GAUGE INVARIANT SURFACE CONTRIBUTION TO THE NUMBER OF PHOTONS INTEGRAL

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The surface contribution to the total number of transverse zero frequency photons is calculated as a gauge invariant surface integral around the tip of the light cone in the momentum space. A similar integral is introduced for nontransverse photons and is shown to be equal to $-Q^2/\pi$, where Q is the total charge of nontransverse photons.

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1. *Introduction*

When a charged particle is scattered by an external field, low frequency radiation is produced. The amplitude $a_\mu(k)$ of the radiation has, for $k^0 \to 0$, the form

$$a_\mu(k) = \frac{e}{2\pi}\left(\frac{p_\mu}{pk} - \frac{q_\mu}{qk}\right). \tag{1}$$

Here e is particle's charge, p is the initial momentum and q is the final momentum. The amplitude (1) has a characteristic homogeneity property: for each real λ

$$a_\mu(\lambda k) = \lambda^{-1} a_\mu(k).$$

In a real scattering process the amplitude (1) is the first term of the asymptotic expansion of the amplitude for $k^0 \to 0$. Disregarding higher terms we can write the low frequency part of the electromagnetic potential produced in a general scattering process as follows

$$A_\mu(x) = \frac{1}{2\pi}\int \frac{d^3k}{k^0}\, a_\mu(k) f(k) e^{-ikx} + \text{c.c.}\,. \tag{2}$$

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Here

$$a_\mu(k) = \frac{1}{2\pi} \sum_s e_s \frac{p_{s\mu}}{p_s k}, \qquad \sum_s e_s = 0.$$

In the sum above terms corresponding to incoming particles are taken with their proper sign while terms corresponding to outgoing particles are taken with the opposite sign. $f(k)$ is a function which cuts off higher frequencies: $f(k) = 1$ for $k^0 = 0$ while for $k^0 > 0$ $f(k)$ tends very rapidly to zero. In other words, the domain in which $f(k)$ differs significantly from zero is an infinitesimal volume around the tip of the future light cone $kk = 0$. Following Zwanziger [1] we call each field of the form (2) a zero frequency field. It turns out that zero frequency fields cause finite observable effects for each $f(k)$ i.e. regardless of how rapidly $f(k)$ falls off to zero.

Units and conventions used in this paper are summarized in the expressions for the action S, the potential $A_\mu(x)$ and the number of photons N:

$$S = -\frac{1}{16\pi} \int d^4x F_{\mu\nu} F^{\mu\nu}, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{3}$$

$$A_\mu(x) = \frac{1}{2\pi} \int \frac{d^3k}{k^0} a_\mu(k) e^{-ikx} + \text{c.c.}, \tag{4}$$

$$N = -\int \frac{d^3k}{k^0} \overline{a_\mu(k) a^\mu(k)}. \tag{5}$$

We shall repeatedly use in this paper the invariant element of integration over the set of null directions of the light cone. The element can be introduced as follows [1]. The element d^3k/k^0 is known to be Lorentz invariant. Let us put

$$\frac{d^3k}{k^0} \equiv \frac{dk^0}{k^0} d^2k.$$

This is the definition of d^2k. Since dk^0/k^0 is obviously Lorentz invariant, d^2k must be also an invariant.

To illustrate these remarks consider the integral which frequently occurs in low frequency calculations. Let p and q be time like vectors. Then

$$\int d^2k \frac{pq}{(pk)(qk)} = \frac{2\pi}{v} \ln \frac{1+v}{1-v},$$

where

$$v = \sqrt{1 - \frac{(pp)(qq)}{(pq)^2}}.$$

The Lorentz invariance od d^2k means that if one integrates an invariant and homogeneous of degree -2 function of k, one obtains a manifestly Lorentz invariant result.

2. *Observable effects produced by zero frequency fields*

Suppose that a test particle moves in the zero frequency field (2). The field is extremely weak and contains only extremely small frequencies. Therefore one can solve the equations of motion in the quasiclassical approximation. The phase S of the wave function has the form $S = -px+\delta$, where p is particle's momentum and δ describes the correction caused by the external field. Putting this into the Hamilton–Jacobi equation

$$g^{\mu\nu}\left(\frac{\partial S}{\partial x^\mu}+eA_\mu\right)\left(\frac{\partial S}{\partial x^\nu}+eA_\nu\right) = m^2$$

and dropping squares of δ and A_μ one has

$$p^\mu\left(\frac{\partial \delta}{\partial x^\mu}+eA_\mu\right) = 0.$$

This equation can be solved in particle's rest frame in which it reads

$$\frac{\partial \delta}{\partial x^0} = -eA_0 = -\frac{e}{\pi}\int \frac{d^3k}{k^0}\, a_0(k) f(k) \cos kx.$$

