CONJUGATIONS AND HERMITIAN OPERATORS IN SPACE TIME

BY G. SOBCZYK

Institute for Theoretical Physics, University of Wrocław*

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In an earlier work a complex vector algebra for space-time was introduced to provide an abstract alternative to the matrix based tensor-spinor formalisms now in use. By means of the notion of a conjugation, reflections as well as Lorentz rotations in spacetime find simple expression. The investigation of Hermitian and complex symmetric operators provides new insight into the principal correlation between the energy-momentum and Weyl tensors.

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1. Introduction

In [1], it was shown how the familiar Gibbs-Heaviside vector algebra can be "complexified" to obtain a powerful complex vector algebra for use in spacetime. The present work is the completion of the work begun in [1] in the sense that it shows how reflections as well as rotations in spacetime find simple geometric expression; thus the full Lorentz group finds more direct expression in complex vector algebra than in any other formalism. The ideas for this work were generated by the author's desire to simplify the algebraic classification of 2nd order tensors of importance in physics, and by the related problem of trying to find an abstract basis for the so-called spinor formalism [2], [3]. Complex vector algebra, and its generalization [4] are interesting in their own right because they provide an alternative conceptual framework upon which can be built the mathematical and physical theories which today depend upon matrix-tensor-spinor formalisms for their expression.

Ref. [1] should be considered a prerequisite for reading the present work, since notation and results from [1] will be used here with little or no comment.

In Section 1, we abstractly define the notion of a conjugation, and find that there are two kinds of conjugations: proper and improper. Each proper conjugation defines a unique inertial observer. The space vectors and space bivectors of an observer are,

* Address: Instytut Fizyki Teoretycznej, Uniwersytet Wrocławski, Cybulskiego 36, 50-205 Wrocław, Poland.

respectively, the real and imaginary vectors with respect to the conjugation which defines that observer. Improper conjugations do not define inertial observers, but generate the space reflections of observers.

In Section 2, it is shown that the composition of two conjugations in spacetime is a Lorentz rotation, which suggests that a conjugation is equivalent to the notion of a reflection in spacetime. The relationship between the composition of conjugations and Lorentz transformations provides a new basis for the classification of the homogeneous Lorentz group [5, p. 21]; for example, the composition of two proper conjugations results in a proper Lorentz rotation, the aggregate of which is a subgroup of the Lorentz group, called the proper orthochronous Lorentz group $SO\uparrow(3,1)$.

In Section 3, the relationship between Hermitian and anti-dual symmetric operators is studied, and in particular the fundamental role played by the notion of a conjugation in their definition. Each proper conjugation defines a unique positive definite Hermitian metric, whereas improper conjugations define indefinite Hermitian metrics. A special situation arises when the operation of conjugation commutes with an antidual symmetric operator, and this is the basis for the so-called principal correlation between the energy--momentum tensor and a Weyl tensor [3].

In Section 4, the complex vector algebra is embedded in a larger algebra, the Dirac--Clifford algebra, in such a way that the elements of the complex vector algebra make-up the even subalgebra, or Pauli algebra, of the abstract Dirac algebra. It is shown that on the level of the Dirac algebra a proper conjugation defines a reflection with respect to a time--like Dirac vector, and an improper conjugation defines a reflection w.r.t a space-like Dirac vector. An anti-dual symmetric operator, on the level of the Dirac algebra, is equivalent to a tracefree vector operator [2, 3]. The fact that this vector operator will always have a real space-like eigenvector is used to prove the existence of a principal correlation; for it is just the improper conjugation generated by this space-like eigenvector which commutes with the anti-dual symmetric operator. The section closes with a discussion of a "boost" in complex vector algebra, and its representation as a 4×4 matrix.

Finally, it is interesting to note that while the above mentioned principal correlation finds its clearest expression on the level of the complex vector algebra, its existence has only been proven on the level of the Dirac algebra. This suggests that perhaps neither level is the more fundamental, but rather that there is a fruitful interdependence that needs to be more carefully examined.

2. *Proper and improper conjugations*

In [1] we defined the conjugation operator of an observer in terms of an orthonormal rest frame $\{E_k\}$. The purpose of this section is to give a generalized abstract definition which later, in Section 4, will lead to the interpretation of a conjugation as a reflection in spacetime.

Recall that the complex vector algebra $\mathscr{P} = \mathscr{C} \oplus \mathscr{B}$ is the formal sum of the space $\mathscr{C}$ of complex numbers of the form $z = x + Iy$, and the complex 3-dim vector space $\mathscr{B}$, and has an abstract algebraic structure which is isomorphic to the algebra of Pauli matrices, as well as to the algebra of complex quaternions. A Pauli number $P = z+B \in \mathscr{P}$ consists of a complex scalar part $z \in \mathscr{C}$ and a complex vector part $B \in \mathscr{B}$.

Definition 1. By a *conjugation* on the complex vector algebra $\mathscr{P}$, we mean an operator, denoted by $\bar{P}$, which satisfies: (*i*) $\bar{z} = x-Iy$, (*ii*) $\overline{P_1 = {}_2P} = \bar{P}_1+\bar{P}_2$, (*iii*) $\overline{P_1P_2} = \bar{P}_2\bar{P}_1$ (*iv*) $\bar{\bar{P}} = P$ for all $P, P_1, P_2 \in \mathscr{P}$.

