BLACK HOLES AND TRAPPED POINTS

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Black holes are defined and their properties investigated without use of any global causality restriction. Also the boundary at infinity of space-time is not needed. When the causal conditions are brought in, the equivalence with the usual approach is established.

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Introduction

Black holes are usually defined as the complements of the past of some suitably defined boundary at infinity of space-time. However, to define the boundary at infinity causal and topological conditions are needed to hold over space-time (see [1], [2] and [12]). Such conditions may restrict severely the applicability of the definition. Thus in the classic approach due to Hawking only isolated gravitating systems are accomodated and hence his method is not useful in cosmological situations.

In this paper we shall try to define black holes and establish their properties without use of the boundary at infinity. Our work is motivated by a theorem proved by the author ([2] theorem 3.1) which characterizes black holes uniquely by trapped points (to be defined below). We do not impose any global causal conditions on space-time.

In Section 1 we define the black hole as a connected component of the set B of all strongly trapped points. We are able to prove the basic properties associated with black holes, namely it is shown that B is a future set, the area theorem holds and for any point not in B there exists a causal curve that attains unbounded values of its generalised affine parameter when maximally extended from the point to the future. In Section 2 we see how the imposition of causal restrictions on space-time makes our approach equivalent to the usual one. Our approach depends essentially on the existence of strongly trapped points, also to prove our theorems we need some additional hypotheses. We investigate these points in Section 3. It is found that strongly trapped points are useful when the Penrose cosmic censor holds and when the singularities are of strong curvature type. Our notation is that of monograph [1].

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SECTION 1

By space-time $(\mathcal{M}, \bar{g})$ we mean a Hausdorff, connected, real C^∞ manifold with a C^2 Lorentz metric $\bar{g}$ everywhere defined.

Definition 1. A space-time point is said to be strongly trapped if for every future--directed causal geodesic λ entering $J^+(p)$ (the causal future of p) and in the causal future of every point q on λ the expansion Θ becomes negative somewhere on λ.

The above concept forms the basis of our theory of black holes. It is a stronger restriction on space-time than the Penrose closed trapped surface concept. Also it is a strengthening of the trapped point concept used in [2]. A similar notion was used by Hawking in one of his singularity theorems ([1] §8 theorem 3).

Definition 2. A black hole in space-time $(\mathcal{M}, \bar{g})$ is a connected component of the set B of all strongly trapped points in $\mathcal{M}$.

Proposition 1. Black holes in space-time $(\mathcal{M}, \bar{g})$ are future sets.

Proof: Let us suppose $p \in I^+(B_1)$ where B_1 is a black hole, then p is in the chronological future of some strongly trapped point and hence from definition it must also be strongly trapped, thus $p \in B_1$. Therefore $I^+(B_1) \subset B_1$ and by definition ([1] p. 186) B_1 is a future set.

The above property means that once an observer enters a black hole it will never be able to escape from it. This property is always attributed to black holes.

Now we shall investigate further the set B and its boundary $\dot{B}$. Since B is a future set its boundary is a closed, imbedded achronal three-dimensional C^{1-} submanifold ([1] prop. 6.3.1). Achronality means that no two points of $\dot{B}$ can be joined by a timelike curve. Points of $\dot{B}$ can be divided into four disjoint subsets $\dot{B}_N, \dot{B}_+, \dot{B}_-, \dot{B}_0$ according to the following criterion: for a point $q \in \dot{B}$ there may or may not exist points p, $r \in \dot{B}$ with $p \in E^-(q)-q$, $r \in E^+(q)-q$. $E^+(p)$ $(E^-(p))$ is that part of the future (past) null cone of p which is generated by null geodesics with endpoints at p. This is shown on the diagram below ([1] p. 187).

$q \in$	$\exists_p$	$\nexists_p$	
	$\dot{B}_N$	$\dot{B}_-$	$\exists_r$
	$\dot{B}_+$	$\dot{B}_0$	$\nexists_r$

We call the boundary $\dot{B}$ the event horizon and we denote it by H. In the following theorem we shall see how the above general structure of the achronal boundary is simplified when it is as event horizon.

