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# STABILITY OF PARTICLE-LIKE SOLUTIONS OF NON-LINEAR KLEIN-GORDON AND DIRAC EQUATIONS

By J. Werle

Institute for Theoretical Physics, University of Warsaw*

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Non-linear Klein-Gordon and Dirac equations in 4 dimensions are investigated with the help of a global method based on the use of variable initial conditions. One arrives at very stringent conditions for the existence and non-existence of stable particle-like solutions of several wide classes of non-linear equations. It is shown that for any polynomial non-linearity there exist finite energy solutions with arbitrarily large sizes. On the other hand suitable fractional non-linearities taken alone or added to polynomials satisfy the necessary conditions for having only confined, non-dissipative solutions for any permissible finite value of energy.

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## 1. Introduction

There is a rapidly growing interest among physicists in various types of non-linear effects. Of particular interest is the existence of solitons, i.e. of peculiar, confined, non-dissipative and shape preserving solutions of certain non-linear field equations (NLFE). The literature concerning the properties of soliton solutions of several two-dimensional NLFE (1 space and 1 time dimension) is already quite large. However, even here our knowledge is quite far from being systematic and comprehensive.

In the last decade particle physicists got interested in soliton-like solutions of classical, relativistic NLFE in the four-dimensional Minkowski space, hoping that they may provide some physically interesting models for extended particles and explain the still rather mysterious phenomenon of confinement. Unfortunately, our knowledge about NLFE in four dimensions is still very scarce [1–4]. Only very few explicit solutions of four-dimensional NLFE are known. Moreover, their stability is sometimes disproved, sometimes it is still not quite sure. There are also several non-existence theorems and a few existence theorems proved for some special types of NLFE. Furthermore, we know several rather weak necessary conditions for the existence of soliton-like solutions [5–11]. General lack of a more systematic and comprehensive approach, which could provide at least

* Address: Instytut Fizyki Teoretycznej UW, Hoża 69, 00-681 Warszawa, Poland.

more general and more restrictive necessary conditions, results in much confusion and misleading or simply false claims.

Among the most widely used methods of investigation of NLFE one should first mention the variational calculus and its simplified versions which are based on the use of some simple distortions of the solutions (if they are known) or on the use of suitable trial functions [5–10]. These methods can provide only necessary conditions for a local extremum of the action functional or a local minimum of the energy functional (EF). For physical reasons, however, we are interested in energetically stable solutions which correspond to a finite absolute minimum of the EF for a fixed value of charge $Q$. The mentioned variational methods are not sufficient to prove energetic stability of a solution.

Some negative conclusions obtained with the help of the variational methods are, nevertheless, of a global character. For example one can exclude, already with the help of so called pseudovirial theorems derived from the variational calculus, the existence of any finite minima of the respective EF. On the other hand positive results concerning the existence of a local minimum are often misleading if they are not supplemented by somewhat more conclusive knowledge about the global behaviour of the EF.

Recently several authors applied the method of trial functions to a global discussion of the behaviour of the EF at arbitrarily large values of suitable distortion parameters [12–14]. Although the results obtained with the help of such global analysis are much more stringent, systematic and illuminating, the method itself has still some drawbacks. First, one always works there with trial functions which are not solutions of the respective NLFE. Second, one is using there only stationary trial functions. Therefore, one may raise several doubts whether the global properties of the EF obtained for such trial functions are also valid for the whole spectrum of true solutions including non-stationary ones. Contrary to the current belief, in the case of certain types of non-linearities, non-stationary solutions may have energies lying below the energies of the stationary solutions with the same value of charge. Some examples of such situations will be shown below.

In this paper another method of a global analysis of the NLFE, announced in a previous communication by the author [15], will be presented in some more detail. It will be applied not only to non-linear Klein-Gordon but also to Dirac equations. The new approach is free from the drawbacks mentioned above. The main idea consists in the use of variable initial values of the field instead of the time-dependent trial functions. Suitable initial values specify completely the solutions of the respective NLFE and, simultaneously, fix the numerical values of the constants of motion: energy $E$, momentum $\boldsymbol{P}$, angular momentum $\boldsymbol{J}$, charge $Q$ etc. In this way one can study the general properties of the energy spectrum of all solutions even without knowing their explicit time and space dependence.