The condition (*i*) in definition 1 shows that conjugation in $\mathscr{P}$ coincides with ordinary complex conjugation when restricted to $\mathscr{C}$. A Pauli number P will be said to be *real* or *imaginary* w.r.t the conjugation $\bar{P}$ if

$$\bar{P} = P \quad \text{or} \quad \bar{P} = -P, \tag{1}$$

respectively. Just as for complex numbers, Pauli numbers can always be decomposed into real and imaginary parts w.r.t a given conjugation. Thus,

$$P = \tfrac{1}{2}(P+\bar{P})+\tfrac{1}{2}(P-\bar{P}) = \tfrac{1}{2}(P+\bar{P})-\tfrac{1}{2}I(IP+\bar{I}\bar{P}) = \langle P\rangle_{re}+I\langle P\rangle_{im}, \tag{2}$$

where

$$\langle P\rangle_{re} \equiv \tfrac{1}{2}(P+\bar{P}) \quad \text{and} \quad \langle P\rangle_{im} \equiv -\tfrac{1}{2}(IP-I\bar{P}).$$

We begin our study of conjugations in $\mathscr{P}$ with the following

Theorem 1. (*i*) $\overline{A \circ B} = \bar{A} \circ \bar{B}$ and (*ii*) $\overline{A \times B} = -\bar{A} \times \bar{B}$.

Proof: Recalling from [1, § 1], the definitions

$$A \circ B \equiv \tfrac{1}{2}(AB+BA) \quad \text{and} \quad A \times B \equiv \tfrac{1}{2}(AB-BA),$$

the proof of the theorem easily follows from the properties (*ii*) and (*iii*) of a conjugation given in definition 1.

Corollary. (*i*) If $\bar{A} = A$ and $\bar{B} = B$ for $A, B \in \mathscr{B}$, then $A \circ B = \langle A \circ B\rangle_{re}$ and $A \times B = I\langle A \times B\rangle_{im}$. (*ii*) If $\bar{A} = -A$, $\bar{B} = B$, then $A \circ B = I\langle A \circ B\rangle_{im}$ and $A \times B = \langle A \times B\rangle_{re}$.

In the above corollary we have assumed that with respect to a given conjugation there will always be real and imaginary vectors. The following lemma guarantees that a conjugation will always have non-trivial real and imaginary vectors.

Lemma 1. There exists non-trivial complex vectors $A, B \in \mathscr{B}$ such that $\bar{A} = A$ and $\bar{B} = -B$.

Proof: Suppose that $\bar{B} \neq B$ for all complex vectors $B \neq 0$. Then for $B \neq 0$,

$$\overline{B+\bar{B}} = \bar{B}+B = B+\bar{B},$$

which implies that $B+\bar{B} = 0$, or $\bar{B} = -B$. By letting $A = IB$ we find that $\bar{A} = \overline{IB} = \bar{B}\bar{I} = IB = A$, and so by our assumption $A = 0$, which in turn implies that $B = -IA = 0$ which is a contradiction.

In [1, thm. 8], we established that a complex null vector can always be expressed in the canonical form

$$N = e^{\phi}(1+E_1')E_2', \tag{3}$$

where ϕ is a real scalar and E_1' and E_2' are orthonormal, i.e.,

$$E_1'^2 = E_2'^2 = 1 \quad \text{and} \quad E_1' \circ E_2' = 0.$$

The following lemma strengthens this result and gives to what extent the canonical form is unique.

Lemma 2. (*i*) There exists an orthonormal frame $\{E_k\}$ such that $N = (1+E_1)E_2$. (*ii*) If $N = (1+E_1)E_2 = (1+E'_1)E'_2$ then $E'_k = e^{\alpha N}E_k e^{-\alpha N}$ for some complex scalar $\alpha \in \mathscr{C}$.

Proof: Using the canonical form (3), define $E_k = e^{\frac{1}{2}\phi E'_1}E'_k e^{-\frac{1}{2}\phi E'_1}$ and then calculate

$$(1+E_1)E_2 = e^{1/2\,\phi E_1'}(1+E'_1)E'_2 e^{-1/2\,\phi E_1'} = e^{\phi E_1'}(1+E'_1)E'_2 = e^{\phi}(1+E'_1)E'_2 = N.$$

The proof of part (*ii*) is a consequence of the fact that $A \times N = 0$ iff $A = \beta N$ for some complex scalar β (see the corollary to thm 3 in [1]). Thus, suppose that $E'_k = e^A E_k e^{-A}$ and that

$$N = (1+E'_1)E'_2 = (1+e^A E_1 e^{-A})e^A E_2 e^{-A} = e^A N e^{-A},$$

or, equivalently, $e^A N = N e^A$. But this last equality is true iff $A = \alpha N$, as follows from the above remark.

Lemma 2 shows that a complex null vector determines an orthonormal frame which is unique up to a light-like Lorentz rotation.

Using the above lemmas, we can now establish two important theorems characterizing the existence of two kinds of conjugations in the complex vector algebra $\mathscr{P}$. But first we give

Definition 2. A conjugation $\bar{P}$ will be said to be *proper* if it has no real null vectors, i.e., $\bar{N} \neq N$ for all complex null vectors $N \in \mathscr{B}$. If the conjugation $\bar{P}$ has a real null vector, $\bar{N} = N$, then the conjugation $\bar{P}$ is said to be *improper*.

