ON GAUGE INVARIANT EXTERNAL SOURCES IN YANG–MILLS THEORY*

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We analyse classical gauge potentials generated by static external sources coupled to gauge invariant, nonlocal operators. For the gauge invariant operators involving fermions we find two sharply distinct cases. In the first one, the external source decouples from Yang–Mills equations what leads to zero gauge potentials. The other one is found to be inconsistent with classical Yang–Mills equations. We consider also nonlocal, gauge invariant operators without fermions. We argue that there exists a possibility that the corresponding classical gauge field configuration is that of the closed magnetic flux tube with quantized energy.

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1. Introduction

The nonlocal, gauge invariant operators (shortly NGIO), constructed from nonabelian gauge potentials, were considered in a number of papers, e.g. [1, 2]. Presently, there exists a hope that such operators provide a string picture of hadrons within the framework of nonabelian gauge theories [2]. They are expected to be directly related to the long distance structure of the nonabelian gauge theory. The elementary fermion and nonabelian gauge fields are not expected to reflect the long distance structure of the theory because of confinement of quarks and gluons.

We shall consider the following examples of NGIO:

$$W(y, x|C) = \bar{\chi}(y)P \exp \left[ig \oint_{C,x}^{y} \hat{A}_\mu dz^\mu\right]\psi(x), \tag{1}$$

where x, y are points in Minkowski space-time, C is a path connecting y and x, P denotes path ordering of exponentials along C, χ, ψ are fermion fields, and $\hat{A}_\mu = A^a_\mu T^a$, where T^a are generators of SU(2) (or its representation). As the fermionless NGIO we take

$$W^0(x, x|C) = \mathrm{Tr}\, P \exp \left[ig \oint_C \hat{A}_\mu dz^\mu\right], \tag{2}$$

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where the trace is with respect to colour indices and C is a closed contour which starts and terminates at the same point x.

In order to calculate S-matrix in terms of NGIO it is necessary to consider Green functions for such operators [3]. As an intuitive starting point for this calculation one could take the Feynman path formula for the generating functional for Green functions. Apart from the gauge fixing and F-P ghost terms which are not important on the classical level, the total action in such a formula would be

$$S = S_{\mathrm{YM}} + S_{\mathrm{F}} + \int JW, \tag{3}$$

where

$$S_{\mathrm{YM}} = -\tfrac{1}{4} F^{a}_{\mu\nu} F^{a\mu\nu}, \qquad a = 1, 2, 3,$$

is the Yang-Mills action, S_F is the Dirac action for fermions, W denotes NGIO and $\int JW$ is specified in Section 2.

In this paper we start an investigation of the static classical approximation to the above sketched problem. Our expectation is that this can be an easy way to get important information about properties of Green functions for NGIO. We restrict ourselves to the most interesting gluonic sector of the theory by neglecting S_F.

The paper contains our results concerning the most interesting question, namely what are the classical stationary points of the action $S_{YM} + \int JW$. As W we take the NGIO (1) and (2). The fermion fields present in (1) are regarded as apriori fixed external fields. In other words, we try to find classical gauge potentials generated by the external source J. This external source is, of course, gauge invariant, as it is coupled to the gauge invariant NGIO.

We observe that such sources imply classical Yang–Mills equations with an external current of colour along the path C. In the case of NGIO given by (1) we show that the Yang–Mills equations are inconsistent, unless the fermion fields satisfy certain condition. When fermions do satisfy the condition, the external current in Yang–Mills equations vanishes and the external source decouples from Yang–Mills equations. This gives zero gauge field for such a source.

The fermionless case (2) is more complicated. The gauge potentials generated by the current of colour flowing along the path C can be easily found. Because the line C has zero thickness, the potentials are singular on C. This is an unpleasant difficulty for the classical approach, because the external current of colour contains explicitly $\hat{A}_\mu(z)$ taken for $z \in C$ and this is infinite. Of course, this difficulty could be resolved by a quantum smearing of the curve C. One should use some smooth $J(x, y|C)$ and to average (2) with it. We say "quantum smearing" because results of papers [2] strongly suggest that $J(x, y|C)$ can be interpreted as a wave functional for a string.

