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BIFURCATION FROM THE MAXWELL SOLUTION IN THE YANG-MILLS THEORY

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The Maxwell solutions to the Yang-Mills SU(2) equations with sources are analyzed from the viewpoint of the bifurcation theory. The necessary and sufficient conditions for the cylindrically symmetric bifurcation are given. An approximate expression for the bifurcating solution is written and its stability is proved.

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1. *Introduction*

The Yang-Mills equations are nonlinear, it seems to be intriguing whether there exist phenomena typical for nonlinear theories. One of such phenomena, namely the loss of stability of solutions at certain values of parameters, was established earlier [1]. A study of the second appearance — local nonuniqueness of the solution — is the aim of this work. This will be done from the standpoint of the bifurcation theory. Precisely, a neighborhood of the Maxwell solutions (i.e. with the abelian holonomy group) will be investigated. It is assumed that potentials and sources belong to a suitable (Hölder, Sobolev) space and satisfy the boundary (say, Dirichlet) conditions.

Bifurcation phenomena are closely related to the so-called zero-mode solution [2, 3]; this connection has been pointed out by Jackiw and Rossi [4]. The existence of the zero-mode solution satisfying certain (linearized) boundary conditions is a hint for bifurcation; this is a necessary but not a sufficient condition.

The problem of the existence of bifurcation points is important e.g. for a semiclassical approximation, since a change in stability and bifurcation are often related [5–7]. If the solution is unstable near the bifurcation point, one must restrict the allowed quantum fluctuations excluding those in instability directions [2, 3]; alternatively one should use the branching solution as a saddle point.

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This paper is an extension of my previous work [8] and contains some results unknown and unpublished earlier. They are mentioned below.

The organization is as follows. Section 2 contains the Yang-Mills equations with sources in a Minkowski space, their linearized form and some remarks about the bifurcation theory.

In Section 3 bifurcation from a cylindrically symmetric Maxwell solution is discussed. The main result is that bifurcation occurs at values of the coupling constant (critical values) corresponding to an odd number of zero-mode solutions. The case of simple multiplicity (simple bifurcation) is presented in more detail afterwards [8], and approximate expressions for branching solutions are given; they are similar to those obtained by Jackiw and Rossi [4]. The simple bifurcation is over-critical. Explicit examples have shown that bifurcation is connected to symmetry breaking.

In Section 4 the criterium of linearized stability [6, 7] is applied to bifurcating solutions in the case of simple bifurcation. They are stable (at least under some assumptions about sources) above the critical values of the coupling constant, where the Maxwell solution is unstable.

2. *Preliminary remarks*

The Yang-Mills equations are

$$\partial_\mu F^a_{\mu\nu} + g\varepsilon_{abc} A^b_\mu F^c_{\mu\nu} = j^a_\nu, \tag{1}$$

where $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\varepsilon_{abc} A^b_\mu A^c_\nu$ — the field strength tensor, A^a_μ — the potential, j^a_ν — sources of class $C^{k+\mu}$ (Hölder space) or W^k (Sobolev space), k — a positive integer, $0 < \mu < 1$. Lower case space-time Greek indices range from 0 to 3 inclusively and lower case space Latin indices range from 1 to 3. The signature is $(-, +, +, +)$. The upper indices range from 1 to 3 and denote the isospin directions (the gauge group is SU(2)). ε_{abc} is the completely antisymmetric tensor.

Because of the nonlinearity of (1) exact analytical solutions for $j^a_\mu \neq 0$ are not known, except the trivial ones with the abelian holonomy group.

I will restrict myself to the simplest possible case, that is, when

$$j^a_\mu = \delta_{a1}\delta_{\mu 0}\varrho, \tag{2}$$

where ϱ — a static source and the boundary conditions admit a Maxwell static solution of the type $A^a_\mu = \delta_{a1}\delta_{\mu 0}\varphi$. It will be shown that even these restrictions allow for nontrivial (i.e. with a nonabelian holonomy group) solutions. Define $\underline{A}^a_0$ by

$$A^a_0 = \delta_{a1}\varphi + \underline{A}^a_0 \tag{3}$$

where φ is the admissible Maxwell static solution. Inserting (3) into the static version of Eqs (1) and using the Coulomb gauge

$$\partial_i A^a_i = 0 \tag{4}$$

and linearizing about $A_i^a = \underline{A}_0^a = 0$, one gets the following equations for linear perturbations δA_i^a, δA_0^a of $A_i^a = 0$, $\underline{A}_0^a = 0$

$$\Delta\delta A_0^a - 2g\varepsilon_{ak1}\delta A_i^k \partial_i \varphi = 0, \tag{5a}$$

$$\Delta\delta A_i^a - 2g\varepsilon_{ak1}\delta A_0^k \partial_i \varphi + g^2(1-\delta_{a1})\varphi^2 \delta A_i^a = 0. \tag{5b}$$