## 2. *Non-linear Klein-Gordon equations* (*NLKGE*)

We shall now describe the general ideas and results of the method for the NLKGE for a complex, scalar or pseudoscalar field in the four-dimensional Minkowski space. The Lagrangian density is taken in the form

$$\mathcal{L} = \partial_\nu \psi^* \partial^\nu \psi - U(\kappa), \tag{1}$$

where $U$ is some real, non-linear function of the real field invariant $\kappa = \psi^*\psi$. The corresponding NLKGE is

$$(\Box + U'(\kappa))\psi = 0 \tag{2}$$

where

$$U'(\kappa) = \frac{dU(\kappa)}{d\kappa}.$$

For a given $U(\kappa)$ a solution of (2) is determined by the following initial values of the field at some fixed time $t_0$:

$$\varphi(x) = \psi(x, t_0), \quad \phi(x) = \left.\frac{\partial\psi(x, t)}{\partial t}\right|_{t=t_0}. \tag{3}$$

In general the functions $\varphi(x)$ and $\phi(x)$ may have different, unrelated forms. However, in the case of stationary solutions they are related by the formula

$$\phi(x) = -i\omega\varphi(x) \tag{4}$$

with real $\omega$. However, it must be noted that (4) is necessary but not sufficient to guarantee the stationary character of the respective solution. A stationary solution $\psi$ must satisfy the much stronger condition

$$\frac{\partial\psi(x, t)}{\partial t} = -i\omega\psi(x, t) \tag{5}$$

valid for all values of $t$ and not only for one value $t = t_0$.

The energy and charge functionals can be now expressed in terms of the initial values (3)

$$E = \int d^3x[|\phi|^2 + |\nabla\varphi|^2 + U(|\varphi|^2)], \tag{6}$$

$$Q = \frac{i}{2}\int d^3x[\varphi^*\phi - \phi^*\varphi]. \tag{7}$$

It follows from (7) that

$$|Q| \leqslant \int d^3x|\varphi^*\phi|, \tag{8}$$

and hence from the Schwartz inequality one obtains

$$Q^2 \leqslant \int d^3x|\varphi|^2 \cdot \int d^3x|\phi|^2. \tag{9}$$

Equality sign in (9) holds if and only if $\varphi$ and $\phi$ are proportional to each other i.e. if (4) is valid. In particular this is also true for stationary solutions which satisfy condition (5).

It can easily be seen that in order to provide finite values of $E$ and $Q$, as well as of other constants of motion, the functions $\varphi(x)$ and $\phi(x)$ cannot be completely arbitrary but must satisfy several rather obvious integrability conditions. Thus $\varphi$ and $\phi$ as well as $\nabla\varphi$ must be square integrable etc. Further restrictions on $\varphi$ will be imposed by the condition of finite value of the last term in (6) etc. We shall always assume that all such conditions are satisfied by the initial values.

Consider now a closed system described by (2) which has a fixed finite value of $Q \neq 0$. How does the energy $E$ behave if one changes the initial values keeping, however,

$Q$ = const? In principle one can calculate the integrals involved in (6) and (7) for arbitrary permissible shapes of $\varphi$ and $\phi$. However, this would be a very tedious and inefficient way. Instead, we shall start from some arbitrary but fixed shapes of $\varphi$ and $\phi$ which fulfill all the necessary integrability conditions, and then we shall introduce the following class of relatively simple but physically interesting finite distortions:

$$\varphi(x) \to \tilde{\varphi}(x) = c\varphi(x/a),$$
$$\phi(x) \to \tilde{\phi}(x) = bc\phi(x/a), \tag{10}$$

where $a$, $b$ and $c$ are positive parameters. It can easily be seen that $a$ determines the size and $c$ the supremum of the absolute magnitude of the initial field, both expressed in some units fixed by the original shape function $\varphi(x)$. Similarly, $b$ determines the rate of change in time of the solution. Obviously, if $\varphi(x)$ and $\phi(x)$ are permissible initial values, then the same applies to $\tilde{\varphi}(x)$ and $\tilde{\phi}(x)$. In order to have the same value of $Q$ the parameters $a$, $b$, $c$ cannot be independent. In fact inserting $\tilde{\varphi}$ and $\tilde{\phi}$ into (7) one obtains the relation

$$a^3bc^2 = 1. \tag{11}$$

Thus, one can eliminate $c$ and for any given $\varphi(x)$ and $\phi(x)$ the energy functional becomes a definite function of $a$ nad $b$ alone. After a simple change of variables one gets

$$E(a, b) = Ab + Ba^{-2}b^{-1} + a^3 \int d^3x U(a^{-3}b^{-1}|\varphi(x)|^2), \tag{12}$$

where

$$A = \int d^3x|\phi|^2 > 0, \quad B = \int d^3x|\nabla\varphi|^2 > 0. \tag{13}$$

The dependence on $a$ and $b$ of the last term cannot be further analysed without knowing something about the shape of $U(\kappa)$. Therefore, let us discuss separately some distinctive classes of non-linear functions $U(\kappa)$.

