NON-VACUUM COSMOLOGIES WITH TOROIDAL TOPOLOGY OF SPACE-SECTIONS

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Cosmological models with toroidal topology of 3-space in the co-moving frame are obtained as exact solutions of the Einstein equations with sensible energy-stress tensor and the Friedmannian behaviour of the scale factor. The red-shift is found to be isotropic in spite of the model anisotropy.

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1. Introduction

The models studied till recently in the general relativistic cosmology have been both isotropic and spatially homogeneous. The condition of isotropy at every point leads uniquely to a certain metric form. This form contains a function $R(t)$ to be determined. For dust it is the Friedman solution. But such universes explain neither the measured isotropy of the black body radiation [1] nor the existence of galaxies of preferred sizes [2]. For these reasons, more general cosmologies have been examined for both their classical and quantum mechanical properties. In this paper we construct explicitly anisotropic, spatially inhomogeneous models with some properties, relating them to the real Universe. At the same time we discuss how to introduce some other nonstandard features which may at first sight contradict one's intuition but need detailed quantitative analyses for final physical judgement.

In these models we impose the toroidal topology of space sections (for the case of the Einstein-Cartan theory a model of the same type was discussed in our group recently [3]). We begin with the geometry of usual 2-surfaces and consider its generalisation to a geometry on toroidal hypersurface in some fictitious four-dimensional flat space E^4 with the coordinates x, y, z, u.

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2. *Model 1*

Using hypertorus equation

$$x^2+y^2+z^2+u^2 = b(x^2+y^2+z^2)^{1/2} \tag{1}$$

we eliminate u from the expression for metric in E^4

$$dl^2 = dx^2+dy^2+dz^2+du^2.$$

The coordinate transformation

$$x = \alpha(\sin R+A)\sin\vartheta\cos\varphi, \qquad y = \alpha(\sin R+A)\sin\vartheta\sin\varphi,$$

$$z = \alpha(\sin R+A)\cos\vartheta, \qquad u = \alpha\cos R, \tag{2}$$

leads to the metric in the physical curved 3-space

$$dl^2 = \alpha^2(dR^2+(\sin R+A)^2\,(d\theta^2+\sin^2\vartheta d\varphi^2)), \tag{3}$$

where $\alpha^2 = b^2/4-a^2$, $A = b/2\alpha$, $0 \leqslant R, \theta \leqslant \pi$, $0 \leqslant \varphi \leqslant 2\pi$. To describe the whole space-time we will use the co-moving frame of reference. Then

$$ds^2 = dt^2-dl^2, \tag{4}$$

provided that units $8\pi G = 1 = c$ have been adopted.

Hypertorus parameters $b/2$ and α (the large and small radii respectively) are in general functions of time. In this and the next models we assume that the scale factor α satisfies the same equation as in the case of the Friedman solution for dust, i.e. $\dot{\alpha}^2+2\alpha\ddot{\alpha}+1 = 0$ ($\dot{} \equiv \partial/\partial t$), while A is taken to be constant, then from the Einstein equations we have

$$\mathscr{E} = T_{(0)(0)} = \frac{\sin R(3\sin R+2A)}{\alpha^2(\sin R+A)^2}+\frac{3\dot{\alpha}^2}{\alpha^2},$$

$$p_2 = T_{(2)(2)} = T_{(3)(3)} = \frac{A}{\alpha^2(\sin R+A)},$$

$$p_1 = T_{(1)(1)} = \frac{A(A+2\sin R)}{\alpha^2(\sin R+A)^2}. \tag{5}$$

The state equation takes the form $p_1+2p_2+\mathscr{E} = 3\,\dfrac{(1+\dot{\alpha}^2)}{\alpha^2}$. Here all the components of energy-stress tensor refer to the orthonormal tetrad frame of basis 1-forms $\theta^{(0)} = dt$, $\theta^{(1)} = \alpha dR$, $\theta^{(2)} = \alpha(\sin R+A)d\vartheta$, $\theta^{(3)} = \alpha\sin\vartheta(\sin R+A)d\varphi$. Energy density is positive for any observer. If $A > 1$ then the minimal value of difference $b/2-\alpha$ remains essentially positive.