We shall see in Section 4 that proper and improper conjugations correspond to reflections w.r.t a time-like and a space-like vector in space-time, respectively.

Theorem 2, A conjugation $\bar{P}$ is improper iff there exists an orthonormal frame $\{E'_k\}$ such that $\bar{E}'_1 = -E'_1$, $\bar{E}'_2 = E'_2$, and $\bar{E}'_3 = -E'_3$.

Proof: The "if" part of the theorem is easily established, for if there exists such an orthonormal frame $\{E'_k\}$, then

$$\bar{N} = \overline{(1+E'_1)E'_2} = \bar{E}'_2(1+\bar{E}'_1) = (1+E'_1)E'_2 = N.$$

To establish the "only if" part, suppose that $N = (1+E_1)E_2$ satisfies $\bar{N} = N$. We will need the following facts, which are easily established: $\bar{E}_1 \circ N = \overline{E_1 \circ N} = 0$, $\bar{E}_1^2 = \bar{E}_2^2 = 1$, $N\bar{E}_1 = \overline{E_1 N} = \bar{N} = N$, and $\bar{\bar{E}}_1 = E_1$. From the first fact it follows that $\bar{E}_1 = \alpha E_1 + \beta N$. Using the 2nd and 3rd facts show that $\alpha = -1$, and applying the fact that $\bar{\bar{E}}_1 = E_1$ shows that $\beta = 2s$ for some real scalar s. We now define $E'_1 = E_1 - sN$, and verify that

$$\bar{E}'_1 = \bar{E}_1 - sN = -E_1 + 2sN - sN = -(E_1 - sN) = -E'_1.$$

Now define $E'_k = e^{\frac{1}{2}sN}E_k e^{-\frac{1}{2}sN}$, and note with help of lemma 2 that

$$N = (1+E_1)E_2 = (1+E'_1)E'_2 = \bar{E}'_2(1+\bar{E}'_1) = \bar{N} = (1+E'_1)\bar{E}'_2.$$

Since $-E'_1 \circ \bar{E}'_2 = \overline{E'_1 \circ E'_2} = 0$, we can apply lemma 2 a second time and conclude that

$$\bar{E}'_1 = e^{1/2\,\omega N}E'_1 e^{-1/2\,\omega N} \quad \text{and} \quad \bar{E}'_2 = e^{1/2\,\omega N}E'_2 e^{-1/2\,\omega N}.$$

But this implies that $E_1' = e^{\omega N}E_1' = E_1' - \omega N$, so that $\omega = 0$ and $\bar{E}_2' = E_2'$. Finally, we check that

$$E_3' = -IE_1'E_2' = I\overline{E_2'E_1'} = \overline{IE_1'E_2'} = -\bar{E}_3'$$

as is required, and the proof is complete.

Corollary. If $\{E_k'\}$ is an orthonormal frame satisfying the conditions of theorem 2, then $N \equiv (1+E_1')E_2'$ satisfies $\bar{N} = N$.

Theorem 3. A conjugation $\bar{P}$ is proper iff there exists an orthonormal frame $\{E_k\}$ such that $\bar{E}_k = E_k$ for $k = 1, 2, 3$.

Proof: Suppose that $\bar{E}_k = E_k$, and let N be any non-trivial null direction. Then we must show that $\bar{N} \neq N$. From [1, thm. 8], we know that there exists a real number ϕ and real (w.r.t P) orthonormal vectors A_1, A_2 such that $N = e^{\phi}(1+A_1)A_2$. We can now check that

$$\bar{N} = e^{\phi}\bar{A}_2(1+\bar{A}_1) = e^{\phi}(1-A_1)A_2 \neq N.$$

To prove the "only if" part of the theorem, assume that $\bar{P}$ is proper. Then lemma 1 guarantees that we can find a complex vector A with $\bar{A} = A$. Furthermore, since $\bar{P}$ is proper, $A^2 \neq 0$, and we can therefore define $E_1 = A/(A^2)^{\frac{1}{2}}$ so that $E_1^2 = 1$ and $\bar{E}_1 = \pm\bar{A}/(\bar{A}^2)^{\frac{1}{2}} = \pm A/(A^2)^{\frac{1}{2}} = \pm E_1$. To complete the construction, define the subspace

$$\mathscr{E}_1^{\perp} = \{B: B \circ E_1 = 0\},$$

and note that $\overline{\mathscr{E}_1^{\perp}} = \mathscr{E}_1^{\perp}$. We can then apply the argument of lemma 1 to the subspace $\mathscr{E}_1^{\perp}$ and conclude there exists a complex vector $A' \in \mathscr{E}_1^{\perp}$ such that $\bar{A}' = A'$. By continuing the argument as above, the construction is completed; we have constructed an orthonormal frame $\{E_k\}$ satisfying the condition $\bar{E}_k = \pm E_k$ for $k = 1, 2, 3$. Finally, we must check the indeterminacy of signs: If the signs were $(-++)$ or $(-+-)$, then the complex null vector $N = (1+E_1)E_2$ would satisfy $\bar{N} = N$ contradicting that $\bar{P}$ is proper. The only other incorrect possibility is $(---)$. For this case we would have $I = E_1E_2E_3 = \bar{E}_3\bar{E}_2\bar{E}_1 = \bar{E}_1\bar{E}_2\bar{E}_3 = \bar{I} = -I$ which is impossible[1]. Thus, we must have $\bar{E}_k = E_k$ for $k = 1, 2, 3$ as required.

