the notation we report on the main steps of the BPHZ procedure. In the classical approach [7, 10] the $\hat{R}$-operation is performed on the expression given in the so-called α-representation i.e. the propagators are represented by the expressions

$$(p^2 - m^2 + i\varepsilon)^{-1} = \int\limits_0^\infty e^{i\alpha(p^2 - m^2 + i\varepsilon)} d\alpha, \tag{2.1}$$

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thanks to which the integrations over the internal momenta are of the gaussian type. For a given diagram with l internal lines, after these integrations we obtain an expression proportional to

$$I(k) = \iint_0^\infty d\alpha_1 \ldots d\alpha_l D^{-2}(\alpha) \exp\left[i\,\frac{A(\alpha, k)}{D(\alpha)} - i \sum_{j=1}^{l} \alpha_j (m_j^2 - i\varepsilon)\right], \tag{2.2}$$

$\alpha \equiv \alpha_1, \ldots \alpha_l$ (the set of variables α_j), k = the set of external invariants, where $D(\alpha)$, as usual, is the sum over all the trees of the diagram from the products of α-s connected with lines being absent in the tree, whereas $A(\alpha, k)$ is the sum over all the 2-trees of the diagram from the products of α-s absent in the 2-trees, each product multiplied by the "squared" sum of external momenta coming into one part of the 2-tree. The ultraviolet divergences are "shifted" to the $d\alpha$-integrations: the denominator $D^2(\alpha)$ in the integrand may possibly lead to some singularity in the lower limit of integration. This singularity may appear when we have to deal with a loop in the diagram. In building any tree we have to drop a line (or lines) from this loop, so in each product in $D(\alpha)$ we have some α-s connected with the lines of the loop. When these α-s are zero simultaneously, $D(\alpha)$ is zero. As we know, this singularity may be ostensible, when the loop contains a sufficiently large number of lines. In every case, when the singularity is essential for the $d\alpha$-integrations we must perform the BPHZ procedure or we have to regularize the integrand.

The BPHZ procedure is based on the following: every differentiation of the integrand of (2.2) with respect to the external momenta k lowers the order of the possible pole in α-s, because the index $A(\alpha, k)D^{-1}(\alpha)$ is a homogeneous function of the α-s of the order $(+1)$ (every tree contains one line more than every 2-tree). Thus, if the singularity appears only when all α-s are zero simultaneously (there are now divergent subdiagrams) it is sufficient to subtract from the integrand some first terms of its Maclaurin expansion with respect to the external momenta. In the case when the diagram contains divergent subdiagrams a similar operation must be first performed with respect to these subdiagrams. So, for $I(k)$ given by (2.2) we have the classical definition of the $\hat{R}$-operation in the form

$$\hat{R}I(k) = [\hat{1} - \hat{M}_{\frac{\omega_1}{2}}(\Gamma_1)] \ldots [\hat{1} - M_{\frac{\omega_k}{2}}(\Gamma_k)] I(k), \tag{2.3}$$

where $\Gamma_1 \ldots, \Gamma_k$ is the set of divergent subdiagrams of the diagram Γ (containing, if necessary, the whole diagram Γ) and the operators $(\hat{1} - \hat{M})$ realize the subtractions mentioned above[1]. The index

$$\frac{\omega_i}{2} = l_i - 2(n_i - 1) \tag{2.4}$$

is the "highest power" of invariants in subtracted terms, where l_i is the number of lines and n_i the number of vertices in Γ_i.

[1] Formally, expression (2.3) is not well defined since $I(k)$ is not defined. Sometimes the regularized expression $I_{reg}(k)$ is considered [11], but the right-hand side of (2.3) may be understood with subtractions performed before integration as well.

The inductive structure of the $\hat{R}$-operation was the difficulty to overcome in proof of the finiteness of the right-hand side of (2.3) [7, 8].