Theorem 1. Suppose B is a black hole in space-time $(\mathcal{M}, \bar{g})$, then the subsets $\dot{B}_0$ and $\dot{B}_+$ of the event horizon H are empty.

Proof: It is instructive to give two proofs of the above theorem.

1. First we note that $\mathrm{int}D^-(\dot{B})$, the interior of the past domain of dependence of $\dot{B}$, must be empty otherwise there would be a point $p \in (\mathcal{M}-\bar{B})$ such that all the future-directed causal curves from p would intersect $\dot{B}$ and enter B by definition of the domain of dependence $D^-(\dot{B})$. Since every point of B is a strongly trapped point, all the causal curves entering $J^+(p)$ would intersect some strongly trapped point hence p would also be a strongly trapped

point. But this implies that p cannot be in $(\mathscr{M}-\bar{B})$ but in B. Now consider the past Cauchy horizon of $\dot{B}$, $H^-(\dot{B})$:

$$\begin{aligned} H^-(\dot{B}) &= \overline{D^-(\dot{B})}-I^-(D^-(\dot{B})) && \text{by definition} \\ &= \dot{B}-I^-(\dot{B}) && \text{since int } D^-(\dot{B}) = 0 \\ &= \dot{B} && \text{by definition} \end{aligned}$$

By [1] prop. 6.5.3 $H^-(\dot{B})$ is generated by null geodesic segments that either have no future endpoints or have endpoints at edge $H^-(\dot{B})$. Since edge $H^-(\dot{B})$ = edge $\dot{B} = 0$ by [1] prop. 6.3.1 the generators of $H^-(\dot{B})$ and therefore of $\dot{B}$ are future endless. In this case $\dot{B}_0 \cup \dot{B}_+$ is empty since $\dot{B}_0$ is spacelike and at $\dot{B}_+$ the generators of $\dot{B}$ leave $\dot{B}$ and thus cannot be future endless.

2. Let $\mathscr{W}$ be a convex normal neighbourhood of a point $q \in \dot{B}$. Consider any point p in $I^-(q)$. There must be a future-directed timelike curve from p to some point of $\mathscr{M}-B-\mathscr{W}$, otherwise all the future-directed non-spacelike curves from p would enter B and by the argument in the first proof p would be a strongly trapped point what would be a contradiction since all the strongly trapped points are in B. Thus $I^-(q) \subset I^-(\mathscr{M}-B-\mathscr{W})$ what implies by [1] lemma 6.3.2 (*ii*) that $q \in \dot{B}_N \cup \dot{B}_-$ i.e. $\dot{B}_0 \cup \dot{B}_+$ is empty.

The above theorem means that H is generated by future endless null geodesics.

Now we shall introduce a hypothesis concerning singularities of space-time. Singularities form a part of the boundary of space-time, so-called b-boundary, which is generated by endpoints of causal curves that are incomplete with respect to their generalized affine parameter (GAP, see [1] p. 259).

Definition 3 (Tipler [8]). A point p on the b-boundary is said to be a strong curvature singularity if for every causal geodesic $\lambda(t)$ which intersects p, all linearly independent vorticity-free Jacobi fields along $\lambda(t)$, which are normal to the tangent vector of $\lambda(t)$, define a volume (or area) element which vanishes as λ approaches p.

Hypothesis. All the b-boundary points are strong curvature singularities.

Strong curvature singularities were introduced by Ellis and Schmidt [7] and their properties were investigated by Tipler [8]. Bielinskii, Khalatnikov and Lifshitz were able to construct a general solution of Einstein's equations around such a singularity [9]. It is clear from the proofs of singularity theorems that space-time can be extended at most as far as the strong curvature singularities, it may happen however that an observer falls into some other singularity before strong curvature singularity could be reached. To prove our hypothesis one must perform an extension of space-time through all singularities that are not of strong curvature type. This, in the author's opinion, can be achieved. A remarkable work on extending space-times was done by Clarke [15].

Now we are in position to prove the main property associated with black holes.

Theorem 2. Let (a) the hypothesis hold, (b) $R_{ab}k^ak^b \geq 0$ for all null vectors $\bar{k}$, then the expansion $\hat{\Theta}$ of null generators of the event horizon H in space-time $(\mathscr{M}, \bar{g})$ is non-negative.