Integrating one has

$$\delta = -\frac{e}{\pi}\int \frac{d^3k}{k^0}\, a_0(k) f(k)\frac{1}{k^0} \sin kx.$$

Integration constant was chosen so that the correction vanishes for $x = 0$. Going back to the original system we have

$$\delta = -\frac{e}{\pi}\int \frac{d^3k}{k^0}\,\frac{p^\mu}{pk}\, a_\mu(k) f(k) \sin kx$$

$$= -\frac{e}{\pi}\int d^2k\,\frac{p^\mu}{pk}\, a_\mu(k)\cdot \int \frac{dk^0}{k^0} f(k) \sin kx.$$

For x at the future or past infinity

$$\int_0^\infty \frac{dk^0}{k^0} f(k) \sin kx = \frac{\pi}{2}\,\mathrm{sign}\, kx;$$

this follows from Dirichlet's lemma.

Summing up we can say that if a test particle moves through the zero frequency field (2), the phase of each plane wave receives a finite shift

$$\delta = -\frac{e}{2}\int d^2k\,\frac{p^\mu}{pk}\, a_\mu(k)\, \mathrm{sign}\, kx.$$

Between plus and minus infinity the shift amounts to

$$\delta(p) = -e \int d^2k \, \frac{p^\mu}{pk} \, a_\mu(k). \tag{6}$$

For a single plane wave the shift is obviously unobservable, but for a normalizable wave packet the shift $\delta(p)$ for each plane wave will produce an observable interference effect.

The result (6) is gauge invariant; performing a gauge transformation $a_\mu(k) \to a_\mu(k) + \varphi(k)k_\mu$ one sees that the corresponding change of $\delta(p)$ does not depend on p and therefore gives an unobservable change of the overall phase of the wave packet.

3. The total number of zero frequency photons

The results of the preceding section show that zero frequency fields are observable. However, they do not belong to the Hilbert space of one photon states: the number of photons (5) computed for the zero frequency field (2) is infinite:

$$N = -\int d^2k a_\mu(k) a^\mu(k) \cdot \int \frac{dk^0}{k^0} f^2(k). \tag{7}$$

The first factor is a well defined, Lorentz and gauge invariant integral but the second factor diverges because $f(k^0 = 0) = 1$.

The problem of finding a norm of one photon states applicable to zero frequency fields as well as to ordinary fields was posed by Zwanziger [1]. He gives an expression (formula A. 19 in Ref. [1]) which does indeed exist for zero frequency fields but is not gauge invariant. This may be seen as follows: for a zero frequency field which is a gradient,

$$a_\mu(k) = \varphi(k)k_\mu,$$

where $\varphi(k)$ is an arbitrary homogeneous of degree -2 function of k, one obtains from Zwanziger's norm

$$N = -\frac{1}{4\pi} \left| \int d^2k \varphi(k) \right|^2 \leqslant 0$$

while, looking at the ordinary norm (5), one would say that the norm of a gradient field should vanish. For this reason we think that the problem of constructing a norm appropriate for zero frequency fields remains open.

We shall argue that the norm of a pure zero frequency field i.e. of a real field $a_\mu(k)$ such that $a_\mu(\lambda k) = \lambda^{-1} a_\mu(k)$ should be defined as

$$N_1 = -\int d^2k a_\mu(k) a^\mu(k). \tag{8}$$

Because of (7) this apparently implies that

$$\int_0^\infty \frac{dk^0}{k^0} f^2(k) = 1. \tag{9}$$

But this equality contains a contradiction and cannot be properly derived. Therefore our argument should be interpreted as supporting the choice (8) rather then the impossible equality (9).

Our starting point is the usual norm

$$N = -\int \frac{d^3k}{k^0} \overline{a_\mu(k)} a^\mu(k),$$

in which

$$a_\mu(k) = \frac{1}{2\pi} \sum_s e_s \frac{p_{s\mu}}{p_s k} f(k).$$

Suppose for a moment that $\sum_s e_s = Q \neq 0$; later we shall come back to the case $Q = 0$. Q is the missing charge i.e. the charge which came in but did not go out. For $Q \neq 0$ the usual norm ceases to be gauge invariant. Moreover, one can choose a gauge so that

$$a_\mu(k) a^\mu(k) \equiv 0.$$

This requirement fixes the gauge uniquely. In this gauge the total number of photons ceases to be infinite and becomes an undefined expression $0 \cdot \infty$.