The most simple proof was given by Zavyalov. It was based on the formula [10, 11]

$$\hat{R}I(k) = \iint_0^\infty d\alpha e^{-i\Sigma \alpha_j(m_j^2 - i\varepsilon)} \hat{M}_\kappa \left[D^{-2}(\beta) e^{i\frac{A(\beta, k)}{D(\beta)}} \right], \tag{2.5}$$

where

$$\hat{M}_\kappa \equiv \frac{1}{\left(\frac{\omega_1}{2}\right)! \cdots \left(\frac{\omega_k}{2}\right)!} \prod_{1 \leq p \leq k} \int_0^1 d\kappa_p (1-\kappa_p)^{\frac{\omega_p}{2}} \frac{\partial^{\frac{\omega_p}{2}+1}}{\partial \kappa_p^{\frac{\omega_p}{2}+1}} \kappa_p^{2(l_p - n_p + 1)} \tag{2.5'}$$

$\beta_i = \alpha_i$ in the case when the line (i) does not enter into the composition of any divergent subdiagram Γ_i. $\beta_i = \kappa_q \ldots, \kappa_r \alpha_i$ when the line (i) belongs to divergent subdiagrams $\Gamma_q \ldots, \Gamma_r$.

In (2.5) the subtractions in the variables $\boldsymbol{k}$ and in similar variables of the subdiagrams are replaced by the subtractions in the variables κ realized by the operator $\hat{M}_\kappa$. Thanks to this the integrations over external momenta of divergent subdiagrams are independent of the subtractions.

A single operator $[\hat{1} - \hat{M}(\Gamma_i)]$ cancels the infinity arising when all parameters α_i connected with the subdiagram Γ_i are zero simultaneously. The action of this operator may be substituted by the following operations[2]: proper regularization of the subdiagram Γ_i, separating and removing the infinite part and finally, by taking off the regularization in Γ_i.

3. *Regularization*

We want to answer a question "What does proper regularization mean?"

We consider a diagram Γ from which all singularities connected with divergent subdiagrams are removed by suitable operators $[\hat{1} - \hat{M}(\Gamma_i)]$ and only the possible singularity connected with the whole diagram Γ remains (Γ may be a part of some bigger diagram). As we know, the singularity, with which we possibly have to deal, arrives when all the parameters α of the diagram Γ are zero simultaneously because of the presence of the term $[D(\boldsymbol{\beta})]^{-2}$ in (2.5). Of course now Γ does not belong to $\Gamma_q, \ldots, \Gamma_r$.

Let us separate from the integrand in (2.5) the part depending on the variables α

$$e^{-\Sigma \alpha_j(m_j^2 - i\varepsilon)} D^{-2}(\boldsymbol{\beta}) e^{i\frac{A(\beta, k)}{D(\beta)}}. \tag{3.1}$$

We perform a standard change of variables

$$\alpha_j = \lambda \xi_j, \quad \lambda \in [0, \infty],$$
$$j = 1 \ldots l, \quad \xi_j \in [0, 1], \quad \sum_j \xi_j = 1, \tag{3.1'}$$
$$\left| \frac{\partial(\boldsymbol{\alpha})}{\partial(\lambda, \xi)} \right| = \lambda^{l-1}.$$

[2] We do not discuss overlaping divergences.

Now, $\lambda = 0$ is our singularity. The function $D(\boldsymbol{\beta})$ is proportional to $\lambda^{(l-n+1)}$ (a tree contains $n-1$ lines)

$$D(\boldsymbol{\beta}) = \lambda^{(l-n+1)}\Delta(\boldsymbol{\kappa}, \boldsymbol{\xi}), \tag{3.2}$$

where $\Delta(\boldsymbol{\kappa}, \boldsymbol{\xi})$ may be zero when some sets of κ-s and ξ-s are zero but, according to our assumption, these singularities are removed by the operator $\hat{M}'_\kappa$ for subdiagrams. $\hat{M}'_\kappa$ is an operator analogous to $\hat{M}_\kappa$ (2.5′), but without Γ in the set $\Gamma_q, \ldots, \Gamma_r$.