R_{ab} is the Ricci tensor, the condition (*b*) is called null convergence condition and by Einstein's equations it means essentially that the energy density is non-negative.

Proof of theorem 2: By theorem 1 we know that H is generated by future endless null geodesics. We shall first prove that all these generators are complete. If not, then some generator of H, say λ, runs into a singularity. Consider now a point p somewhere on λ, $p \in \dot{B}_N$. All the future-directed timelike geodesics from p enter B since H is a boundary of a future set. Now consider the future-directed null geodesics originating from p: since they all must intersect at p all of them must enter B except for λ. Thus all the future-directed causal geodesics that enter $J^+(p)$ intersect some strongly trapped point eficept for λ. However, by hypothesis λ runs into a strong curvature singularity thus the expansion $\hat{\Theta}$ on λ becomes negative to the future of every point on λ. Therefore it is clear that p must be a strongly trapped point. This is a contradiction since all the strongly trapped points belong to B by definition.

Suppose that $\hat{\Theta}$ is negative on some null generator of H at a point p. Since the generators of H are future endless and complete, by Raychaudhuri equation and the null convergence condition there is a focal point on H to the future of p. This is a contradiction since at a focal point null generators of H intersect and enter B what is impossible by Theorem 1.

Theorem 2 is a generalization of the famous Hawking's area theorem ([1] prop. 9.2.7). Thus we have established for our set B all the main properties associated with black holes.

To have a clear correspondence with the standard approach to black holes we would like any observer outside the set B to be able to avoid black holes and "escape to infinity". This is established by the following theorem:

Theorem 3. Let B be the set of all strongly trapped points in $(\mathcal{M}, \bar{g})$ and let the hypothesis holds, then for every point $p \in \bar{P} = \mathcal{M} - B$ there is a future-endless causal curve from p that attains unbounded values of its generalized affine parameter (GAP).

Proof: Suppose that there is no future-directed causal curve originating from p that attains unbounded values of its GAP, then all the future-directed causal geodesics entering $J^+(p)$ must be incomplete and thus they must all run into a strong curvature singularity. Hence it follows from definitions that p is a strongly trapped point but this is impossible since $p \in \mathcal{M} - B$.

We see that using straightforward arguments we were able to establish all the essential properties of black holes. However it is clear that our method depends heavily on the existence of the strongly trapped points and validity of our hypothesis; we shall investigate this matter in Section 3.

SECTION 2

Now we shall relate our approach to the black holes with the classic one proposed by Hawking [1]. In his method the event horizon H is defined as the boundary $\dot{I}^-(\mathcal{I}^+, \mathcal{M})$ of the past of some suitable boundary at infinity of space-time. The black holes are three--dimensional sets $\mathcal{B}(\tau)$ defined by $\mathcal{B}(\tau) = \mathcal{S}(\tau) - \bar{I}^-(\mathcal{I}^+, \mathcal{M})$, where $\mathcal{S}(\tau)$ is an appropriately constructed foliation of the space-time by spacelike non-intersecting surfaces parametrized by the parameter τ.