Note that Eqs (1) are elliptic in the static case, as well as Eqs (5a, b). It is well known that the imposed boundary conditions upon solutions to nonlinear elliptic equations do not insure a unique solution in general, although it is possible for particular classes of equations [9]. It is rather difficult to study the global structure of the solution space, but locally (in a suitable functional norm) this may be done quite easily.

In the first step the linearized equations (5a, b) are analyzed. Let us recall that the solution of the full nonlinear equations (i.e. the static version of (1)) $A_i^a = 0$, $\underline{A}_0^a = 0$ may bifurcate at this value of the coupling constant g_0 (or the charge q) — called from now on the critical value — at which the linearized equations (5a, b) possess nontrivial solutions satisfying the homogeneous boundary conditions [6, 7, 10, 11]. Note that this fact, although necessary, is not in general sufficient [6, 7, 10, 11].

The sufficiency may be proved by using an analytical method, a version of the Lyapunov-Schmidt procedure [11]. This was done in [8]. The crucial theorem in this procedure asserts that the bifurcating solutions are in a one to one correspondence with the small solutions to the Lyapunov-Schmidt equations. (The solution $u(g)$ near $u(g_0)$ is said to be small if $u(g)$ tends to $u(g_0)$ when g tends to g_0). The Lyapunov-Schmidt equations are elements of an algebraic finite-dimensional system of equations for some (finite) number of unknown parameters. This fact allows one also to use qualitative criteria for a sufficiency based on the Morse theorem [7, 10] or degree theory [7, 10]; one of them will be applied below.

The reduction from the infinite-dimensional differential problem to the finite-dimensional algebraic system of equations is sketched in the Appendix, since it is generally unknown to physicists.

3. The cylindrically symmetric bifurcation

Let us differentiate covariantly both sides of Eqs (1) and sum after repeating indices. Then

$$D_\mu j^\mu = 0, \tag{6}$$

where $D_\mu = \partial_\mu + g[A_\mu, \cdot]$. Hence, taking into account (2) one gets $A_0^{2,3}(\vec{x}) = 0$ whenever $\varrho(\vec{x}) \neq 0$. Let us assume $A_0^{2,3} = 0$ throughout the entire domain. Since the vector potentials obey the Coulomb gauge condition div $\vec{A}^a = 0$, they may be expanded as follows

$$A_i^a = \sum_{JM} f_{JM}^a(r)\varepsilon_{ikl}x_k\partial_l Y_{JM}(v, \varphi), \tag{7}$$

(Note that the vector harmonics Y_{JM}^{1+J}, Y_{JM}^{-1+J}, Y_{JM}^0 forms the complete base for vector functions of angle variables [13], but only the magnetic vector harmonics Y_{JM}^0, whose components were written above, are transverse, div $Y_{JM}^0(v, \varphi) = \partial_i(\varepsilon_{ikl}x_k\partial_l Y_{JM}(v, \varphi)) = 0$).

Eqs (1) for $A_0^{2,3}$ may be treated as additional constraints upon the space of solutions to the rest of Eqs (1). There exists at least one set of solutions trivially satisfying these constraints [12], namely that formed by a cylindrically symmetric component A_0^1 and cylindrically symmetric vector potentials A_i^a (which are defined by an expansion analogous to (7)):

$$A_i^a = \sum_J f_J^a(r)\varepsilon_{ikl}x_k\partial_l Y_{J0}(v). \tag{7a}$$

(7a) can be rewritten as below

$$A_i^a = \sum_J f_J^a(r)\varepsilon_{ikl}x_k\partial_l Y_{J0}(v) = \varepsilon_{i3j}\frac{x_j}{\varrho}\sum_J f_J^a(r)\partial_v Y_{J0}(v) = \varepsilon_{i3j}\frac{x_j}{\varrho}f^a(\varrho, z). \tag{8}$$

This is the Sikivie-Weiss cylindrically symmetric form.