From (2.5), (2.5′), (3.1′) and (3.2) we have

$$\hat{R}'I(\boldsymbol{k}) = \iint d\xi_1 \ldots d\xi_{l-1}\hat{M}'_\kappa[\Delta^{-2}(\zeta) \int_0^\infty d\lambda f(\zeta, \boldsymbol{k}, \lambda)], \tag{3.3}$$

where

$$f(\zeta, \boldsymbol{k}, \lambda) = \frac{1}{\lambda^{\frac{\omega_\Gamma}{2}+1}} e^{i\frac{A(\zeta, \boldsymbol{k})}{D(\zeta)}\lambda} e^{-i\lambda\sum_j \zeta_j(m_j^2 - i\varepsilon)} \tag{3.3′}$$

and the set of variables $\boldsymbol{\zeta}$ is created from $\boldsymbol{\xi}$ in the same manner, as the variables $\boldsymbol{\beta}$ are created from $\boldsymbol{\alpha}$ (see the definition after (2.5)).

The integral over λ is singular in the lower limit if $\omega_\Gamma \geq 0$.

From the definition of $A(\zeta, \boldsymbol{k})$ it follows that every differentiation of $f(\zeta, \boldsymbol{k}, \lambda)$ with respect to some invariant of the set $\boldsymbol{k}$ introduces λ into the numerator of the integrand, so the expression

$$\partial^{\boldsymbol{t}}_{\boldsymbol{k}} f(\zeta, \boldsymbol{k}, \lambda) \equiv \frac{\partial^{t_1}}{(\partial k_1)^{t_1}} \cdots \frac{\partial^{t_m}}{(\partial k_m)^{t_m}} f(\zeta, \boldsymbol{k}, \lambda),$$

$$\sum_{i=1}^{m} t_i \geq \frac{\omega_\Gamma}{2} + 1, \tag{3.4}$$

($k_1 \ldots, k_m$ denote any subset of invariants $\boldsymbol{k}$), is integrable over λ, which is the basis of BPHZ procedure.

The alternative solution (with respect to BPHZ) is regularization. We can look for some function $\Phi(\zeta, \boldsymbol{k}, \lambda, \gamma)$, where

$$\lim_{\gamma \to \gamma_0} \phi(\zeta, \boldsymbol{k}, \lambda, \gamma) = f(\zeta, \boldsymbol{k}, \lambda) \tag{3.5}$$

(at least in the sense of point-wise convergence with respect to the variable λ for $\lambda \neq 0$), and the pole $\lambda = 0$ becomes integrable (e.g. the analytical [3, 5] or dimensional [4, 6] regularization) or is removed from the domain of integration for $\gamma \neq \gamma_0$.

After integration over λ is performed

$$\int_0^\infty d\lambda \phi(\zeta, \boldsymbol{k}, \lambda, \gamma) = F(\zeta, \boldsymbol{k}, \gamma), \tag{3.6}$$

we have to separate in $F(\zeta, \boldsymbol{k}, \gamma)$ the "infinite part" $S(\zeta, \boldsymbol{k}, \gamma)$ singular for $\gamma = \gamma_0$

$$F(\zeta, \boldsymbol{k}, \gamma) = S(\zeta, \boldsymbol{k}, \gamma)+a_0(\zeta, \boldsymbol{k})+N(\zeta, \boldsymbol{k}, \gamma), \tag{3.7}$$

where

$$N(\zeta, \boldsymbol{k}, \gamma)|_{\gamma=\gamma_0} = [a_1(\zeta, \boldsymbol{k})(\gamma-\gamma_0)+a_2(\zeta, \boldsymbol{k})(\gamma-\gamma_0^2)+ \dots]_{\gamma=\gamma_0} = 0, \tag{3.7'}$$

and to remove it (formally, by introducing a suitable counterterm into the lagrangian). Putting $\gamma = \gamma_0$ gives the finite part $a_0(\zeta, \boldsymbol{k})$.