I. Polynomial non-linearity:

$$U(\kappa) = \kappa \sum_{n=0}^{m} \alpha_n \kappa^n \tag{14}$$

where $\alpha_n$ are real constants and $\alpha_m \neq 0$ for some $m \geqslant 1$. In the polynomial case as well as in other cases we assume $U(0) = 0$ in order to have $E = 0$ for the vacuum solution $\psi = 0$. Of course, one can easily remove this restriction.

For the polynomial non-linearity (14) one gets

$$E(a, b) = Ab + Ba^{-2}b^{-1} + b^{-1} \sum_n \alpha_n C_n a^{-3n} b^{-n} \tag{15}$$

where

$$C_n = \int d^3x|\varphi|^{2n+2} > 0. \tag{16}$$

General discussion of the form of (15) is greatly facilitated by the knowledge of its limits when $a$ and $b$ are tending to their extremal values zero and $+\infty$. Taking account of (13) and (16) one finds for $E(a, b)$ given by (15):

$$\lim_{a\to 0} E(a, b) = +\infty \cdot \operatorname{sign} \alpha_m.$$

$$\lim_{a\to\infty} E(a, b) = Ab+\alpha_0 C_0 b^{-1},$$

$$\lim_{b\to 0} E(a, b) = +\infty \cdot \operatorname{sign} \alpha_m,$$

$$\lim_{b\to\infty} E(a, b) = +\infty. \tag{17}$$

Thus, a finite and positive lower energy bound can exist only if

$$\alpha_0 > 0, \quad \alpha_m > 0. \tag{18}$$

For $a \to \infty$ the limit is still depending on $b$, but for $\alpha_0 > 0$ its infimum is finite

$$\inf\,[\lim_{a\to\infty} E(a, b)] = 2\sqrt{\alpha_0 C_0 A} \geqslant 2m|Q| \tag{19}$$

The last inequality follows from (9), (13), (16) after putting the coefficient at the linear term $\alpha_0 = m^2$. The equality sign is valid if and only if relation (4) holds.

One can now draw several general conclusions concerning polynomial non-linearities. The necessary conditions for the existence of a finite lower energy bound given by (18) are still insufficient for the existence of a minimum at finite values of $a$ and $b$. In order to have a minimum of $E(a, b)$, the coefficients $\alpha_n$ cannot be all non-negative. Because of (18) the NLKGE with the simplest non-linearity of the form

$$U = \alpha_0 \psi^* \psi + \alpha_1 (\psi^* \psi)^2 \tag{20}$$

cannot produce any minimum and hence cannot have stable, particle-like solutions at all. The lowest degree of the polynomial in $\kappa$ which can form an absolute minimum for a suitable set of coefficients $\alpha_n$ is three. For example, if

$$\alpha_0 > 0, \quad \alpha_1 < 0, \quad \alpha_2 > 0, \quad \alpha_n = 0 \quad \text{for} \quad n > 2 \tag{21}$$

and $|\alpha_1|$ is large enough, a minimum can appear.

Since (19) holds for any polynomial, one can see immediately that even if for some polynomial non-linearities there exist stable, confined solutions, there exist for the same value of $Q$ also unconfined (dissipative?) solutions of arbitrarily large sizes and finite energies. Such states of the field can be produced e.g. at collisions between two confined solitons. However, for physical reasons we are interested in finding such NLKGE which have only confined, non-dissipative solutions. If follows from the above global analysis that no polynomial non-linearity can fulfill this requirement.

II. Logarithmic non-linearity:

$$U(\kappa) = \alpha_0 \kappa - \alpha\kappa \ln \kappa. \tag{22}$$

This non-linearity is of particular interest because it is the only case for which explicit, stationary, confined solutions are known for arbitrary $\alpha_0 > 0$, $\alpha > 0$ and for the whole spectrum of frequencies $-\infty < \omega < +\infty$. The stationary solutions of the Gaussian form

$$\psi(x, t) = N \exp\{-\omega^2/2\alpha - \alpha/2r^2 - i\omega t\} \tag{23}$$

and their Lorentz transforms describing freely moving solitons have been found quite long ago by Rosen [16]. The factor $N$ in (23) is a constant independent of $\omega$ but depending on the particular values of $\alpha_0$ and $\alpha$. Similar Gaussian solutions of the non-linear Schrödinger equation of quantum mechanics with the same non-linearity (22) have been extensively studied by Birula-Białynicki, Mycielski et al. [17].