Now, if the probability current in the k-space vanishes at each point, the probability current in the x-space is a divergence and the total number of photons can be unambiguously computed by means of the Gauss–Ostrogradski theorem as a surface integral at spatial infinity. We shall now perform this calculation.

Since the functional form of $f(k)$ is not relevant, we choose

$$f(k) = e^{-vk^0},$$

where v is an extremely large distance. Integrating over k^0 in (2) we have

$$A_\mu(x) = \frac{1}{2\pi} \int d^2k a_\mu(k) (vk^0 + ikx)^{-1} + \text{c.c.}\,.$$

The total number of photons equals

$$N = \frac{i}{4\pi} \int d^3x \{A_\mu^{(-)} \partial_0 A^{\mu(+)} - \text{c.c.}\},$$

where $A_\mu^{(\pm)}$ are, respectively, positive and negative energy parts of A_μ. Hence

$$N = -\frac{1}{16\pi^3} \int d^2k a_\mu(k) \int d^2l a^\mu(l) \int_{x^0=0} d^3x (vk^0 + ikx)^{-1} l^0 (vl^0 - ilx)^{-2} + \text{c.c.}\,.$$

The integral over the volume $x^0 = 0$ is not absolutely convergent. If, however, one integrates over angular variables first, one obtains a well defined result

$$\int_{x^0=0} d^3x(vk^0+ikx)^{-1}l^0(vl^0-ilx)^{-2} = \frac{2\pi^2}{kl}.$$

Therefore

$$N = -\frac{1}{4\pi}\int\frac{d^2kd^2l}{kl}\, a^\mu(k)g_{\mu\nu}a^\nu(l). \tag{10}$$

Note, that the gauge condition $a_\mu(k)a^\mu(k) \equiv 0$ is necessary and sufficient for the existence of the last integral.

The expression (10) is a part of Zwanziger's generalized norm [1]. Actually, it is the part responsible for the undesirable property of Zwanziger's norm which has been indicated above: for $a_\mu(k) = \varphi(k)k_\mu$ equation (10) gives

$$N = -\frac{1}{4\pi}\left[\int d^2k\varphi(k)\right]^2 \leqslant 0.$$

Our calculation shows the physical meaning of (10): it is the value of the integral

$$N = \int d^3xj_0(x),$$

where

$$j_\lambda = \frac{i}{4\pi}A_\mu^{(-)}\partial_\lambda A^{\mu(+)}+\text{c.c.}, \tag{11}$$

calculated in the gauge in which the integrand is a divergence.

For a transverse field ($Q = 0$) the integral (10) does not exist. However, the integral (10) is actually a surface contribution and we can change it by adding a divergence to the current (11), for example a term like

$$\frac{i}{4\pi}\partial^\mu(A_\lambda^{(-)}A_\mu^{(+)}-A_\mu^{(-)}A_\lambda^{(+)}). \tag{12}$$

The additional surface term should be chosen so that the total number of particles vanishes for a gradient field.

Investigating the surface contribution to N resulting from (12) we were led to the conclusion that the right additional term is chosen by the following Ansatz: the tensor $g_{\mu\nu}$ in (10) should be replaced by

$$g_{\mu\nu}+k_\mu\frac{\partial}{\partial k^\nu}-k_\nu\frac{\partial}{\partial k^\mu},$$

where the differential operators are supposed to act on $a^{\mu}(k)$:

$$N_1 = -\frac{1}{4\pi}\int \frac{d^2k d^2l}{kl}\, a^{\nu}(l)\left(g_{\mu\nu}+k_{\mu}\frac{\partial}{\partial k^{\nu}}-k_{\nu}\frac{\partial}{\partial k^{\mu}}\right)a^{\mu}(k). \tag{13}$$

The last integral has two important properties: it does exist for a transverse field and it vanishes for a gradient field.

The integral (13) is always equal to the integral (8). One can prove this calculating both integrals for the most general infrared field

$$a_{\mu}(k) = \frac{1}{2\pi}\sum_{s} e_s \frac{p_{s\mu}}{p_s k}, \qquad \sum_s e_s = 0.$$

The proof relies on the identity

$$\int d^2k \frac{1}{(pk)^2}\ln\frac{pk}{qk} + \int d^2k \frac{pq}{pkqk} = 4\pi,$$

valid for each pair of unit, future oriented time like vectors p, q; the identity is easily established by integration over angular variables.

Summing up we can say that the integral (13) (or (8)) gives the gauge invariant surface contribution to the total number of zero frequency photons.