Without any additional requirements, the division of F into the singular S and non-singular a_0+N parts is non-uniquely defined. Namely, we can write

$$F = S+n(\zeta, \boldsymbol{k})-n(\zeta, \boldsymbol{k})+a_0+N, \tag{3.8}$$

where $n(\zeta, \boldsymbol{k})$ is an arbitrary function, then remove $S' = S+n$ and obtain a new finite part a_0-n. These additional requirements can be found on the grounds of the BPHZ procedure, what will be discussed in Section 4.

4. *Correctness and uniqueness of regularization*

The regularization leads to a proper result if two requirements are satisfied:

1. the method of regularization (the function ϕ) is chosen properly,
2. the separation of the infinite part is properly realized.

We want to show, how the BPHZ procedure fixes the criteria of satisfying these two requirements.

The effects of regularization and BPHZ procedure coincide if

$$\partial_k^t a_0(\boldsymbol{k}) = \int_0^\infty d\lambda \partial_k^t f(\boldsymbol{k}, \lambda), \tag{4.1}$$

where ∂_k^t is defined by (3.4).

Let us take $\gamma \neq \gamma_0$[3] and differentiate both sides of (3.7)

$$\partial_k^t S(\zeta, \boldsymbol{k}, \gamma)+\partial_k^t a_0(\zeta, \boldsymbol{k})+\partial_k^t N(\zeta, \boldsymbol{k}, \gamma) = \int_0^\infty d\lambda \partial_k^t \phi(\zeta, \boldsymbol{k}, \lambda, \gamma). \tag{4.2}$$

Let us assume also

$$\lim_{\gamma\to\gamma_0} \int_0^\infty d\lambda \partial_k^t \phi = \int_0^\infty d\lambda \partial_k^t f. \tag{4.3}$$

Then

$$\lim_{\gamma\to\gamma_0} \partial_k^t S(\zeta, \boldsymbol{k}, \gamma)+\partial_k^t a_0(\zeta, \boldsymbol{k}) = \int_0^\infty d\lambda \partial_k^t f(\zeta, \boldsymbol{k}, \lambda) \tag{4.4}$$

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[3] Writing $\gamma \neq \gamma_0$ we mean γ belongs to the area, where the integral (3.6) is well defined.

and to satisfy (4.1) the necessary and sufficient condition is

$$\lim_{\gamma \to \gamma_0} \partial_k^t S(\zeta, \boldsymbol{k}, \gamma) = 0 \tag{4.5}$$

and the sufficient condition is

$$\partial_k^t S(\zeta, \boldsymbol{k}, \gamma) \equiv 0. \tag{4.5'}$$

So, if (4.3) is satisfied, the regularization is correct and the separation (3.7) is fixed by (4.5) or (4.5′) up to some additive function $n(\zeta, \boldsymbol{k})$ (cf (3.8)) satisfying

$$\partial_k^t n(\zeta, \boldsymbol{k}) \equiv 0$$

for every operator ∂_k^t defined by (3.4).

However, in the case when (4.3) is not satisfied, no prescription for separating can be given — the regularization is incorrect.

To satisfy (4.3) it is sufficient (but not necessary) for the function $\partial_k^t \phi$ (treated as a function of λ) to be uniformly convergent to $\partial_k^t f$ when $\gamma \to \gamma_0$.

5. Examples

5.1. The analytical regularization

The analytical regularization applied to propagators $\Delta_F(p)$

$$\Delta_F(p) = \frac{1}{p^2 - m^2 + i\varepsilon} = \int_0^\infty d\alpha e^{i(p^2 - m^2 + i\varepsilon)\alpha}, \tag{5.1}$$

$$\Delta_{F\,\mathrm{reg}}(p) = \frac{1}{(p^2 - m^2 + i\varepsilon)^\gamma} = \frac{i^{-\gamma}}{\Gamma(\gamma)} \int_0^\infty d\alpha \alpha^{\gamma - 1} e^{i\alpha(p^2 - m^2 + i\varepsilon)} \tag{5.1'}$$