3. *Model 2(a)*

Using another version of hypertorus equation

$$x^2+y^2+z^2+u^2 = b(x^2+y^2)^{1/2} \tag{6}$$

with the previous expression for metric in E^4 (1a) and transforming coordinates

$$x = \alpha(A+\sin\chi\sin\vartheta)\cos\varphi, \quad y = \alpha(A+\sin\chi\sin\vartheta)\sin\varphi, \quad z = \alpha\sin\chi\cos\vartheta,$$

$$u = \alpha\cos\chi, \tag{7}$$

we obtain

$$dl^2 = \alpha^2(d\chi^2+\sin^2\chi d\vartheta^2+(A+\sin\chi\sin\vartheta)^2 d\varphi^2), \tag{8}$$

where $0 \leqslant \chi \leqslant 2\pi$, $0 \leqslant \vartheta \leqslant \pi$, $0 \leqslant \varphi \leqslant 2\pi$.

Spacetime in the co-moving frame of reference is again described by the metric (4), and under the previous assumptions on the time dependence of α and $b/2$, it follows from the Einstein equations that

$$T_{(\alpha)(\beta)} = (\mathscr{E}+p)u_{(\alpha)}u_{(\beta)}-p(\delta_{\alpha\beta}+V_{(\alpha)}V_{(\beta)}), \tag{9}$$

where $u_{(\alpha)} = \delta^0_\alpha$, $V_{(\alpha)} = \delta^3_\alpha$, and

$$p = \frac{A}{\alpha^2(A+\sin\vartheta\sin\chi)}, \quad \mathscr{E} = \frac{3\dot{\alpha}^2}{\alpha^2}+\frac{(A+3\sin\vartheta\sin\chi)}{\alpha^2(A+\sin\vartheta\sin\chi)}. \tag{10}$$

All components refer to the orthonormal frame $\theta^{(0)} = dt$, $\theta^{(1)} = \alpha d\chi$, $\theta^{(2)} = \alpha\sin\chi d\vartheta$, $\vartheta^{(3)} = \alpha(A+\sin\chi\sin\vartheta)d\varphi$. If $A > 3$ (and it is just a restriction to hypertori with certain relation of the radii) then local energy density is always positive. So the satisfaction of the weak energy condition ensures that the minimal value of difference $b/2-\alpha$ remains positive ($\alpha(A-1) > 0$) and it allows the existence of only sufficiently extended tori (i.e. $b/2 > 3\alpha$).

4. *Model 2(b)*

In the scope of the previous model, we consider instead of (7) the following coordinate transformations

$$x = \alpha(A+\cos\varrho)\cos\varphi,\ y = \alpha(A+\cos\varrho)\sin\varphi,\ z = \alpha\sin\varrho\sin\xi,\ u = \alpha\sin\varrho\cos\xi. \tag{11}$$

The spacetime metric is rewritten as

$$ds^2 = dt^2-\alpha^2(t)(d\varrho^2+\sin^2\varrho d\xi^2+(\cos\varrho+A)^2 d\varphi^2). \tag{12}$$

We allow now the coordinates to range as follows: $0 \leqslant \varrho,\ \xi,\ \varphi \leqslant 2\pi$.

The metric may be rewritten as

$$ds^2 = dt^2-e^{2\beta_{ij}}\tilde{\sigma}^i\tilde{\sigma}^j,$$

where $\tilde{\sigma}^3 = (\cos\varrho + A)d\varphi$, $\tilde{\sigma}^2 = \sin\varrho d\xi$, and $\tilde{\sigma}^1 = d\varrho$ and β_{ij} are functions of t only. The β matrix is diag ($\ln\alpha$, $\ln\alpha$, $\ln\alpha$). In this form the metric resembles that of the Bianchi--type universes. But it admits an Abelian group G_2 (as in the Gowdy three-torus and three--handle vacuum universes [4]) acting on spacelike 2-surfaces (with Killing vectors ∂_ξ and ∂_φ) whereas homogeneous universes admit a group G_3 simply transitive on spacelike hypersurfaces.

From the previous analysis and requirements the topology of physical 3-space does not follow uniquely. Local arguments do not give any information regarding the connectivity of spacetime (M, g) at large (see e.g. [5]). Any spacetime (M, g) is obtained from its universal covering manifold $\widetilde{M}$ by identifying suitably the points in $\widetilde{M}$ (and such identification usually lowers the dimension of its group of isometries) [6].

In order to understand the possible topology of the model with metric (12) we consider metric of 2-torus with α and αA being the small and large radii respectively

$$dl^2 = \alpha^2(d\varrho^2 + (\cos\varrho + A)^2 d\varphi^2), \tag{13}$$

where $0 \leqslant \varphi$, $\varrho \leqslant 2\pi$ and the circles $\varrho = 0$ and $\varrho = 2\pi$, as well as $\varphi = 0$ and $\varphi = 2\pi$, are identified (φ changes along the large and ϱ along the small circumferences).