It is to be noted that all the stationary solutions (23) have the same size which is independent of $\omega$. However, the supremum of $|\psi|$ as well as the values of rest energy and charge depend on $\omega$. Putting for the sake of simplicity $\alpha_0 = 1$ and $\alpha = 1$ we find

$$E(\omega) = N_E(2\omega^2+1)\exp(-\omega^2),$$

$$Q(\omega) = N_Q\omega\exp(-\omega^2), \tag{24}$$

where $N_E$ and $N_Q$ are some positive constants. It is interesting to note that the values of rest energy $E(\omega)$ and charge $Q(\omega)$ cannot exceed certain limits. Thus

$$0 < E(\omega) \leqslant 2N_E\exp(-\tfrac{1}{2})$$

$$-N_Q 2^{-1/2}\exp(-\tfrac{1}{2}) \leqslant Q(\omega) \leqslant N_Q 2^{-1/2}\exp(-\tfrac{1}{2}). \tag{25}$$

To each permissible value of $Q \neq 0$ lying within these mimits there are two frequencies $\omega_1, \omega_2$ such that

$$Q(\omega_1) = Q(\omega_2) = Q \tag{26}$$

Hence, for each permitted value of $Q$ there are two different stationary solutions with different energies $E(\omega_1)$ and $E(\omega_2)$. Only the solution corresponding to the lowest value of energy satisfies the necessary condition of energetic stability and provides also the lower bound for the energy of a single Gausson with charge $Q$.

What can be said about the energy spectrum of other solutions of the NLKGE with logarithmic non-linearity (22)? In order to answer this question let us apply our global analysis to the present case. One finds

$$E(a, b) = Ab + Ba^{-2}b^{-1} + C_0[\alpha_0 + \alpha(3\ln a + \ln b)]b^{-1}$$

$$-\alpha b^{-1}\int |\varphi|^2 \ln|\varphi|^2 d^3x. \tag{27}$$

Hence

$$\lim_{a\to 0} E(a, b) = +\infty, \quad \lim_{a\to\infty} E(a, b) = +\infty\cdot\operatorname{sign}\alpha,$$

$$\lim_{b\to 0} E(a, b) = -\infty\cdot\operatorname{sign}\alpha, \quad \lim_{b\to\infty} E(a, b) = +\infty. \tag{28}$$

For any choice of $\alpha$ one of the limits is equal to $-\infty$. Therefore, the energy functional for the logarithmic non-linearity has no finite energy bound for any value of $Q$. Thus we see that the solutions of Rosen do not satisfy the condition of energetic stability. This instability is caused by the existence of some non-stationary solutions corresponding to arbitrary low negative values of $E$. For $\alpha > 0$, which is the necessary condition for the existence of confined Gaussian solutions, the troublesome solutions are characterized by very slow rate of change in time of the initial field, or simply speaking by small $|\tilde{\phi}|$ and large $|\tilde{\varphi}|$ An example of a limiting case, namely an explicit solution for $Q = 0$, displaying some unphysical behaviour has been given by Schick [18].

III. Fractional non-linearities

Some authors claimed that NLKGE with

$$U(\kappa) = \kappa(\alpha_0 - \alpha\kappa^u), \tag{29}$$

where $\alpha_0 > 0$, $\alpha > 0$ and the exponent

$$0 < u < 2/3, \tag{30}$$

have stable, confined solutions [24]. Let us calculate the corresponding expression for $E(a, b)$:

$$E(a, b) = Ab + Bb^{-1}a^{-2} + \alpha_0 C_0 b^{-1} - \alpha C_u a^{-3u} b^{-u-1}. \tag{31}$$

For the respective limits of (31) one gets

$$\lim_{a\to 0} E(a, b) = +\infty, \quad \lim_{a\to\infty} E(a, b) = Ab + \alpha_0 C_0 b^{-1}$$
$$\lim_{b\to 0} E(a, b) = -\infty, \quad \lim_{b\to\infty} E(a, b) = +\infty. \tag{32}$$

It follows that the energy is not bounded from below and, therefore, the solutions of the respective NLKGE do not satisfy the requirement of energetic stability.