Theorem 3 tells us that a proper conjugation in spacetime determines the rest frame of an observer, see [1, §4]. The meaning of this last statement will be further discussed in Section 4.

3. Conjugations and Lorentz rotations

In [1, thm. 9], it was established that a Lorentz rotation can always be put into the canonical form

$$L(B) = e^{C}Be^{-C} \tag{4}$$

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[1] If condition (*i*) of definition 1 is not assumed, the case $(---)$ becomes possible (when $\bar{I} = I$), and can be used to define the operation of *space reversion.* If a Pauli number is unchanged under space reversion, it is said to possess *Time Symmetry.*

for some $C \in \mathscr{B}$. When a conjugation $\bar{P}$ is specified, we can decompose the Lorentz rotation given in (4) into the composition of an imaginary Lorentz rotation followed by a real Lorentz rotation. Precisely, we have

Theorem 4. A Lorentz rotation e^C can always be written in the form $e^C = e^A e^B$ where $\bar{A} = A$ and $\bar{B} = -B$.

Proof: Since the composition of Lorentz rotations is itself a Lorentz rotation, and since the square root of a Lorentz rotation is always well defined, we can define the real part of the Lorentz rotation e^C by the equation $e^A = [e^C e^{\bar{C}}]^{\frac{1}{2}}$, and then set $e^B = e^{-A} e^C$. We then calculate $e^{2A} = e^C e^{\bar{C}}$, from which it follows that

$$e^{2\bar{A}} = \overline{e^C e^{\bar{C}}} = e^{\bar{\bar{C}}} e^{\bar{C}} = e^{2A},$$

so that $\bar{A} = A$ as is required. To complete the proof, note that

$$e^B e^{\bar{B}} = e^{-A} e^C e^{\bar{C}} e^{-A} = 1,$$

which implies that $\bar{B} = -B$ as is required.

It is a well known fact that the composition of two reflections in Euclidean space generates a rotation; we see in the next theorem that the composition of two conjugations in spacetime is a Lorentz rotation.

Theorem 5. If $\bar{P}$ and $\tilde{P}$ are conjugations in the complex vector algebra $\mathscr{P}$, then $L(B) \equiv \tilde{\bar{B}}$ defines a Lorentz rotation.

Proof: By definition [1, def. 9], a Lorentz rotation preserves the complex inner product $A \circ B$. Thus, we need only verify the steps

$$L(A) \circ L(B) = \tilde{\bar{A}} \circ \tilde{\bar{B}} = \overline{\bar{A} \circ \bar{B}} = \overline{\overline{A \circ B}} = A \circ B$$

by using the properties of a conjugation given in definition 1.

Corollary. There exists a complex vector C such that (*i*) $\tilde{\bar{B}} = e^C B e^{-C}$, (*ii*) $\tilde{B} = e^C \bar{B} e^{-C}$ and (*iii*) $\bar{C} = \tilde{C} = e^{I\theta} C$.

Proof: Since $\tilde{\bar{B}}$ is a Lorentz rotation, (*i*) follows immediately. (*ii*) follows by substituting $\bar{B}$ for B in part (*i*). To see (*iii*), first set $B = C$ in part (*i*), which shows that $\tilde{\bar{C}} = C$. Taking the conjugation of this last equality w.r.t $\tilde{P}$ gives $\bar{C} = \tilde{C}$. Setting $B = C$ in part (*ii*), gives

$$e^C \bar{C} e^{-C} = \tilde{C} = \bar{C} \quad \text{so that} \quad e^C \bar{C} = \bar{C} e^C,$$

from which it follows that $\bar{C} = \alpha C$ for some complex scalar α. Finally, by calculating $C = \bar{\bar{C}} = \overline{\alpha C} = \bar{\alpha}\alpha C$, we can conclude that $\alpha = e^{I\theta}$.

The following theorem tells to what extent a Lorentz rotation e^C relating the conjugations $\bar{P}$ and $\tilde{P}$ is arbitrary. But first we will need the

Lemma. If $\bar{C} = \tilde{C} = -C$ and $\tilde{\bar{B}} = e^C B e^{-C}$, then $C = \frac{1}{2}\pi I A$ for some $A \in \mathscr{B}$ where $A^2 = 1$ and $\bar{A} = \tilde{A} = A$.

Proof: $\tilde{B} = e^C \bar{B} e^{-C}$ implies that

$$B = \tilde{\tilde{B}} = e^C \tilde{\bar{B}} e^{-C} = e^{2C} B e^{-2C}.$$

But this is possible only when $e^{2C} = \pm 1$ or, equivalently, when

$$\cosh(2C) = \pm 1 \quad \text{and} \quad \sinh(2C) = 0,$$

which has the solutions $C = \frac{1}{2} k\pi IA$ for integers k and where $A^2 = 1$. Without loss of generality, we may choose the non-trivial solution $k = 1$. We then check that $\bar{C} = -\frac{1}{2}\pi I\bar{A} = -\frac{1}{2}\pi IA$, so that $\bar{A} = \tilde{A} = A$.