4. *The case of nonvanishing missing charge*

Suppose that $Q = \sum_s e_s \neq 0$. Processes with $Q \neq 0$ are impossible in the Maxwell electrodynamics but are possible in the generalized Dirac-Fock-Podolsky electrodynamics [2, 3] (see also the Appendix). For $Q \neq 0$ the integral (13) does not exist. Now, the integral (10), which does not exist even in the Maxwellian case $Q = 0$ is made convergent by the substitution

$$g_{\mu\nu} \to g_{\mu\nu}+k_{\mu}\frac{\partial}{\partial k^{\nu}}-k_{\nu}\frac{\partial}{\partial k^{\mu}}$$

and, after the substitution, gives the reasonable result (8). The integral (13) does exist for a transverse field but does not exist for a nontransverse field. One might therefore try to make it convergent by applying the substitution twice. Thus we are led to the integral

$$N_2 = -\frac{1}{4\pi}\int \frac{d^2k d^2l}{kl}\left\{\left(g_{\mu\nu}+k_{\nu}\frac{\partial}{\partial k^{\mu}}-k_{\mu}\frac{\partial}{\partial k^{\nu}}\right)a^{\nu}(k)\right.$$
$$\left.\times\left(g^{\mu\lambda}+l^{\lambda}\frac{\partial}{\partial l_{\mu}}-l^{\mu}\frac{\partial}{\partial l_{\lambda}}\right)a_{\lambda}(l)\right\}. \tag{14}$$

The last integral does indeed exist and equals

$$N_2 = -\frac{Q^2}{\pi},$$

where $Q = \sum_s e_s$ is the missing charge. In fact, putting

$$a_\mu(k) = \frac{1}{2\pi}\sum_s e_s \frac{p_{s\mu}}{p_s k}$$

into the integral (14) one has

$$N_2 = -\frac{1}{4\pi}\int d^2k \int d^2l \frac{\partial a^\mu(k)}{\partial k^\mu}\frac{\partial a^\nu(l)}{\partial l^\nu}$$

$$= -\frac{1}{4\pi}\left[\int d^2k \frac{\partial a^\mu(k)}{\partial k^\mu}\right]^2 = -\frac{Q^2}{\pi}.$$

5. The total action of a zero frequency field

We shall calculate now the total action

$$S = -\frac{1}{16\pi}\int d^4x(F_{\mu\nu}F^{\mu\nu}+2F^2),$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad F = \partial^\mu A_\mu,$$

and $A_\mu(x)$ has the form (2). If equations of motion hold, the integrand is a divergence:

$$F^{\mu\nu}F_{\mu\nu}+2F^2 = 2\partial^\mu(F_{\mu\nu}A^\nu + FA_\mu)$$

and the total action can be computed as a surface integral over a suitably chosen surface which subsequently will tend to infinity. Following Gervais and Zwanziger [4] we choose as a surface of integration the four-dimensional pseudosphere $|xx| = R^2 =$ const. We assume that R is so large that $f(k)$ in (2) can be replaced by $f(k^0 = 0) = 1$. In this way we have

$$A_\mu(x) = \int d^2k a_\mu(k)\delta(kx).$$

One sees that $A_\mu(x)$ vanishes for time like x and the integration reduces to the time like hyperboloid $xx = -R^2$:

$$S = -\frac{1}{8\pi}\int_{xx=-R^2} dS_\mu(F^{\mu\nu}A_\nu + FA^\mu)$$

$$= -\frac{1}{8\pi}\int d^2k\int d^2l\{[k^\mu a^\nu(k) - k^\nu a^\mu(k)]a_\nu(l) + k^\nu a_\nu(k)a^\mu(l)\}\int_{xx=-R^2} dS_\mu \delta'(kx)\delta(lx).$$

The integral over the surface $xx = -R^2$ is not absolutely convergent. If, however, one introduces the spherical coordinates

$$x^0 = R \operatorname{sh} \psi,$$

$$x^1 = R \operatorname{ch} \psi \sin \vartheta \cos \varphi,$$

$$x^2 = R \operatorname{ch} \psi \sin \vartheta \sin \varphi,$$

$$x^3 = R \operatorname{ch} \psi \cos \vartheta,$$

and integrates over φ and ϑ first, one obtains a well defined result

$$\int\limits_{xx=-R^2} dS_\mu \delta'(kx)\delta(lx) = \frac{\partial}{\partial k^\mu} \frac{2\pi}{kl}.$$

Therefore

$$S = \tfrac{1}{4} \int \frac{d^2k d^2l}{kl} a^\nu(l) \left(g_{\mu\nu} + k_\mu \frac{\partial}{\partial k^\nu} - k_\nu \frac{\partial}{\partial k^\mu} \right) a^\mu(k).$$

Comparing with (13) we see that

$$S = -\pi N_1.$$

If N_1 is interpreted as the number of photons, the last equality says that the total action is a multiple of a half of the old Planck constant, which sounds reasonable and supports the idea that

$$N_1 = - \int d^2k a_\mu(k) a^\mu(k)$$

should be interpreted as the number of zero frequency photons.