However, there still exists a possibility of a classical description. Namely, one could think of such a wave functional $J(x, y|C)$ that it can be described classically by some very complicated curve C, so complex that $\hat{A}_\mu$ will be finite on C. In fact, in the quantum theory the curve C (being the shape of the string) strongly fluctuates, and therefore there is no reason why the best classical description should be given by geometrically simplest lines.

We consider a curve C that fills in a torus, the twodimensional manifold. Such a curve could be considered intuitively as a limiting case of a curve winding around some given circle, when the number of windings increases to infinity. Continuous curves filling more than onedimensional manifolds are known in mathematics, e.g., Sierpiński curve [4]. The corresponding solution of Yang–Mills equations is given by some colour magnetic field restricted to the inside of the torus and zero electric field. The classical selfenergy of the source is, unexpectedly, quantized through a selfconsistency condition. For a thin torus the energy spectrum is linear. Such a toroidal magnetic flux tube we would like to interpret as a classical picture of a glueball.

Two remarks are in order:

1. We consider the simplest, so called abelian, solutions of Yang–Mills equations. It is known, [5], that for a given external source Yang–Mills equations admit also other types of solutions. A similar phenomenon should be expected also in the case of the gauge invariant external sources.
2. Note that the action S is not in general real because of the term $S_{\text{ext}} = \int JW$. It is interesting that in spite of this, one could choose an overall constant in S_{ext} in such a way that the stationary points are given by real gauge potentials A^a_μ.

In the next Section we shall present an analysis of Yang–Mills equations for the case of NGIO (1). In Section 3 we analyse the case of NGIO given by (2). In Section 4 we make some ending remarks.

2. *Yang Mills equations with the gauge invariant sources involving fermions*

We search for a stationary point of the action

$$S = -\tfrac{1}{4}\int d^4x F^a_{\mu\nu}F^{a\mu\nu} + S_{\text{ext}}, \tag{4}$$

where

$$S_{\text{ext}} = \int d^4x d^4y \int [dc_\mu] J(y, x|C) W(y, x|C), \tag{5}$$

$W(y, x|C)$ being given by (1). Here $[dc_\mu]$ denotes a functional measure in a space of paths C, connecting the points x, y in Minkowski space-time. Observe that the terms $\bar{\psi}\gamma A\psi$, present in the neglected S_F, would act as an additional gauge noninvariant external source for $\hat{A}_\mu$. Such sources are already well investigated [5]. In this paper we are not concerned with them.

In the following we assume that the external source J is strictly localised in three-space and that it is static, i.e.,

$$J(y, x|C) = q\delta(\vec{x}-\vec{x}_a)\delta(\vec{y}-\vec{x}_b)\,[\delta(\vec{C}-\vec{C}^{(0)})]\,[\delta(C_0-x_0)]\delta(x_0-y_0), \tag{6}$$

where $[\delta(\vec{C}-\vec{C}^{(0)})]$, $[\delta(C_0-x_0)]$ are the functional delta functions, $\vec{x}_a$, $\vec{x}_b$ are fixed points in three-space and $C^{(0)}$ is a curve connecting x_a, x_b. The two deltas, $\delta(x_0-y_0)$ and $[\delta(C_0-x_0)]$ make the configuration to be equal time configuration. The fact that x_0 is unspecified implies that the configuration is the static one. We assume that $\bar{\chi}$, ψ are constant in time. q is a constant characterising the strength of the external source.

The action (4) implies the following equations for the gauge potentials

$$D_\mu \hat{F}^{\mu\nu} = -\hat{j}^{\nu}_{\rm ext}, \tag{7}$$

where

$$j^{a\nu}_{\rm ext}(x) = \frac{\delta S_{\rm ext}}{\delta A^a_\nu(x)}, \tag{8}$$

and $D_\mu = \partial_\mu + ig[\hat{A}_\mu, \cdot\,]$.