Consider now another type of fractional non-linearity given by

$$U(\kappa) = \alpha\kappa^{1-u} \tag{33}$$

where $\alpha > 0$ and the fractional exponent $u$ satisfies the condition

$$0 < u < 1/2. \tag{34}$$

For this case one finds

$$E(a, b) = Ab + Ba^{-2}b^{-1} + \alpha C_{-u} a^{3u} b^{u-1}. \tag{35}$$

where

$$C_{-u} = \int d^3x |\varphi|^{2-2u} > 0 \tag{36}$$

For the respective limits one gets:

$$\lim_{a\to 0} E(a, b) = +\infty, \quad \lim_{a\to\infty} E(a, b) = +\infty,$$
$$\lim_{b\to 0} E(a, b) = +\infty, \quad \lim_{b\to\infty} E(a, b) = +\infty. \tag{37}$$

Thus we see that for $\alpha > 0$ all four limits are equal to $+\infty$ and, therefore, the function $E(a, b)$ must have a positive absolute minimum at some finite values of $a$ and $b$. Since (37) holds for any permissible choice of the shapes of $\varphi(x)$ and $\phi(x)$, we infer that the respective NLKGE should have only confined, non-dissipative solutions. Both squeezing the size to a point and expanding it to infinity require infinite amount of energy.

IV. Polynomial + fractional non-linearity

The last — highly encouraging result — can be generalized to more complex non--linearities of the form

$$U(\kappa) = \kappa(\sum_{n=0}^{m} \alpha_m \kappa^n + \sum_{i=1}^{k} \beta_i \kappa^{-u_i}), \tag{38}$$

where

$$0 < u_1 < \ldots < u_k < 1/2 \tag{39}$$

and $\alpha_n$, $\beta_i$ are some real constants. It can easily be checked that if $\alpha_m > 0$ and $\beta_k > 0$, then again all the four limits are equal to $+\infty$. Therefore, we can draw exactly the same conclusions about the confined and non-dissipative character of all solutions of the respective NLKGE corresponding to finite values of $E$ and $Q$ like in the previous case.

Explicit solutions of NLKGE belonging to this type with $k = 2$, $m = 0$ have been found by the author [22]. They have very interesting bag-like (or droplet-like) forms with sharp boundaries. It can easily be seen that, although $U(\kappa)$ given by (33) and (38) is finite at small $\kappa$ and $U(0) = 0$, the first derivative $U'(\kappa)$ which appears in the equations of motion becomes infinite for $\kappa \to 0$:

$$\lim_{\kappa \to 0} U'(\kappa) = +\infty. \tag{40}$$

Since $U'$ plays the role of the squared effective mass, it is evident that in the case of functions satisfying (40) the effective mass tends to infinity when the absolute magnitude of the field decreases to zero. $U'(\kappa)$ can also be interpreted as the potential of forces. It generates very strong (confining) forces which become infinite on the surface separating the field from the surrounding vacuum. Some mechanism of this kind has been postulated on intuitive grounds by physicists working on the bag model [25]. Here it is explained as the effect of addition of a fractional non-linearity which satisfies condition (40).

Morris [20] has then given another proof, based on Coleman's definition of dissipative solutions [1], that the NLKGE with a fractional non-linearity belonging to this class cannot have dissipative solutions at all (strictly speaking: uniformly dissipative solutions). The strong confining properties of fractional non-linearities have been confirmed by the numerical studies of collisions between two respective solitons in two-dimensional Minkowski space [21].

### 3. *Non-linear Dirac equations* (*NLDE*)

Consider Lagrangian densities of the form

$$\mathscr{L} = i/2(\bar{\psi}\gamma_\nu\partial^\nu\psi - (\partial^\nu\bar{\psi})\gamma_\nu\psi) - W(\kappa)\bar{\psi}\psi, \tag{41}$$

where $\psi$ denotes now the four-component Dirac spinor field. The non-linear term is assumed to depend on the simplest scalar field invariant $\bar{\psi}\psi$ which may, however, have either sign. In order to have invariance under charge conjugation, the function $W$ must be even in $\bar{\psi}\psi$, i.e. it may depend on the absolute magnitude

$$\kappa = |\bar{\psi}\psi|. \tag{42}$$

The respective NLDE have the following form

$$(-i\gamma_\nu\partial^\nu + V(\kappa))\psi = 0 \tag{43}$$

with

$$V(\kappa) = W(\kappa) + \kappa W'(\kappa). \tag{44}$$

Since Dirac equations are of the first order in the time derivative, the solution of (43) will be specified by the initial value of the field:

$$\varphi(\boldsymbol{x}) = \psi(\boldsymbol{x}, t_0). \tag{45}$$

The initial value $\varphi$ fixes also the numerical values of all the constants of motion. For the energy and charge functionals one has:

$$E = i/2 \int d^3x[(\bar{\varphi}\gamma \cdot \nabla\varphi - (\nabla\bar{\varphi}) \cdot \gamma\varphi) + W(\kappa)\bar{\varphi}\varphi], \tag{46}$$

$$Q = \int d^3x \varphi^\dagger \varphi. \tag{47}$$

For any stationary solution $\psi$ of (43):

$$(-i\gamma \cdot \nabla + V(\kappa))\psi = \omega\gamma_0\psi. \tag{48}$$

The initial field $\varphi(\boldsymbol{x})$ generating a stationary solution must, therefore, satisfy the condition

$$(-i\gamma \cdot \nabla + V(\kappa))\varphi = \omega\gamma_0\varphi. \tag{49}$$

Obviously, (49) is a weaker condition than (48), so it is only the necessary but not sufficient condition for the stationary character of the respective solution.

The initial field $\varphi(\boldsymbol{x})$ cannot be arbitrary but must satisfy several rather obvious integrability conditions in order to give finite values of $E$, $Q$ and other constants of motion. Furthermore, it is well known that in semiclassical theory of Dirac fields the energy can have both signs. Thus, if the initial field $\varphi(\boldsymbol{x})$ generates a positive energy solution, then the charge conjugate field $\varphi^c(\boldsymbol{x})$ generates the corresponding negative energy solution. For the free linear Dirac field the positive part of the energy spectrum is separated from the negative part by a finite gap. Moreover, one assumes that all the negative energy states are occupied and form the physical vacuum. We shall assume that the same should apply to physically permissible NLDE. In other words the positive part of the energy spectrum must have a finite and positive lower bound. This is obviously a condition imposed on the functions $W(\kappa)$. Functions $W(\kappa)$, for which the positive and negative energy spectra are not separated, should be rejected.

We shall also make the tentative assumption that for the positive energy solutions of physically acceptable NLDE the kinetic energy term is also positive (like for the linear case):

$$B = i/2 \int d^3x(\bar{\varphi}\gamma \cdot \nabla\varphi - (\nabla\varphi) \cdot \gamma\varphi) > 0. \tag{50}$$

Suppose now that we have some initial spinor field $\varphi(\boldsymbol{x})$ which fulfils all the integrability conditions and gives a finite value of $Q$ and positive and finite values of $E$ and $B$. We shall again study the energy spectrum of all solutions of (43) applying the method of variable initial conditions. Let us consider the following simple class of charge preserving distortions of $\varphi(\boldsymbol{x})$:

$$\varphi(\boldsymbol{x}) \rightarrow \tilde{\varphi}(\boldsymbol{x}) = a^{-3/2}\varphi(\boldsymbol{x}/a). \tag{51}$$

After substitution (51) the energy (46) becomes (for any fixed shape of $\varphi(\boldsymbol{x})$) a function of the size parameter $a$:

$$E(a) = Ba^{-1} + \int d^3x W(a^{-3}\kappa)\bar{\varphi}\varphi. \tag{52}$$

Further discussion of the dependence of (52) on $a$ is not possible without some specification of the form of $W(\kappa)$.

I. Polynomial non-linearity:

$$W(\kappa) = \sum_{n=0}^{m} \alpha_n \kappa^n, \quad \alpha_m \neq 0, \quad m \geqslant 1. \tag{53}$$

The function $E(a)$ has now the form

$$E(a) = Ba^{-1} + \sum_{n=0}^{m} \alpha_n \Gamma_n a^{-3n}, \tag{54}$$

where

$$\Gamma_n = \int d^3x \kappa^n \bar{\varphi}\varphi. \tag{55}$$

One can see immediately that

$$\lim_{a\to 0} E(a) = +\infty \cdot \operatorname{sign} \alpha_m \Gamma_m, \quad \lim_{a\to\infty} E(a) = \alpha_0 \Gamma_0. \tag{56}$$

Therefore, the necessary conditions for the existence of a positive lower energy bound are:

$$\alpha_0 \Gamma_0 > 0, \quad \alpha_m \Gamma_m > 0. \tag{57}$$

If all the terms in (54) are positive and $B > 0$, no minimum can appear on the curve $E(a)$. Because of (57), NLDE with the simplest polynomial non-linearity $W = \alpha_0 + \alpha_1\kappa$ cannot have any stable, confined and non-dissipative solutions. For higher polynomials involving also negative terms $\alpha_n \Gamma_n < 0$ some minima at finite values of $a$ may appear. Changing then the shape functions $\varphi(x)$ we could find the lowest positive value of $E$ which would then correspond to solutions having the property of energetic stability.