When ξ = const in (12) in a fixed moment of time the metric has the form (13). If we fix φ instead of ξ, the remaining 2-metric may be interpreted as the metric on 2-sphere

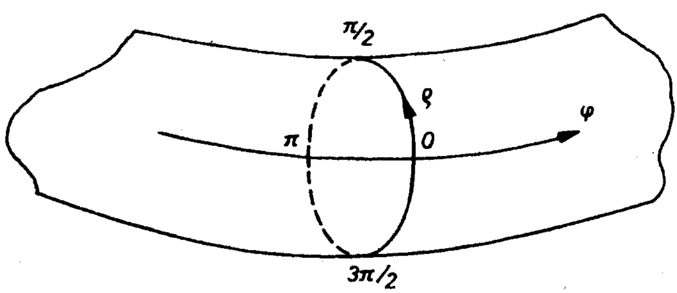

Fig. 1. A 3-surface with toroidal topology embedded in E^4 with ξ suppressed. The 2-surfaces of fixed ϱ (which look like circles in the picture, because one dimension is suppressed) are actually 2-spheres of surface area $4\pi^2\alpha^2\sin^2\varrho$. The space-sections with ξ suppressed are 2-tori. The length of circles ϱ = const is $2\pi(A + \cos\varrho)$

with radius α and longitude ξ (ϱ being considered as latitude, running from one of the poles, and passing at $\varrho = \pi$ into another sheet). Thus we may allow ξ to range from 0 to 2π. The hypersurfaces $\varphi = 0$ and $\varphi = 2\pi$ as well as $\xi = 0$ and $\xi = 2\pi$, and $\varrho = 0$ and $\varrho = 2\pi$, are identical. (See Figs 1 and 2.)

From the Einstein equations for the spacetime metric (12) we obtain $T_{(\mu)(\nu)}$ in the form (9) where

$$p = \frac{A}{\alpha^2(A + \cos\varrho)}, \quad \mathscr{E} = \frac{3\alpha^2}{\alpha^2} + \frac{A + 3\cos\varrho}{\alpha^2(A + \cos\varrho)}. \tag{14}$$

Under the previous conditions (including $A > 3$) $T_{(\mu)(\nu)}$ satisfies the weak energy condition. In 2(*b*) the components of $T_{(\mu)(\nu)}$ refer to the orthonormal frame $\theta^{(0)} = dt$, $\theta^{(1)} = \alpha d\varrho$, $\theta^{(2)} = \alpha \sin \varrho d\xi$, $\theta^{(3)} = \alpha(A+\cos \varrho)d\varphi$.

The state equation takes the form

$$\mathscr{E}+2p = 3\,\frac{1+\dot{\alpha}^2}{\alpha^2} \tag{15}$$

$\left(\text{compare with the Friedman case, } \mathscr{E}_{FR} = 3\,\frac{1+\dot{\alpha}^2}{\alpha^2}\right)$. It is clear that $T_{(\mu)(\nu)}$ satisfies the strong energy condition as a consequence of the state equation (15). When $A \gg 1$, $\mathscr{E}$ and p

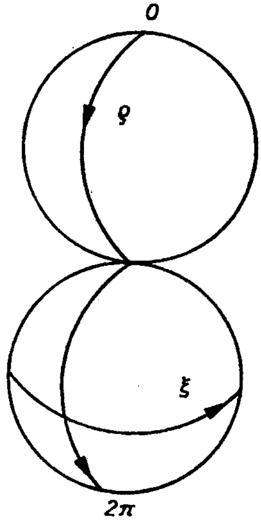

Fig. 2. Another representation of the 3-surface with toroidal topology. The 3-surface embedded in E^4 with φ suppressed consists of two balls. The 2-surfaces of ϱ = const. (which look like circles again) are actually 2-tori of surface area $4\pi\alpha^2(A+\cos \varrho)^2$. As ϱ ranges from 0 to 2π one moves outward from the "north pole" of the hypersurface, through successive 2-tori ("shells"). The 2-surfaces $\varrho = 0$ and $\varrho = 2\pi$ are identical

do not depend on t at all and in this case we come to the energy-stress tensor of anisotropic fluid (probably better to be called a circular string to which the hypertorus is reduced).

In spite of the anisotropy of the space and the presence of pressure, the matter in the model is moving geodesically (along the time-lines of synchronous frame of reference), i.e. from $T^{\mu\nu}_{;\nu} = 0$ it follows that $u^\nu u^\mu_{;\nu} = 0$.