In order to calculate explicitly $j^{a\nu}_{\rm ext}$ we parametrize the line $C^{(0)}$: $\lambda \in [0, 1]$, $C^{(0)}_\mu = C^{(0)}_\mu(\lambda) \equiv z_\mu(\lambda)$, $C^{(0)}_\mu(0) = x_{a\mu}$, $C^{(0)}_\mu(1) = x_{b\mu}$. Of course, $x_{a0} = x_{b0} = z_0(\lambda)$. Then

$$P \exp\left[ig \int_{x_a}^{x_b} A_\mu dz^\mu\right] = P \exp\left[ig \int_0^1 d\lambda v^\mu A_\mu\right],$$

where $v^\mu = \dfrac{dz^\mu}{d\lambda}$, and the variational derivative in (8) yields after a straightforward calculation

$$j^{a0}_{\rm ext} = 0,$$

$$j^{ai}_{\rm ext} = gq \int_0^1 d\lambda v^i(\lambda) I^a(\vec{z}(\lambda))\delta(\vec{x}-\vec{z}(\lambda)), \tag{9}$$

where

$$I^a(\vec{z}(\lambda)) = i\bar{\chi}(\vec{x}_b)V_C(\vec{x}_b, \vec{z}(\lambda))T^a V_C(\vec{z}(\lambda), \vec{x}_a)\psi(\vec{x}_a), \tag{10}$$

and

$$V_C(\vec{x}, \vec{y}) = P \exp\left[ig \int_{C,\vec{y}}^{x} \hat{A}_i dz^i\right].$$

Thus, $j^{a\mu}_{\rm ext} = 0$, except for the line $\vec{C}^{(0)}$, along which there is a flow of colour charge.

It is well-known [6] that Yang Mills equations (7) imply the constraint

$$D_\mu j^{a\mu}_{\rm ext} \equiv \partial_\mu j^{a\mu}_{\rm ext} - g\varepsilon^{abc}A^b_\mu j^{c\mu}_{\rm ext} = 0 \tag{11}$$

for any current on the r.h.s. of them. For gauge noninvariant external sources this constraint is a nontrivial condition to be satisfied. For the gauge invariant sources, the current (9) satisfies the constraint identically on the whole $\vec{C}^{(0)}$, excluding the end points $\vec{x}_a$, $\vec{x}_b$. At these points the constraint is not satisfied unless the fermion wave functions $\bar{\chi}$, ψ obey certain condition. Namely, from (9) we get

$$\partial_\mu j^{a\mu}_{\rm ext} = (\partial_i I^a)j^i + I^a \partial_i j^i, \tag{12}$$

where $j^i(\vec{x}) = gq \int_0^1 d\lambda v^i(\lambda)\ \delta(\vec{x}-\vec{z}(\lambda))$ is the usual current obeying the static continuity equation $\partial_i j^i = 0$ for $\vec{x} \neq \vec{x}_a, \vec{x}_b$. Thus, the last term on the r.h.s. of (12) vanishes for

$\vec{x} \neq \vec{x}_a, \vec{x}_b$. It is easy to verify that the first term on the r.h.s. of (12) cancels with the term $\varepsilon^{abc} A^b_\mu j^{\mu c}_{\rm ext}$ present on the l.h.s. of the constraint (11). For $\vec{x} = \vec{x}_a$, $\vec{x} = \vec{x}_b$ we have $\partial_i j^i \sim \delta(\vec{x}-\vec{x}_{a,b})$ and therefore the constraint (11) implies

$$I^a(\vec{x}_a) = I^a(\vec{x}_b) = 0. \tag{13}$$

However, from (10) it follows that

$$I^a(\vec{z}) = D^{ab}(V_C) I^b(\vec{x}_b),$$

where $D^{ab}(V_C)$ is the matrix of the adjoint representation of SU(2), corresponding to the group element $V_C(\vec{z}(\lambda), \vec{x}_b)$. Therefore the condition (13) implies $I^a(\vec{x}) = 0$ for all $\vec{x} \in \vec{C}^{(0)}$, i.e. the external source decouples from Yang–Mills equations. This means that the external source does not generate any gauge field. In particular, it has zero classical selfenergy.

To summarize, either the external source decouples from classical Yang–Mills equations or it is not consistent with them. We would like to interpret this result as an indication that NGIO for which $I^a(x) = 0$ are, in some sense, favored by the nonabelian gauge theory.