gives

$$I_{\mathrm{reg}}(\boldsymbol{k}) = \frac{i^{-\gamma l}}{[\Gamma(\gamma)]^l} \iint_0^\infty \frac{d\alpha_1}{\alpha_1^{1-\gamma}} \cdots \frac{d\alpha_l}{\alpha_l^{1-\gamma}}$$

$$\times D^{-2}(\boldsymbol{\alpha}) \exp\left[i \frac{A(\boldsymbol{\alpha}, \boldsymbol{k})}{D(\boldsymbol{\alpha})} - i \sum \alpha_j (m_j^2 - i\varepsilon) \right] \tag{5.2}$$

instead of (2.2).

Changing variables according to (3.1′) we obtain

$$I_{\mathrm{reg}}(\boldsymbol{k}) = \frac{i^{-\gamma l}}{[\Gamma(\gamma)]^l} \iint \frac{d\lambda d\xi}{\lambda^{l(1-\gamma) + \frac{\omega}{2} + 1} \xi_1^{1-\gamma} \cdots \xi_l^{1-\gamma}}$$

$$\times D^{-2}(\xi) \exp\left[i \frac{A(\xi, \boldsymbol{k})}{D(\xi)} \lambda - i\lambda \sum \xi_j (m_j - i\varepsilon) \right]. \tag{5.3}$$

We assume $\omega_\Gamma \geqslant 0$. For $\operatorname{Re}\gamma > 1$ the term $\lambda^{\gamma-1}$ removes the singularity from the integration over λ whereas the product $\xi_1^{\gamma-1} \ldots, \xi_l^{\gamma-1}$ is responsible for removing the singularities from subdiagrams of the diagram Γ.

We have to take off the regularization for each subdiagram separately to avoid "interference" of terms proportional to $(1-\gamma)$ with the singular terms of different subdiagrams.

We assume our regularization to be correct for all divergent subdiagrams and replace the product $\xi_1^{\gamma-1} \ldots, \xi_l^{\gamma-1}$ by the operator M'_κ (3.3), which results in

$$I_{\text{reg}}(\boldsymbol{k}) = \frac{i^{-\gamma l}}{[\Gamma(\gamma)]^l} \int d\xi \hat{M}'_\kappa \left[\Delta^{-2}(\zeta) \int_0^\infty d\lambda \phi(\zeta, \boldsymbol{k}, \lambda, \gamma) \right], \tag{5.4}$$

where

$$\phi(\zeta, \boldsymbol{k}, \lambda, \gamma) = \lambda^{l(\gamma-1)-\frac{\omega}{2}-1} \exp\left[i \frac{A(\zeta, \boldsymbol{k})}{D(\zeta)} \lambda - i\lambda \sum \xi_j(m_j - i\varepsilon) \right]. \tag{5.4'}$$

The integral with which we have to deal is of the form

$$\int_0^\infty d\lambda \phi = \int_0^\infty \frac{d\lambda}{\lambda^{1-\delta}} e^{i(K+i\varepsilon)\lambda}, \tag{5.5}$$

where

$$\delta = l(\gamma-1) - \frac{\omega_\Gamma}{2}, \qquad K = \frac{A(\zeta, \boldsymbol{k})}{D(\zeta)} - \sum \xi_j m_j. \tag{5.5'}$$

As we know, the integral (5.5) is definite for $\operatorname{Re}\delta > 0 \Leftrightarrow \operatorname{Re}\omega_\Gamma < 0$ (with $\gamma = 1$) and in these cases the regularization is not necessary: after the possible singularities in subdiagrams are removed, the whole diagram is finite.