Theorem 6. If $\tilde{B} = e^{C}\bar{B}e^{-C}$, then either C is of the form (i) $C = \phi A$ or (ii) $C = (\phi + \frac{1}{2}\pi I)A$ where ϕ is real and $\bar{A} = \tilde{A} = A$ with $A^2 = 1$, or, (iii) $C = N = \bar{N} = \tilde{N}$ where $N^2 = 0$.

Proof: From the corollary to theorem 5, $\bar{C} = e^{I\theta}C = \tilde{C}$. Defining $C_{\|} = e^{\frac{1}{2}I\theta}C$, we can easily check that

$$\bar{C}_{\|} = e^{-1/2\, I\theta}\bar{C} = e^{1/2\, I\theta}C = C_{\|}.$$

By an argument similar to that used to prove the previous lemma, it can be shown that C is either of the forms (i) or (ii), or, (iii) $C = N$ where $\bar{N} = \tilde{N} = N$ and $N^2 = 0$.

Theorem 6 can be used as basis for the classification of the Lorentz group [5, p. 21]. To gain more insight into the relationship between the conjugations $\bar{P}$ and $\tilde{P}$, we prove

Theorem 7. Let $\tilde{B} = e^{\phi A}\bar{B}e^{-\phi A}$ for real ϕ and where $\bar{A} = \tilde{A} = A$, $A^2 = 1$. Suppose that $D \in \mathscr{B}$, and define $D' = e^{\frac{1}{2}\phi A}De^{-\frac{1}{2}\phi A}$. Then $\bar{D} = D$ iff $\tilde{D}' = D'$.

Proof: Suppose that D' is given as in the theorem. Then it is easy to calculate

$$\tilde{D}' = e^{\phi A}\bar{D}'e^{-\phi A} = e^{\phi A}e^{-1/2\,\phi A}\bar{D}e^{1/2\,\phi A}e^{-\phi A} = e^{1/2\,\phi A}\bar{D}e^{-1/2\,\phi A},$$

from which the theorem easily follows.

Corollary. The conjugation $\tilde{P}$ is proper (improper) iff the conjugation $\bar{P}$ is proper (improper).

Proof: If $\{E_k\}$ is an orthonormal frame of the conjugation $\bar{P}$ satisfying $\bar{E}_k = \pm E_k$ for $k = 1, 2, 3$, then $\{E_k'\}$ is an orthonormal frame of the conjugation $\tilde{P}$ satisfying $\tilde{E}_k' = \pm E_k'$ where $E_k' = e^{\frac{1}{2}\phi A}E_k e^{-\frac{1}{2}\phi A}$.

Definition 3. If $\bar{P}$ and $\tilde{P}$ are the proper conjugations of inertial observers, then $e^{\phi A}$ is said to be the *proper Lorentz rotation* relating the observers $\bar{P}$ and $\tilde{P}$. The aggregate of all proper Lorentz rotations form a subgroup of the Lorentz group, called the proper orthochronous Lorentz group SO(3,1). (Note that proper conjugations are never related by Lorentz rotations of the form e^{N} where $\bar{N} = \tilde{N} = N$ and $N^2 = 0$).

We state without proof

Theorem 8. Let $\tilde{B} = e^{\frac{1}{2}I\pi A}\bar{B}e^{-\frac{1}{2}I\pi A} = A\bar{B}A$ where $\bar{A} = \tilde{A} = A$ and $A^2 = 1$. Suppose that $D \in \mathscr{B}$, and define $D' = ADA$. Then $\bar{D} = D \Leftrightarrow \tilde{D}' = D' \Leftrightarrow D' = D$.

Corollary. $\tilde{P}$ is proper (improper) iff $\bar{P}$ is improper (proper). Theorem 8 and its corollary show that the *space reflections* of an observer $\bar{P}$ are generated by improper conjugations $\tilde{P}$,

$$\tilde{\bar{B}} = ABA. \tag{5}$$

It follows from (5) that the space reflections of an observer $\bar{P}$ are Lorentz rotations in spacetime.

4. Hermitian operators

Let $\bar{P}$ be a conjugation in the complex vector algebra $\mathscr{P}$.

Definition 4. $(A, B) \equiv A \circ \bar{B}$ for all $A, B \in \mathscr{B}$, is said to be the *Hermitian metric* of the conjugation $\bar{P}$. (See Ref. [6, p. 243].)

Theorem 9. (A, B) is positive definite (indefinite) iff $\bar{P}$ is proper (improper).

Proof: Suppose $\bar{P}$ is proper. Then there exists an orthonormal frame $\{E_k\}$ such that $\bar{E}_k = E_k$ (recalling thm. 3). Letting $A = \alpha^k E_k$ we see that

$$(A, A) = A \circ \bar{A} = \alpha^k \bar{\alpha}_k > 0 \quad \text{for} \quad A \neq 0.$$

Suppose now that $\bar{P}$ is improper. Then, by theorem 2, there exists an N such that $\bar{N} = N$. But then we have

$$(N, N) = N \circ \bar{N} = N \circ N = 0 \quad \text{for} \quad N \neq 0,$$

so (A, B) is indefinite.