3. The gauge invariant sources without fermions

Now we shall consider the action (4), (5) with W replaced by $W^0(x, x|C)$ given by (2) and $J = J(x|C) = q\delta(\vec{x}-\vec{x}_a)\,[\delta(\vec{C}-\vec{C}^{(0)})]\,[\delta(C_0-x_0)]$. In the corresponding Yang–Mills equations (7) the external current $j^{a\nu}_{\rm ext}$ has the form (9), where now

$$I^a(\vec{z}) = i\,\mathrm{Tr}\,[\tilde{V}_C(\vec{x}_a, \vec{x}) T^a V_C(\vec{x}, \vec{x}_a)]. \tag{14}$$

$\tilde{V}_C$ is calculated along the other arc of $\vec{C}^{(0)}$ than that used in V_C. One can verify that the external current satisfies the constraint (11) on the whole $\vec{C}^{(0)}$.

At first sight Yang–Mills equations look as very complicated integro-differential nonlinear equations. Still one could find a solution for them. Namely, we observe that because gauge transformations just rotate the colour spin vector (I^a), one can perform such a gauge transformation that the resulting (I^a) will point in the 3-rd direction for all $\vec{z} \in \vec{C}^{(0)}$, i.e. $I^a = Ie^a_z$, where $e^a_z = \delta^{a3}$, [7]. Then, the Ansatz $A^a_\mu(x) = \delta^{a3}\, A^\mu(x)$ reduces Yang Mills equations to ordinary Maxwell equations for $A_\mu(x)$ with the external static current

$$j^i_{\rm ext}(x) = Igq \int_0^1 d\lambda v^i \delta(\vec{x} - \vec{C}^{(0)}(\lambda))$$

along the line $\vec{C}^{(0)}$. The corresponding solutions are known from a text-book electrodynamics. As the next step, we insert the solutions for A_μ on the r.h.s. of (14). Because the l.h.s. is given ($I^a = \delta^{a3} I$), this yields a consistency condition from which one could try to determine some constants present in A_μ.

Unfortunately, because of zero thickness of the line $\vec{C}^{(0)}$, $\hat{A}_\mu$ has a logarithmic singularity on $\vec{C}^{(0)}$ and we meet the difficulty mentioned in Introduction. As it is explained there, we assume that $\vec{C}^{(0)}$ is at least a twodimensional structure. The simplest possibility is to assume that $\vec{C}^{(0)}$ is a torus, i.e. the current $j^\nu_{\rm ext}$ forms a torus-like coil. This assumption is not as peculiar as it may look at first sight. Firstly, we recall that in Yang's formulation of nonabelian gauge theories, [1], one uses the exponentials exp $[ig\hat{A}_\mu dx^\mu]$ independently at each space-time point x. There is no reason why one should arrange these infinitesimal exponentials just along the simplest lines and to neglect more refined possibilities. Secondly, some support comes also from energy considerations. Namely, the classical selfenergy of an infinitely thin, static, linear current diverges logarithmically. On the other hand, classical selfenergy of the current forming the torus is finite, equal to the energy of magnetic field inside the torus.

Thus, we assume that $\vec{C}^{(0)}$ forms a torus-like coil. The solution of the Maxwell equations is given by some magnetic field inside the coil. It remains to check the consistency condition. Because we have $A^{ai} = \delta^{a3}A^i$, $T^a = \frac{1}{2}\sigma^a$, σ^a — Pauli matrices, then

$$V_C(\vec{x}, \vec{x}_a) = \mathrm{Tr}\exp\Big[-ig\int_{\vec{C}^{(0)},\vec{x}_a}^{\vec{x}} A^i dx^i T^3\Big] = \exp[-igT^3\Phi d(\vec{x}, \vec{x}_a)\kappa],$$

and

$$\tilde{V}_C(\vec{x}_a, \vec{x}) = \exp[-igT^3\Phi\tilde{d}(\vec{x}_a, \vec{x})\kappa].$$

Here Φ is the flux of the magnetic field through the torus, $d(\vec{x}, \vec{x}_a)$ is the length of the torus between the points $\vec{x}_a$, $\vec{x}$ (that is the distance along the big circle of the torus between the points obtained by perpendicular projections of the points $\vec{x}_a$, $\vec{x}$ on the big circle), $\tilde{d}(\vec{x}_a, \vec{x})$ also is the distance between the projections of $\vec{x}_a$ and $\vec{x}$ but taken along the other arc of the big circle of the torus, κ is an unknown coefficient describing the density of windings of the current around the torus. The path ordering was dropped out because now the exponentials commute. From (14) we obtain that