We have to check, whether the condition (4.3) is satisfied. In the notation of (5.5) the function f (cf. (3.3')) is of the form

$$f(\zeta, \boldsymbol{k}, \lambda) = \frac{1}{\lambda^{1+\frac{\omega}{2}}} e^{i(K+i\varepsilon)\lambda}. \tag{5.6}$$

The differentiation $\partial_{\boldsymbol{k}}^t$ would introduce $\lambda^{\frac{\omega}{2}+1}$ to the numerator of (5.6) — the singularity in $\lambda = 0$ would disappear. The differential $\partial_{\boldsymbol{k}}^t \phi$ is also a regular function of λ and tends uniformly to $\partial_{\boldsymbol{k}}^t f$ when $\delta \to -\frac{\omega}{2}$ ($\operatorname{Re}\delta > 0$), so the condition (4.3) is satisfied.

In order to separate the infinite part S, the expression (5.5) must be analytically continued to $\operatorname{Re}\delta > -\frac{\omega}{2}-1$

$$\int_0^\infty d\lambda\lambda^{\delta-1}\exp[i(K+i\varepsilon)] = \frac{(-iK)^{\frac{\omega}{2}+1}}{\delta(\delta+1)\dots\left(\delta+\frac{\omega}{2}\right)}\int_0^\infty \lambda^{\delta+\frac{\omega}{2}}\exp[i(K+i\varepsilon)\lambda]d\lambda. \tag{5.7}$$

The singularity $\delta = -\frac{\omega}{2}$ $(\gamma = 1)$ is a pole of the first order, whereas the integral (5.7) is already regular at this point and may be expanded in the Taylor series around it

$$\int_0^\infty \lambda^{\delta+\frac{\omega}{2}}\exp[i(K+i\varepsilon)\lambda]d\lambda$$

$$= (-iK)^{-1}+\left[\int_0^\infty \ln\lambda d\lambda e^{i(K+i\varepsilon)\lambda}\right]\left(\delta+\frac{\omega}{2}\right)+O\left[\left(\delta+\frac{\omega}{2}\right)^2\right]. \tag{5.8}$$

The first term of this expansion together with the singular term before the integral (5.7) gives the singular part $S(\zeta, \mathbf{k}, \delta)$

$$S(\zeta, \mathbf{k}, \delta) = \frac{(-iK)^{\frac{\omega}{2}}}{\delta(\delta+1)\dots\left(\delta+\frac{\omega}{2}\right)} \tag{5.9}$$

which satisfies the condition (4.5′) because of (3.4), (5.5′) and the definition of $A(\mathbf{k}, \xi)$.

5.2. Shifting the pole

Another possibility is to remove the pole in λ from the domain of integration. In this case we have, instead of (5.5),

$$\int_0^\infty d\lambda\phi(\zeta, \mathbf{k}, \lambda, \eta) = \int_0^\infty \frac{d\lambda}{(\lambda+\eta)^{\frac{\omega}{2}+1}}e^{i(K+i\varepsilon)\lambda} \tag{5.10}$$

and the criterion about uniform convergence is easy to verify.

The separation of S is not so simple as in the previous case. Let us write

$$\int_0^\infty d\lambda\phi = \int_0^1 d\lambda\phi+\int_1^\infty d\lambda\phi. \tag{5.11}$$

The second integral of (5.11) is regular for $\eta = 0$, and the singular part must be found in the first one. For $n = \omega/2+1$ we can write

$$\int_0^1 d\lambda\phi = \int_0^1 d\lambda \exp\left[-i(K+i\varepsilon)\eta\right]\left\{\sum_{l=0}^{n} \frac{(iK)^{n-l}}{(\lambda+\eta)^l(n-l)!} + O[(\lambda+\eta)^{n+1}]\right\}$$

$$= e^{-iK\eta}\int_0^1 d\lambda \sum_{l=0}^{n} \frac{(ik)^{n-l}}{(\lambda+\eta)^l(n-l)!} + \alpha(\eta, K), \tag{5.12}$$

where $\alpha(\eta, K)$ is finite for $\eta = 0$. After integration over λ we have