APPENDIX

An interpretation of the electrodynamics of Dirac, Fock and Podolsky

As it is well known, one of the Maxwell equations does not contain the time derivative; this causes a considerable trouble in the quantum version of the theory. Dirac, Fock and Podolsky [2] introduced a modified theory of a tensor field $F_{\mu\nu} = -F_{\nu\mu}$ and a scalar field F which fulfil the equations

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0,$$

$$\partial^\mu F_{\mu\nu} + \partial_\nu F = 0. \tag{A1}$$

A natural question arises: what is the physical interpretation of the additional field F? We give such an interpretation below; the interpretation may sound naive and it might be even physically incorrect but it at least helps to organize imagination.

At the beginning of this century it was obvious that electron's charge cannot be kept in a small volume without some nonelectromagnetic forces which would counterbalance the electric repulsion; these forces were called Poincaré stresses. Let us imagine that the Poincaré stress is suddenly removed by some external agent; then the charge will be suddenly released and will start moving in outward directions as a free dynamical system. We propose to imagine that the Dirac-Fock-Podolsky equations (A1) describe this phase of motion i.e. they describe a charge let loose by removal of the Poincaré stresses.

Usually an interpretation of a physical theory is not relevant. For example, Maxwell thought that his equations describe phenomenologically a more complicated microscopic motion. Nowadays we prefer to think that the electromagnetic field is fundamental, which means that there is nothing ontologically deeper than the electromagnetic field. But this difference is not really relevant as long as we use the equations which, quite rightly, are called Maxwell's equations. It turns out, however, that our interpretation of the Dirac-Fock-Podolsky equations is relevant because it diminishes the linear space of solutions: there are well behaved, for example analytic, finite energy solutions of equations (A1) which are not compatible with our interpretation. The point is that the total charge

$$Q = -\frac{1}{4\pi}\int d^3x\,\frac{\partial F}{\partial x^0}$$

does not have to be a Lorentz scalar; in general this integral will depend on the reference frame in which it is computed. This, however, will not be the case if the charge is released from an initially small volume. We shall now prove this assertion.

Since $\Box F = 0$, we can write

$$F(x) = \frac{1}{2\pi}\int \frac{d^3k}{k^0} f(k)e^{-ikx} + \text{c.c.}\,.$$

The function $f(k)$ can be expressed by means of the Cauchy data for $F(x)$ at $x^0 = 0$:

$$f(k) = \frac{i}{8\pi^2}\int\limits_{x^0=0} d^3x\{e^{ikx}\partial_0 F(x) - F(x)\partial_0 e^{ikx}\}.$$

If the charge expands from an initially small volume, both $F(x)$ and $\partial F(x)/\partial x^0$ are functions of compact support. Therefore we can expand the exponent

$$e^{ikx} = 1 + ikx + \ldots$$

and integrate term by term. In this way we obtain

$$f(k) = -\frac{1}{2\pi}\left(iQ + k_\lambda Q^\lambda + \frac{i}{2}k_\lambda k_\varrho Q^{\lambda\varrho} + \ldots\right), \tag{A2}$$

where

$$Q = -\frac{1}{4\pi}\int dS_\mu \partial^\mu F,$$

$$Q^{\lambda} = \frac{1}{4\pi}\int dS_{\mu}(x^{\lambda}\partial^{\mu}F - F\partial^{\mu}x^{\lambda}),$$

$$Q^{\lambda\varrho} = \frac{1}{4\pi}\int dS_{\mu}(x^{\lambda\varrho}\partial^{\mu}F - F\partial^{\mu}x^{\lambda\varrho}),$$

with

$$x^{\lambda\varrho} = x^{\lambda}x^{\varrho} - \tfrac{1}{4}\, xxg^{\lambda\varrho};$$

all integrations are over the volume $x^0 = 0$.

Thus our interpretation implies the existence of the multipole expansion (A2) which in general does not exist. In general

$$\lim_{k^0 \to 0} f(k)$$

can be an arbitrary function of angular variables; in this case the total charge will depend on the reference frame in which it is computed.

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