Unfortunately, because for any polynomial the second limit (56) is always finite, it follows that for any value of $Q$ there exist unconfined solutions of finite energy but arbitrarily large sizes. Thus no NLDE with purely polynomial nonlinearity can have only confined, non-dissipative solutions of finite sizes.

II. Logarithmic non-linearity

$$W(\kappa) = \alpha_0 - \alpha \ln \kappa. \tag{58}$$

The function $E(a)$ has now the form:

$$E(a) = Ba^{-1} + \Gamma_0(\alpha_0 + 3\alpha \ln a) - \alpha \int d^3x \bar{\varphi}\varphi \ln \kappa. \tag{59}$$

Hence

$$\lim_{a\to 0} E(a) = +\infty \cdot \operatorname{sign} B, \qquad \lim_{a\to\infty} E(a) = +\infty \cdot \operatorname{sign} \alpha\Gamma_0. \tag{60}$$

Thus the necessary conditions for the existence of a finite, positive energy bound are

$$B > 0, \qquad \alpha\Gamma_0 > 0 \tag{61}$$

for any $\varphi(\boldsymbol{x})$ which generates positive energy solutions. Since both the limits (60) are then equal to $+\infty$, these are also necessary conditions for the respective NLDE to have only confined and non-dissipative solutions.

III. Fractional non-linearity

$$W(\kappa) = \alpha\kappa^{-u} \tag{62}$$

with

$$0 < u < 1/2.$$

In this case one finds

$$E(a) = Ba^{-1} + \alpha\Gamma_{-u}\, a^{3u}, \tag{63}$$

where

$$\Gamma_{-u} = \int d^3x \bar{\varphi}\varphi\kappa^{-u}. \tag{63'}$$

It follows that

$$\lim_{a\to 0} E(a) = +\infty \cdot \operatorname{sign} B, \qquad \lim_{a\to\infty} E(a) = +\infty \cdot \operatorname{sign} \alpha\Gamma_{-u}. \tag{64}$$

The necessary conditions for the existence of a positive lower energy bound are

$$B > 0, \qquad \alpha\Gamma_{-u} > 0 \tag{65}$$

which must be valid for any initial shape functions $\varphi(\boldsymbol{x})$ which generate positive energy solutions. Again, since both limits are then equal to $+\infty$, these are also necessary conditions for the respective NLDE to have only confined, non-dissipative solutions.

IV. Polynomial + fractional non-linearities

$$W(\kappa) = \sum_{n=0}^{m} \alpha_n \kappa^n + \sum_{i=1}^{k} \beta_i \kappa^{-u_i}, \tag{66}$$

where

$$\alpha_m \neq 0, \quad \beta_k \neq 0, \quad m > 0; \quad 0 < u_1 < \ldots < u_k < 1/2. \tag{66'}$$

From the respective form of $E(a)$ one gets

$$\lim_{a \to 0} E(a) = +\infty \cdot \operatorname{sign} \alpha_m \Gamma_m; \qquad \lim_{a \to \infty} E(a) = +\infty \cdot \operatorname{sign} \beta_k \Gamma_{-u_k}. \tag{67}$$

Thus the necessary conditions for the existence of a positive lower energy bound are

$$\alpha_m \Gamma_m > 0, \qquad \beta_k \Gamma_k > 0. \tag{68}$$

These are also the necessary conditions for the respective NLDE to have only confined, non-dissipative solutions. Like in the analogous case of NLKGE we see that addition of a suitable fractional non-linearity to a polynomial can drastically change the properties of solutions.

There is one delicate point in the above analysis of NLDE. In fact all the necessary conditions for the existence of a lower positive energy bound can be satisfied with fixed coefficients $\alpha_n$ and $\beta_i$ if the integrals $\Gamma_n$ and $\Gamma_{-u_i}$ defined by (55) and (63') have the same sign for all permissible values of $\varphi(x)$ which generate positive energy solutions. It is by no means clear that this condition of sign stability is always satisfied. Perhaps it should be regarded as an additional restriction imposed on the type of non-linearity.

Some explicit solutions of NLDE with a fractional nonlinearity of the form $W = \alpha_0 + \alpha\kappa^{-u}$ have been found by the author [19]. Like in the similar case of NLKGE these solutions have bag-like (or droplet-like) forms with sharp boundaries. The rest frame solutions, which are stationary, have spherically symmetric charge distributions. For these explicit solutions the invariant $\bar{\varphi}\varphi$ is everywhere non-negative and vanishes in the same region of space-time where the charge density $\varphi^+\varphi$ is zero. For these explicit solutions the integrals $\Gamma_0$ and $\Gamma_{-u}$ as well as $B$ are positive, so (65) implies that the constant $\alpha$ must be positive. Unfortunately, these conclusions apply only to particular solutions of particular NLDE and do not allow for any immediate generalizations.