By the *unitary space* of an observer with the proper conjugation $\bar{P}$, we mean the vector space $\mathscr{B}$ together with the pos. def. Hermitian metric $(A, B) \equiv A \circ \bar{B}$.

We wish now to establish the connection between anti-dual symmetric operators and Hermitian operators [6, p. 268]. Anti-dual symmetric operators were studied in [2, 3], and are defined by the property

$$T(A) \circ B = \overline{A \circ T(B)} \quad \text{for} \quad A, B \in \mathscr{B}. \tag{6}$$

Definition 5. By the *Hermitian operator* of the anti-dual symmetric operator $T(B)$, we mean the operator $H(B) \equiv T(\bar{B})$. Note that in the definition of an Hermitian operator we have used the conjugation $\bar{P}$; a different Hermitian operator would be specified by $H'(B) \equiv T(\tilde{B})$, where we are using the conjugation $\tilde{P}$. To see that $H(B)$ is Hermitian in the usual sense, we check that

$$(H(A), B) = T(\bar{A}) \circ \bar{B} = \overline{\bar{A} \circ T(\bar{B})} = A \circ \overline{T(\bar{B})} = (A, H(B)), \tag{7}$$

so that $H = H^*$, where H^* is the adjoint of H w.r.t the Hermitian metric $(A, B) = A \circ \bar{B}$.

The Hermitian operator $H(B)$ is (self)dual in the sense that

$$H(\beta B) = T(\bar{\beta}\bar{B}) = \beta T(\bar{B}) = \beta H(B) \quad \text{for} \quad \beta \in \mathscr{C}.$$

Dual operators, and in particular, dual symmetric operators were systematically studied in [2], resulting in the equivalent of what is known as the Petrov classification of the Weyl tensor [7]. Dual symmetric (w.r.t the complex metric $A \circ B$) operators satisfy the property that

$$S(A) \circ B = A \circ S(B) \quad \text{for all} \quad A, B \in \mathscr{B}, \tag{8}$$

which is the analogue of the condition (6) for an anti-dual symmetric operator.

The equivalent of an anti-dual symmetric operator, the energy-momentum tensor, has been studied in the literature in a variety of matrix, tensor, and spinor formalisms, for example [8], [9]. However, a new result uncovered in [3] is that the algebraic classification of an anti-dual symmetric operator is equivalent to the classification of a dual symmetric operator once a *principal correlation* has been found. The notion of a principal correlation of an anti-dual symmetric operator $T(B)$ is intrinsically tied up with the notion of a conjugation.

Definition 6. A conjugation $\tilde{P}$ is said to be *compatible* with $T(B)$ if $\widetilde{T(B)} = T(\tilde{B})$ for all $B \in \mathscr{B}$.

Definition 7. If a conjugation $\tilde{P}$ is compatible with $T(B)$, then the Hermitian operator $H(B) \equiv T(\tilde{B})$ is said to be a *principal correlation* of $T(B)$.

It will be shown in the next section that there always exists an improper conjugation $\tilde{P}$ which is compatible with $T(B)$, and hence $T(B)$ will always have at least one principal correlation $H(B) = T(\tilde{B})$. A ppl. cor. is always dual symmetric, i.e.,

$$H(A) \circ B = T(\tilde{A}) \circ B = \widetilde{\tilde{A} \circ T(B)} = A \circ \widetilde{T(B)} = A \circ T(\tilde{B}) = A \circ H(B).$$

Thus, H not only has the symmetry of a Hermitian operator (7), but is also symmetric w.r.t the complex metric (8). Translating this into a statement about the eigenvectors and values of $H(B)$, we find that

$$(\lambda_i - \lambda_j) C_i \circ C_j = (\lambda_i - \bar{\lambda}_j) C_i \circ \bar{C}_j = 0, \tag{9}$$

where $H(C_k) = \lambda_k C_k$ for $k = 1, 2, 3$ (the eigenvectors and values of $H(B)$ need not be all distinct [2]). Finally, note the relationship

$$T(C_k) = H(\tilde{C}_k) = \bar{\lambda}_k C_k, \tag{10}$$

between the eigenvectors and values of T as an anti-dual symmetric operator, and the eigenvectors and values of H. The relationships (9) and (10) are the basis for the algebraic classification of T by way of its ppl. cor. H, and (10) corrects an oversight made in [3, p. 598].

5. *The Dirac-Clifford splitting of spacetime*

As has been pointed out in [1, §4], the complex vector algebra can be embedded in a larger algebra, the Dirac-Clifford algebra, in such a way that the elements of $\mathscr{P}$ become the even subalgebra, or the Pauli subalgebra of the Dirac algebra $\mathscr{D}$. The best way to carry out this embedding is to express some orthonormal basis $\{E_k\}$ of the complex vector algebra in terms of the orthonormal basis $\{e_u\}$ of a Dirac-Clifford algebra, i.e.,

$$E_k = e_k e_0 = e_k \wedge e_0 \text{ for } k = 1, 2, 3 \text{ and } I = E_1 E_2 E_3 = e_0 e_1 e_2 e_3, \tag{11}$$

where the Minkowski metric of spacetime is determined by the condition

$$e_0^2 = 1 = -e_1^2 = -e_2^2 = -e_3^2.$$

Relationship (11) in effect "splits" or factors the 8(real)-dim. complex vector algebra $\mathscr{P}$ into a larger 16(real)-dim. Dirac-Clifford algebra $\mathscr{D}$. It is well-known that a Clifford algebra can be embedded as an even subalgebra in the next higher dimensional Clifford algebra, but the theory is usually shrouded in matrix representations [10]. This reliance on matrix representations is a heavy notational burden which clouds rather than enlightens the abstract essence of a theory which is fully capable of standing on its own feet without the matrix crutch [4].