The volume of 3-space of the model

$$V = 4\int_0^{2\pi}\int_0^{2\pi}\int_0^{\pi/2} \alpha^3 \sin \varrho(\cos \varrho + A)d\varphi d\xi d\varrho = 16\pi^2\alpha^3(\tfrac{1}{2}+A) \tag{16}$$

(the volume of 3-space in the closed Friedman solution is $V_{FR} = 2\pi^2\alpha^3$).

5. *Open models*

Similarly we may introduce open models. Here we will consider only the case corresponding to (12). For simplicity we just transform $\pi/2-\varrho \to i\varrho$, $\alpha \to i\alpha$, $\xi \to i\xi$, and the metric turns into

$$ds^2 = dt^2 - \alpha^2(t)(d\varrho^2 + \mathrm{sh}^2\varrho d\xi^2 + (A+\mathrm{ch}\varrho)^2 d\varphi^2). \tag{17}$$

Thus, the embedding of the corresponding 3-space into E is realized by transformations

$$u = \alpha\mathrm{ch}\varrho \cos\xi, \quad z = \alpha\mathrm{ch}\varrho \sin\xi, \quad x = \alpha(A+\mathrm{sh}\varrho)\cos\varphi, \quad y = \alpha(A+\mathrm{sh}\varrho)\sin\varphi. \tag{18}$$

The intermediate case is realized by transformations

$$x = \alpha(A+\varrho)\cos\varphi, \quad y = \alpha(A+\varrho)\sin\varphi, \quad u = \alpha\cos\xi, \quad z = \alpha\sin\xi, \tag{19}$$

which lead to

$$ds^2 = dt^2 - \alpha^2(d\varrho^2 + d\xi^2 + (\varrho+A)^2 d\varphi^2), \tag{20}$$

but this is just the metric of the Friedman model with flat space-sections in cylindrical coordinates $\{\varrho+A, \xi, \varphi\}$.

6. *Red-shift*

Cosmological red-shift may be calculated by means of the general formula of Schrödinger-Brill [7]:

$$r = \frac{(U_1 \cdot K)}{(U_2 \cdot K)}. \tag{21}$$

Here

$$U_1 = U_2 = \partial_t \tag{22}$$

are vectors of 4-velocities of a source and a detector of radiation, which are co-moving with matter, K is null vector connecting corresponding point on timelike lines. The latter vector must satisfy the geodesic equation and be gradient of a scalar, because it represents a generator of light cone, i.e. it belongs to a (null) normal congruence. Let us find all such vectors which are on the cone with apex at $\varrho = 0$. It is clear that in this case $\xi = \text{const.}$ and

$$K_\mu dx^\mu = F(t)dt + f(\varrho)d\varrho + d\varphi. \tag{23}$$

Since K is a null vector we have

$$K \cdot K = F^2 - \alpha^{-2}[f^2 + (A+\cos\varrho)^{-2}], \tag{24}$$

and from this one concludes that

$$f^2 + (A+\cos\varrho)^{-2} = N^2 = \text{const.},$$

$$F = \alpha^{-1}N. \tag{25}$$

Different values of the constant N correspond to different choices of all possible initial directions of the light ray from the point $\varrho = 0$, $\varphi = \varphi_0 = \text{const}$. Substituting this result to (21) we find finally:

$$r = \frac{\alpha(t_2)}{\alpha(t_1)} \tag{26}$$

being formally the same expression as in the Friedman universe. Thus we may conclude that there is an effective isotropy of the red-shift in spite of anisotropy and toroidal symmetry of our models.

7. *Reducing to Friedman's case*

Formal substitution $A = 0$ in (12) gives the Friedman metric for non-coherent dust, but in nonstandard coordinates (S^3 in such coordinates have been discussed in [8])

$$ds^2 = dt^2 - \alpha^2(d\varrho^2 + \sin^2 \varrho d\xi^2 + \cos^2 \varrho d\varphi^2). \tag{27}$$

The usual Friedman expression in spherical coordinates may be obtained from this metric using transformations

$$\varrho = \arccos(\sin\chi \sin\vartheta),$$

$$\xi = \arcsin \frac{\sin\chi \cos\vartheta}{(\cos^2\theta + \cos^2\chi \sin^2\vartheta)^{1/2}}.$$

However the spacetime (12) with $A = 0$ is only locally equivalent to that of Friedman coinciding with it after identification of the hypersurfaces $\pi - \varrho$ and ϱ. Similar transition (and transformations) may be realized for the open models.

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