$$e^{-iK\eta}\left[\frac{(iK)^n}{n!}(\lambda+\eta) + \frac{(iK)^{n-1}}{(n-1)!}\ln(\lambda-\eta)\right.$$

$$\left.+\sum_{l=2}^{n} \frac{1}{1-l}(\lambda+\eta)^{1-l}(iK)^{n-l}\cdot\frac{1}{(n-l)!}\right]_{\lambda=0}^{\lambda=1} + \alpha(\eta, K)$$

$$= \bar{S}(K, \eta)+\alpha(K, \eta)+e^{-iK\eta}\frac{(iK)^n}{n!} = \bar{S}+\alpha+\beta, \tag{5.13}$$

where

$$\bar{S}(K, \eta) = e^{-iK\eta}\left[\frac{(iK)^{n-1}}{(n-1)!}\ln\frac{1+\eta}{\eta} + \sum_{l=2}^{n}\frac{1}{1-l}[(1+\eta)^{1-l}-\eta^{1-l}](iK)^{n-l}\right. \tag{5.14}$$

and β is finite for $\eta = 0$.

The expression $\bar{S}(\eta, K)$ still does not satisfy (4.5′) because of the presence of $\exp(-i(K+i\varepsilon)\eta)$. Expanding the exponent about $\eta = 0$ and bringing together the suitable terms of this expansion with the contents of brackets in (5.14) we obtain

$$\bar{S}(K, \eta) = \frac{-(iK)^{n-1}}{n-1}\ln\eta - (iK)^{n-1}\sum_{s=1-n}^{-1}\eta^s(iK)^s\sum_{t=0}^{s}\frac{1}{(s-t)!(n+t+1)!} + \gamma(K, \eta)$$

$$\equiv S(K, \eta)+\gamma(\eta, K), \tag{5.15}$$

where $\gamma(\eta, K)$ is finite for $\eta = 0$ and $S(\eta, K)$ satisfies (4.5′). The finite part is the sum of the second integral in (5.11), $\alpha(\eta, K)$, $\beta(\eta, K)$ and $\gamma(\eta, K)$ taken for $\eta = 0$.

5.3. Incorrect regularization

Finally, we give an example of a regularization similar to the analytical one, but not satisfying the criterion (4.3).

Let the singular integral be of the form

$$I(k^2) = \int_0^\infty \frac{d\lambda}{\lambda} e^{-\lambda k^2} \equiv \int_0^\infty d\lambda f(\lambda, k^2). \tag{5.16}$$

Let us take

$$I_{\text{reg}}(k^2) = \int_0^\infty \frac{d\lambda}{\lambda^{1-\gamma}} (k^2)^\gamma e^{-\lambda k^2} \equiv \int_0^\infty d\lambda \phi(\lambda, k^2, \gamma). \tag{5.17}$$

The condition (3.5) is satisfied for (5.17), but

$$\lim_{\gamma\to 0} \int_0^\infty \frac{d}{dk^2} \phi d\lambda = \lim_{\gamma\to 0} \int_0^\infty \lambda^{\gamma-1}\gamma(k^2)^{\gamma-1} e^{-\lambda k^2} + \int_0^\infty d\lambda \frac{d}{dk^2} f(\lambda, k^2) \tag{5.18}$$

and first term in (5.18) is not zero, because the integral

$$\int_0^\infty \lambda^{\gamma-1} e^{-\lambda k^2} d\lambda \overset{(\text{Re}\,\gamma>0)}{=} \frac{k^2}{\gamma} \int_0^\infty \lambda^\gamma e^{-\lambda k^2} \tag{5.19}$$

defines a function with a pole of the first order in $\gamma = 0$, so

$$\lim_{\gamma\to 0} \gamma(k^2)^{\gamma-1} \int_0^\infty \lambda^{\gamma-1} e^{-\lambda k^2} d\lambda = \int_0^\infty e^{-\lambda k^2} d\lambda \neq 0 \tag{5.20}$$

and the condition (4.3) is not satisfied.

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