Let us now show that when the complex vector algebra $\mathscr{P}$ is considered to be a subalgebra of the Dirac algebra $\mathscr{D}$, then a conjugation $\bar{P}$ in $\mathscr{P}$ can be uniquely expressed in terms of an inner automorphism in $\mathscr{D}$.

Theorem 10. (*i*) If $\bar{P}$ is a proper conjugation, then there exists a time-like Dirac vector $a_0 \in \mathscr{D}_1$ with $a_0^2 = 1$, such that $\bar{B} = -a_0 B a_0$ for all $B \in \mathscr{B}$. (*ii*) If $\tilde{P}$ is an improper conjugation, then there exists a space-like Dirac vector $a_1 \in \mathscr{D}_1$ with $a_1^2 = -1$, such that $\tilde{B} = a_1 B a_1$.

Proof: (*i*) Suppose $\bar{P}$ is a proper conjugation. Then by theorem 3, there exists an orthonormal frame $\{E_k\}$ such that $\bar{E}_k = E_k$ for $k = 1, 2, 3$. Let us now split $\mathscr{P}$ along the orthonormal vectors E_k by writing

$$E_k = a_k a_0 = a_k \wedge a_0 \text{ where } a_0^2 = 1 = -a_1^2 = -a_2^2 = -a_3^2,$$

to get an orthonormal frame $\{a_\mu\}$ of the Dirac algebra $\mathscr{D}$, the same as was done in (11). Then for all $B = \beta^k E_k \in \mathscr{B}$, it is easy to check that

$$\bar{B} = -a_0 B a_0 = -a_0 \beta^k E_k a_0 = -\bar{\beta}^k a_0 a_k a_0 a_0 = \bar{\beta}^k E_k,$$

as is required.

(*ii*) Suppose now that an improper conjugation $\tilde{P}$ is given. Then, by theorem 2, there exists an orthonormal frame $\{E_k\}$ such that $\tilde{E}_1 = E_1$, $\tilde{E}_2 = -E_2$, and $\tilde{E}_3 = -E_3$. Splitting $\mathscr{P}$ along E_k gives an orthonormal frame $\{a_\mu\}$ of $\mathscr{D}$ such that $E_k = a_k a_0 = -a_0 a_k = a \wedge a_0$. For $B = \beta^k E_k$, we find that

$$\tilde{B} = a_1 B a_1 = \beta^1 E_1 - \beta^2 E_2 - \beta^3 E_3,$$

as is required.

If, following Markus [10, p. 279], we define the mapping $r: \mathscr{D}_1 \to \mathscr{D}_1$,

$$r(v) \equiv -ava^{-1}, \text{ where } \quad a \in \mathscr{D}_1 \quad \text{and} \quad a^2 \neq 0, \tag{12}$$

then a reflection in $\{a\}^\perp$ is the negative of an inner automorphism in the Clifford algebra $\mathscr{D}$. To relate the action of the mapping $r(v)$ to the conjugation $\bar{P}$ that it defines, write $B = b_1 \wedge b_2 \in \mathscr{B}$. Then

$$\bar{B} = -r(b_1) \wedge r(b_2) = -\langle ab_1 a^{-1} ab_2 a^{-1} \rangle_2 = -a(b_1 \wedge b_2)a^{-1} = -aBa^{-1}. \tag{13}$$

A conjugation $\bar{P}$ can also be considered to be the main anti-automorphism on an appropriately defined Clifford algebra: In the case of a proper conjugation $\bar{P}$, the Clifford algebra is generated by an orthonormal frame $\{E_k\}$ of vectors satisfying $\bar{E}_k = E_k$ for $k = 1, 2, 3$ and

$$E_1^2 = E_2^2 = E_3^2 = 1.$$

In the case of an improper conjugation $\tilde{P}$, the Clifford algebra is generated by an orthonormal frame $\{F_k\}$ where $\tilde{F}_1 = F_1 = E_1$, $\tilde{F}_2 = F_2 = IE_2$, and $\tilde{F}_3 = F_3 = IE_3$, and

$$F_1^2 = 1 = -F_2^2 = -F_3^2,$$

[10, p. 199]. Note that a conjugation in $\mathscr{P}$ is not the main anti-automorphism when considered on the level of the Dirac algebra $\mathscr{D}$[2].

Let us now consider the anti-dual symmetric operator $T(B)$, given in (6), on the level of the Dirac algebra $\mathscr{D}$. First, note that we can define a trace-free symmetric $t: \mathscr{D}_1 \to \mathscr{D}_1$ by

$$t(a) \equiv \partial_v \cdot T(v \wedge a), \tag{14}$$

as discussed in [2, §3], and where we are using the notion of the vector derivative ∂_v, as discussed in [3]. The operator $T(B)$ can be expressed in terms of $t(b)$ by the relationship

$$T(a \wedge b) = \tfrac{1}{2}(a \wedge b) \cdot \partial_v t(v) \equiv \tfrac{1}{2}[t(a) \wedge b + a \wedge t(b)], \tag{15}$$

and thus the study of anti-dual symmetric operators $T(B)$ on $\mathscr{B}$ is equivalent to the study of trace-free symmetric vector operators $t(b)$ on $\mathscr{D}_1$.

