QUASI-CLASSICAL VACUA FOR GAUGE GROUPS SU($N \to \infty$)

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The problem of finding the covariantly constant chromomagnetic field, which minimizes the real part of the energy density, as obtained in the one-loop approximation, is solved for the SU(N) gauge groups in the $N \to \infty$ limit.

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1. Introduction

It has been pointed out [1] that the real part of the energy density of a covariantly constant Yang-Mills field, when calculated in the one-loop approximation, may be negative, i.e. lower than the corresponding quantity for the perturbative vacuum. The physical relevance of this observation is not clear (cf. e.g. Ref. [2] and the caveats in Ref. [3]), but work has been done looking for covariantly constant fields minimizing the real part of the energy density in the one-loop approximation. Savvidy [1] found the solution for the SU(2) gauge group. Flyvbjerg [4] derived some auxiliary formulae for arbitrary gauge groups, solved the SU(3) and SU(4) cases and pointed out a difficulty, which occurs for SU(N) groups with $N > 4$.

In this paper we find the solution for the SU($N \to \infty$) groups. As a byproduct we also find the solutions for $N = 6$, 8, 12 and 20.

2. Minimization of the one-loop energy density

As shown in Ref. [4], the real part of the energy density of a covariantly constant chromomagnetic field for the SU(N) group, when evaluated in the one-loop approximation with a suitable renormalization prescription, reads:

$$E = \sum_{\alpha} H_{\alpha}^2 \left(\frac{1}{Ng^2} + \frac{11}{48\pi^2} \ln \frac{H_{\alpha}}{\mu^2} \right). \tag{2.1}$$

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Here the summation extends over all pairs of charge conjugated gluons. Selfconjugated gluons are excluded. The parameter H_α may be interpreted as the field seen by the gluon pair α. For $N \leqslant 4$ the $N(N-1)/2$ parameters H_α may be chosen independently and

$$E = \frac{N(N-1)}{2} E_0, \tag{2.2}$$

where

$$E_0 = -\frac{11}{96\pi^2} H_0^2 \tag{2.3}$$

is the minimum of a single term in the sum (2.1). The corresponding field

$$H_0 = \mu^2 \exp\left(-\tfrac{1}{2} - \frac{48\pi^2}{11Ng^2}\right). \tag{2.4}$$

For $N > 4$ formula (2.2) breaks down, because the parameters H_α are no more independent [4]. We will translate the problem of finding the parameters H_α minimizing expression (2.1) with the constraints imposed by the covariant constancy of the field into a problem in mechanics. There is a one to one correspondence between the gluon pairs included in the summation (2.1) and positive weight vectors of the gauge group. Each weight vector may be written in the form

$$\boldsymbol{\alpha} = \sum_i x_{\alpha,i} \boldsymbol{e}_i, \tag{2.5}$$

where $x_{\alpha,i}$ are the additive quantum numbers (corresponding to a complete set of commuting generators) and $\boldsymbol{e}_i$ are Cartesian unit vectors. For a covariantly constant field (cf. [4])

$$\boldsymbol{H}_\alpha = \sum_i x_{\alpha,i} \boldsymbol{H}_i, \tag{2.6}$$

where the vectors $\boldsymbol{H}_i$ may be arbitrarily chosen. For the SU(N) group,however, it is possible to introduce N quarks so that each gluon may be considered as a $q\bar{q}$ pair. Since the quantum numbers x_i are additive and change sign under charge conjugation, each vector α from formula (2.5) can be considered as the difference of two out of N vectors corresponding to quarks. The vectors corresponding to quarks

$$\boldsymbol{q} = \sum_i x_{q,i} \boldsymbol{e}_i \tag{2.7}$$

are related only by the constraint

$$\sum_i \boldsymbol{q}_i = 0. \tag{2.8}$$

Since it is always possible to make the substitution $\boldsymbol{q}_j \to \boldsymbol{q}_j + \boldsymbol{a}$ to enforce (2.8) without affecting the differences $\boldsymbol{H}_\alpha$, we may chose N independent vectors $\boldsymbol{q}_j$ so as to minimize expression (2.1). Thus the problem of finding the set of vectors $\boldsymbol{H}_\alpha$ satisfying the constraints (2.6) and minimizing expression (2.1) reduces to the following problem in mechanics:

Find the equilibrium distribution of N points in ordinary three dimensional space, if each pair contributes to the total energy

$$V(r) = \mu^4 r^2 \left(\frac{1}{Ng^2} + \frac{11}{48\pi^2} \ln r \right), \tag{2.9}$$

where r is the distance between the two points in the pair.