However, there is a way to ensure that all the integrals $\Gamma_n$, and $\Gamma_{-u}$ are positive. In fact, one can apply the strategem used by Johnson in his field theoretical formulation of the bag model [23]. Thus instead of the Lagrangian density $\mathscr{L}$ given by (41) one can use a modified Lagrangian $\hat{\mathscr{L}}$ obtained by multiplication of $\mathscr{L}$ by the step function $\theta(\bar{\psi}\psi)$:

$$\hat{\mathscr{L}} = \theta(\bar{\psi}\psi)\mathscr{L}(\psi). \tag{69}$$

The Euler-Lagrange equations of motion which follow from (69) are:

$$\theta(\bar{\psi}\psi)\left[-i\gamma_\nu\partial^\nu\psi + V(\kappa)\psi\right] + \delta(\bar{\psi}\psi)\left[\mathscr{L}(\psi)\psi + i/2\partial_\nu(\bar{\psi}\psi)\gamma^\nu\psi\right] = 0. \tag{70}$$

This equation will be satisfied if and only if the factors in brackets, which multiply $\theta(\bar{\psi}\psi)$ and $\delta(\bar{\psi}\psi)$, vanish separately. The first equation obtained in this way coincides with our previous equation of motion (43) supplemented by the subsidiary condition $\bar{\psi}\psi > 0$. The second equation has the form of a boundary condition

$$\lim_{\bar{\psi}\psi\to 0^+} [\mathscr{L}(\psi)+i/2\partial_\nu(\bar{\psi}\psi)\gamma^\nu]\psi = 0. \tag{71}$$

Of course, one should take in (71) the value of $\mathscr{L}(\psi)$ which is acquired for the solution $\psi$ of the respective equation of motion (43). Taking (43) into account one gets

$$\mathscr{L} = \kappa W'(\kappa)\bar{\psi}\psi. \tag{72}$$

For all the types of non-linearities considered in this paper

$$\lim_{\bar{\psi}\psi\to 0^+} [\kappa W'(\kappa)\bar{\psi}\psi] = 0. \tag{73}$$

Hence the boundary condition (71) simplifies then to the form:

$$\lim_{\bar{\psi}\psi\to 0^+} [\partial_\nu(\bar{\psi}\psi)\gamma^\nu\psi] = 0 \tag{74}$$

which is independent of the particular choice of $W(\kappa)$. The time derivative appearing in (74) can be eliminated with the help of the equations of motion. Thus we arrive at the following boundary condition for the initial values $\varphi(\boldsymbol{x})$:

$$\lim_{\bar{\varphi}\varphi\to 0} \{[(\nabla\varphi^\dagger)\cdot\gamma\varphi-\varphi^\dagger\gamma\cdot\nabla\varphi]\gamma_0\varphi+\nabla(\varphi^\dagger\varphi)\cdot\gamma\varphi\} = 0 \tag{75}$$

which must be satisfied by any initial value.

Using the modified Lagrangian density $\tilde{\mathscr{L}}$ we arrive at the following new form of the energy functional

$$E = \int d^3x\theta(\bar{\varphi}\varphi)\,[i/2(\bar{\varphi}\gamma\cdot\nabla\varphi-(\nabla\bar{\varphi})\cdot\gamma\varphi)+W(\kappa)\bar{\varphi}\varphi]. \tag{76}$$

The initial field must satisfy the condition $\bar{\varphi}\varphi > 0$ and the boundary condition (75) on the surface $\bar{\varphi}\varphi = 0$. It can easily be seen that if these conditions are satisfied by $\varphi(\boldsymbol{x})$, they are also satisfied by the changed initial values $\tilde{\varphi}(\boldsymbol{x})$ defined by (51). The analysis of the modified energy functional (76) runs completely parallel to that performed before, but all the integrals $\Gamma_n$, $\Gamma_{-u}$ are now positive definite. In this way the mentioned difficulty with signs of these integrals is removed.

It can easily be seen that the method of variable initial conditions can be used also for the investigation of NLKGE and NLDE with more complicated non-linearities involving transcendental functions. E.g. in the case of superpositions of exponential functions multiplied by polynomials one arrives at results similar to those obtained above for pure polynomials. Applications of the method of variable initial conditions to other fields different from complex scalars and Dirac spinors, as well as to some systems of various interacting fields will be presented in another paper.

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