A well-known property of $t(v)$ is that $t(v)$ will always have at least one real space-like eigenvector, that is

$$t(a_1) = \tau_1 a_1 \text{ where } a_1 \in \mathscr{D}_1 \text{ and } a_1^2 = -1,$$

[8]. This property can be used to establish

Lemma. Let $r(v)$ be the reflection generated by the eigenvector a_1 as defined in (12). Then $tr(v) = rt(v)$ for all $v \in \mathscr{D}_1$.

Proof: First note that since t is symmetric, t: $\{a_1\}^\perp \to \{a_1\}^\perp$, as follows from

$$b \cdot a_1 = 0 \Leftrightarrow t(b) \cdot a_1 = b \cdot t(a_1) = \tau_1 b \cdot a_1 = 0.$$

Thus, for $v = v_{\|} + v_{\perp} \in \mathscr{D}_1$, we find that

$$tr(v) = tr(v_{\|}) + tr(v_{\perp}) = -t(v_{\|}) + t(v_{\perp}),$$

and, similarly

$$rt(v) = rt(v_{\|}) + rt(v_{\perp}) = -t(v_{\|}) + t(v_{\perp}),$$

and hence $tr(v) = rt(v)$ for all $v \in \mathscr{D}_1$ as was to be proved.

We can now prove, as promised at the end of the last section, that $T(B)$ always has a principal correlation.

Theorem 11. Let $\tilde{P}$ be the improper conjugation in $\mathscr{P}$ generated by a space-like eigenvector a_1 of t, as given in thm. 10 (*ii*). Then $H(B) = T(\tilde{B})$ is a principal correlation of T.

[2] The operation of space reversion, discussed in the preceding footnote, is the main anti-automorphism when considered on the level of the Dirac algebra.

Proof: We need only show that the conjugation $\tilde{P}$ is compatible with T, i.e., we must show that $\widetilde{T(B)} = T(\tilde{B})$. Let $B = a \wedge b \in \mathscr{D}_2 \equiv \mathscr{B}$. Then, by using (15), (13), and the above lemma, we can verify the steps

$$\widetilde{T(a \wedge b)} = -T[r(a) \wedge r(b)] = -\tfrac{1}{2}[tr(a) \wedge r(b) + r(a) \wedge tr(b)]$$

$$= -\tfrac{1}{2}[rt(a) \wedge r(b) + r(a) \wedge rt(b)] = \tfrac{1}{2}[\widetilde{t(a) \wedge b} + \widetilde{a \wedge t(b)}]$$

$$= \widetilde{T(a \wedge b)}.$$

Finally, let us see how the usual representation of a "boost" as a 4×4 matrix $\Lambda_u{}^v$ can be obtained using complex vector algebra. Let $\bar{P}$ and $\tilde{P}$ be the proper conjugations defining two inertial observers, and let $\{E_k\}$ be an orthonormal rest frame of $\bar{P}$, i.e., $\bar{E}_k = E_k$ for $k = 1, 2, 3$. Then by using theorem 7 and its corollary, the conjugation $\bar{P}$ and $\tilde{P}$ are related by

$$\tilde{B} = e^{\phi A}\bar{B}e^{-\phi A} \text{ where } \tilde{A} = \bar{A} = A \text{ and } A^2 = 1,$$

and an orthonormal rest frame of $\tilde{P}$ is given by

$$E_k' \equiv L(E_k) = e^{1/2\,\phi A}E_k e^{-1/2\,\phi A}.$$

Without loss of generality we can choose $E_1 = A$.

To find the matrix representation of the boost in the Pauli algebra $\mathscr{P}$, we first calculate

$$L(E_1) = E_1, \quad L(E_2) = e^{\phi E_1}E_2 = \cosh(\phi)E_2 + IE_3\sinh(\phi)$$

and

$$L(E_3) = e^{\phi E_1}E_3 = \cosh(\phi)E_3 - IE_2\sinh(\phi),$$

from which it follows that

$$L_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh(\phi) & I\sinh(\phi) \\ 0 & -I\sinh(\phi) & \cosh(\phi) \end{pmatrix} \equiv L(E_i) \circ E_j.$$

To find the matrix representation of the boost in the Dirac algebra $\mathscr{D}$, note that $e_u' = e^{\frac{1}{2}\phi E_1}e_u e^{-\frac{1}{2}\phi E_1}$, where $E_k = e_k e_0$ and $E_k' = e_k' e_0$, and calculate

$$e_0' = e^{\phi E_1}e_0 = \cosh(\phi)e_0 + \sinh(\phi)e_1,$$

$$e_1' = e^{\phi E_1}e_1 = \cosh(\phi)e_1 + \sinh(\phi)e_0,$$

$$e_2' = e_2, \quad \text{and} \quad e_3' = e_3,$$

from which it follows that

$$\Lambda_u^{\ v} = \begin{bmatrix} \cosh\phi & \sinh\phi & 0 & 0 \\ \sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \equiv e'_u \cdot e^v.$$

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