The potential (2.9) has a minimum at $r = H_0$ given by formula (2.4). Therefore, the equilibrium configuration for $N = 2, 3, 4$ corresponds to setting respectively: the two points at the ends of a section of length H_0, the three points at the vertices of an equilateral triangle with side length H_0 and the four points at the vertices of a regular tetrahedron with edge length H_0. In each case all the distances are H_0 and formula (2.2) holds. For $N > 4$ it is no more possible to have all the distances equal H_0.

For N large it is convenient to use a density of points in space. The normalization is

$$\int \varrho(\boldsymbol{r}) d^3 r = N. \tag{2.10}$$

The minimum of

$$E = \tfrac{1}{2} \int \varrho(\boldsymbol{r}_1) \varrho(\boldsymbol{r}_2) V(\boldsymbol{r}_1 - \boldsymbol{r}_2) d^3 r_1 d^3 r_2 \tag{2.11}$$

with the constraint (2.10) implies that the potential

$$\overline{V}(\boldsymbol{r}) = \int \varrho(\boldsymbol{r}_1) V(\boldsymbol{r}_1 - \boldsymbol{r}_2) d^3 r_1 \tag{2.12}$$

must be constant in all the region, where $\varrho(\boldsymbol{r}) > 0$ and be not smaller than this constant in the remaining part of space. A distribution, which satisfies this condition is[1]

$$\varrho(\boldsymbol{r}) = \frac{N}{4\pi r_0^2} \delta(r - r_0), \tag{2.13}$$

where

$$r_0 = \tfrac{1}{2} e^{1/4} H_0. \tag{2.14}$$

The corresponding real part of the energy density is

$$E = \frac{\sqrt{e}}{2} \frac{N^2}{2} E_0 = 0.824 \frac{N^2}{2} E_0. \tag{2.15}$$

The distribution of the vectors $\boldsymbol{H}_\alpha$ is isotropic in space, with the distribution in length

$$\frac{dn}{dH_\alpha} = \frac{N}{2r_0^2} H_\alpha \quad \text{for} \quad 0 \leqslant H_\alpha \leqslant 2r_0 \tag{2.16}$$

and zero otherwise.

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[1] Analogous solutions can be found for some finite N. In particular, putting the N points at the centres of the faces of a regular polyhedron, we can adjust the size of the polyhedron so that the resultant radial force on one of the points equals zero. Then by symmetry the radial forces acting on all the other points and all the tangential forces vanish and the configuration is stationary. Besides the known solution for $N = 4$ this approach yields configurations of minimum energy for $N = 6$, 8, 12 and 20.

3. Discussion

The solution (2.13)–(2.16) yields by construction a minimum of the one-loop energy density. We have no complete proof that this is an absolute minimum, but there are strong arguments in favour of this conjecture. Using the fact that the force derived from the potential (2.9) is a concave function of x at fixed y and z, one shows easily that in equilibrium all the N points must be within a sphere of radius H_0. Thus the result

$$\varrho(\boldsymbol{r}) = 0 \quad \text{for} \quad |\boldsymbol{r}| > H_0 \tag{3.1}$$

holds rigorously for arbitrary N. We have also proved that for $N \to \infty$ nothing is gained by changing from a spherically symmetrical distribution to a distribution with axial symmetry only. Among spherically symmetrical distributions vanishing for $|\boldsymbol{r}| > a$, where a is some constant, we found that for $\varrho \sim r^{\alpha}$ the minimum corresponds to $\alpha \to \infty$ and $a \to r_0$ thus to solution (2.13). Also substituting for ϱ a linear combination of two δ-functions nothing is gained. It is plausible that the only distribution, which is not better approximated by a combination of two delta functions than by one delta function is a delta function.

On the other hand, very different distributions correspond to only slightly higher values of the energy density. Thus e.g. putting $N/4$ points in each vertex of a regular tetrahedron with edge length H_0, which corresponds to

$$\frac{dn}{dH_{\alpha}} = \frac{N}{4}\delta(0) + \frac{3N}{4}\delta(H_{\alpha} - H_0), \tag{3.2}$$

one finds

$$E = 0.750\,\frac{N^2}{2}E_0, \tag{3.3}$$

which agrees with (2.15) within ten per cent. Consequently, quantum fluctuations are likely to mix widely different background fields configurations.

After this paper has been written, the preprint [5] has reached us, where the same minimum for $N \to \infty$ was found. What remains new in our paper, is the mechanical analogy, which we found of great heuristic value, and the arguments in favour of the absolute character of the minimum.

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