$$I^1 = I^2 = 0, \qquad I^3 = I = \sin\tfrac{1}{2}g\Phi l\kappa,$$

where $l = d(\vec{x}, \vec{x}_a) + \tilde{d}(\vec{x}_a, \vec{x})$ is the perimeter of the torus along its big circle. Because for the thin, toroidal coil $\Phi \cong qIg\kappa S_\perp$, where $S_\perp$ is the area of the perpendicular crossection of the torus, we get the consistency condition

$$I = \sin[\tfrac{1}{2}(g\kappa)^2 IqV], \tag{15}$$

where V is the volume of the torus. Of course, our considerations require $\kappa \to \infty$. In order to obtain finite results we parallelly take $g \to 0$, in such a way that $g\kappa \equiv \lambda = \mathrm{const}$.

The condition (15) can be read over in at least two ways. Straightforwardly, it can be regarded as an equation for I, in which V, λ, q are fixed parameters describing the given torus. The energy of the torus is $E = \frac{1}{4}H^2V = \frac{1}{4}q^2\lambda^2 I^2 V$. For sufficiently large $\lambda^2 qV$ there are many values of I obeying (15).

However, the condition (15) can be interpreted also as a quantisation condition for $\lambda^2 V$. Namely, when I is apriori fixed, (15) implies

$$\lambda^2 V = \frac{2}{Iq}(\arcsin I + k\pi), \qquad k - \text{integer}. \tag{16}$$

Observe that (15) implies also that q is a positive number, $q > 0$. Thus we have to assume $k = 0, 1, 2, \ldots$, in order to ensure $\lambda^2 V > 0$. Then, the energy is

$$E = \frac{1}{2} qI(\arcsin I + k\pi), \tag{17}$$

where $k = 0, 1, 2, \ldots$ Observe that the spectrum of energy is linear and that it depends only on the strength of the external source q and on the constant I. The constant I in this case is not determined. However, it is natural to assume that I characterises directly the representation used for T^a, e.g., $I(I+1) = T^a T^a$ (for $T^a = \frac{1}{2}\sigma^a$ this gives $I = 1/2$).

4. *Final remarks*

Our results indicate that classical gauge fields can be created in a gauge invariant manner on the classical level only if the accompanying fermions satisfy the condition (13). Then, the external current vanishes and the external source decouples from Yang–Mills equations. In particular, this means that the classical selfenergy of such a source is given entirely by the classical selfenergy of the set of accompanying fermions. This selfenergy can be calculated from Yang–Mills equations with the external current $\bar{\psi} T^a \psi$, [5, 7], (this current, coupled directly to $\hat{A}_\mu$, is a gauge noninvariant external source of gauge fields and therefore it was neglected in our considerations). Unfortunately, we cannot relate the condition (13) to the common requirement that the fermions should form a colour singlet state.

In the case without fermions, the classical approach seems to require rather complicated objects instead of a simple, closed contour C. We have considered C to be a torus. The question arises whether the resulting magnetic flux tube is stable. Presently we have no answer to this question. It seems that there is no reason for a topological stability. On the other hand, the spectrum (17) is bounded from below and, intrigueingly, does not depend on the size of the torus–this suggests an energetic stability.

The above results can be easily extended to SU(n) groups. Of course, the number of types of NGIO then increases.

REFERENCES

[1] T. T. Wu, C. N. Yang, *Phys. Rev.* **D12**, 3845 (1975).
[2] J. L. Gervais, A. Neveu, *String States in QCD*, Cargése Lectures (1979); Y. Nambu, *Phys. Lett.* **80B**, 372 (1979); A. M. Polyakov, *Phys. Lett.* **82B**, 247 (1979).
[3] T. Matsumoto, preprint UT-Komaba 78–2, March 1978.
[4] K. Kuratowski, *Introduction to the Set Theory and Topology*, Ch. 18, Sect. 5. Polish Scientific Publishers, Warsaw 1972.
[5] R. Jackiw, L. Jacobs, C. Rebbi, *Phys. Rev.* **D20**, 474 (1979); P. Sikivie, N. Weiss, *Phys. Rev.* **D18**, 3809 (1978).
[6] W. Marciano, H. Pagels, *Phys. Rep.* **36C**, 137 (1978).
[7] H. Arodź, *Phys. Lett.* **78B**, 129 (1978).