By Theorem 3 from every point $p \in \bar{P} = \mathcal{M} - B$ there is a future-directed causal curve from p of unbounded GAP. The past $I^-(\lambda)$ of this curve can be thought of as the past of a point not in space-time but belonging to the boundary at infinity $\mathscr{I}^+$ of $\mathscr{M}$. This way of defining the boundary of space-time, relying only on the causal structure of $(\mathscr{M}, \bar{g})$, was developed by Geroch, Kronheimer and Penrose [10]. However, their approach can be applied when space-time is past- and future-distinguishing i.e. $I^-(p) = I^-(q) \Leftrightarrow I^+(p) = I^+(q) \Leftrightarrow p = q$, in other words, when every point of $\mathscr{M}$ is uniquely determined by its past and future. This is a global causality condition and it implies that there are no closed causal curves. Pasts (futures) of future (past) endless curves are called TIPs (TIFs). Hence under the above mentioned causality condition every point of the set $\bar{P} = \mathscr{M} - B$ is in the closure of some TIP which is the past of a causal curve of unbounded GAP. Thus we can write $\bar{P} = \bar{I}^-(\mathscr{I}^+, \mathscr{M}) := \bigcup_{p \in P} \bar{I}^-(\lambda_p)$, where λ_p is a future-endless causal curve of unbounded GAP originating from p. To be able to define three-dimensional black holes we must impose the condition of stable causality which means that there are no closed causal curves and that a slight perturbation of light cones does not introduce such curves. Stable causality is equivalent to the foliation of $\mathscr{M}$ by non-intersecting spacelike surfaces. Thus we can define a black hole as a connected component of the set $\mathscr{B}(\tau) = \mathscr{S}(\tau) - \bar{I}^-(\mathscr{I}^+, \mathscr{M})$ where τ parametrizes a spacelike foliation $\mathscr{S}(\tau)$ of $\mathscr{M}$ (see [1] §9). Thus once global causality conditions are brought in, our approach can be related to the standard one.

SECTION 3

In this section we shall investigate the existence of strongly trapped points what is crucial for the applicability of our definition. We shall indicate the connection of our approach with the validity of the cosmic censor hypothesis.

First of all we prove the following theorem:

Theorem 4. Let F be a future set in spacetime $(\mathscr{M}, \bar{g})$.

Suppose that

(a) the hypothesis holds,

(b) there is no future-complete causal geodesics in F, then every point of F is a strongly trapped point.

Proof: Let $p \in F$, then every future-directed causal geodesic λ entering $J^+(p)$ is contained in F and hence λ is future-incomplete. Thus by our hypothesis the expansion Θ becomes negative on λ to the future of every point q on λ. Thus by definition p in a strongly trapped point.

The above theorem establishes existence of the strongly trapped points.

According to the Penrose cosmic censor hypothesis ([4], [16]), from the initial regular data no space-time singularity can arise that is visible from the infinity.

We adopt the following definition of the cosmic censor:

Definition 4 (see [2] def. 1.3). The strong cosmic censorship holds in space-time $(\mathscr{M}, \bar{g})$ if

(a) there are no future-incomplete endless curves in the complement of the set B of all black holes in space-time $\mathscr{M}$.

(b) all the causal geodesics entering B are future-incomplete.

Since $\mathscr{M}-B$ can be thought of as the past of some boundary at infinity of space-time $\mathscr{M}$, condition (a) excludes the possibility to see the singularity from that infinity. Condition (b) excludes the situation occuring in the Kerr and Reissner-Nordstrom space-times where there are observers that enter the black hole and which can obtain information about the singularity and are able to carry it to a component of the boundary at infinity of space-time.

It is clear from Theorem 4 that when the cosmic censor holds and the hypothesis is valid the strongly trapped points exist.

From the definition 4 it follows that when the cosmic censor holds, a causal geodesic is future-incomplete if and only if it enters B. Hence we have the following lemma:

Lemma 1. A causal geodesic is future-incomplete if and only if it intersects a strongly trapped point.

The above lemma can be adopted as the definition of the cosmic censor.

Another convenient formulation of the cosmic censor is that space-time is globally hyperbolic (see [4]). In this connection we note the following theorem which indicates once again the relation between the cosmic censor and the concept of strongly trapped points.

Theorem 5. Let (a) space-time be globally hyperbolic,

(b) $R_{ab}k^a k^b \geqslant 0$ for every non-spacelike vector $\bar{k}$,

then every causal geodesic intersecting a strongly trapped point is future-incomplete.

The proof of the above theorem is given in paper [2] Theorem 2.1. Finally we remark that some promising attempts to prove the cosmic censor have already been made [13].

Conclusions

We were able to define black holes and investigate their properties without use of any global causality condition. Also we were able to show that the additional hypotheses needed for our approach were not too restrictive. From our work it is clear that once the cosmic censor and the hypothesis are proved, black holes are uniquely characterized by trapped points. However in the author's opinion the investigation of these hypotheses will require different techniques than those of Penrose and Hawking exploited above, for example the global analysis methods developed by Fischer and Marsden [14] could be useful.

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