Recall that under the above symmetry assumptions one obtains from the Yang-Mills equations [12]

$$\vec{A}^1 = \vec{0}, \quad \vec{A}^2 = \vec{A}^3. \tag{9}$$

The constraining equations for second and third isospin components A_0^2, A_0^3 are then satisfied identically [12]. Taking (9) into account one gets substantially simplified Yang-Mills equations

$$\Delta A + g^2(-2\varphi A_i A_i - 2AA_iA_i) = 0, \tag{10a}$$

$$\Delta A_i + g^2 A_i(A+\varphi)^2 = 0, \tag{10b}$$

where $A = A_0^1$, $A_i = A_i^2 = A_i^3$, φ — the known abelian solution. The Frechet derivative [7, 10, 11] at $A = A_i = 0$ is

$$F = \begin{pmatrix} \Delta, & 0 \\ 0, & (\Delta + g^2\varphi^2)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{pmatrix}. \tag{11}$$

F is a hermitian elliptic operator in the (Sobolev, Hölder) space, so $\ker F = \ker F^* = \operatorname{coker} F$, $\dim\ker F < \infty$ and its index is equal to zero, $\operatorname{ind} F = \dim\ker F - \dim\ker F^* = 0$. The linearized equations (5a, b) are as follows

$$F\begin{pmatrix} \delta A \\ \overrightarrow{\delta A} \end{pmatrix} = \begin{pmatrix} 0 \\ \vec{0} \end{pmatrix} \tag{12}$$

with the boundary conditions

$$\begin{pmatrix} \delta A \\ \overrightarrow{\delta A} \end{pmatrix}_{\partial\Omega} = \begin{pmatrix} 0 \\ \vec{0} \end{pmatrix}. \tag{13}$$

The following is now proved.

Theorem 1. The values g_0^2, for which the number of solutions of (12), (13) is odd, are the bifurcation points.

Proof. We will apply the following theorem (Th. 4.2.3. in [10]). Suppose that the operator $(1+\lambda_0 L)X+T(X, \lambda) = 0$ satisfies the following hypothesis.

i) $(1+\lambda_0 L)$ is a Fredholm operator of index zero,

ii) dimker $(1+\lambda_0 L)$ is odd,

iii) $\ker(1+\lambda_0 L) \cap \mathrm{Range}(1+\lambda_0 L) = \{0\}$,

iv) the nonlinear part $T(X, \lambda)$ is a C^1 mapping with $T(0, \lambda_0) = \frac{\partial}{\partial X} T(0, \lambda_0) = 0$.

Then λ_0 is a point of bifurcation for this equation.

We must show the validity only of *i*), *iii*) and *iv*), since *ii*) was assumed. Note that in our notation

$$(1+\lambda_0 L)(y) = \int_\Omega G(x, y)F(\varphi(x), g_0^2)dx = \begin{pmatrix} 1, & 0 \\ 0, & (1+g_0^2 \int_\Omega \varphi^2(x)G(x, y)dx) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{pmatrix},$$

where $G(x, y)$ is the Green function for the Laplace operator Δ vanishing on the boundary $\partial\Omega$, and $g_0^2 = \lambda_0$. (This representation is defined by the Schauder inversion [1, Ch. 2.2D]). Then *i*) and *iii*) are consequences of the hermiticity and ellipticity of F. The nonlinear part in (10a, b) and their first Frechet derivative vanish at $A = 0$, $\vec{A} = 0$, so *iv*) is satisfied. That proves theorem 1.

We obtain more information in the case of a simple eigenvalue g_0^2 using the Lyapunov--Schmidt method. One can then get [8] in Vajnberg-Trenogin's notation [11, Ch. 21–25]

$$L_{0n} = 0, \quad n \text{ – a positive integer number,}$$

$$L_{20} = 0,$$

$$L_{30} \neq 0, \quad L_{11} \neq 0.$$

The reduced Lyapunov-Schmidt equation is

$$\xi(L_{30}\xi^2 + L_{11}\lambda) = 0, \tag{14}$$

where λ is defined by $g^2 = \lambda + g_0^2$ (g_0^2 is the critical value corresponding to the one-dimensional kernel of F). The small solutions of (14) are as follows

$$\xi_1 = 0, \quad \xi_{2,3} = \pm\sqrt{\frac{-L_{11}\lambda}{L_{30}}}. \tag{15}$$

The Lyapunov-Schmidt theorem assures us now that there are two bifurcating nonabelian solutions, approximate expressions for them can be obtained from the construction of branching equations (14) [8].

Hence in the case of simple bifurcation we have the trivial solution

$$A_\mu^a = \delta_{a1}\delta_{\mu 0}\varphi \tag{16}$$

and the nonabelian, bifurcating from φ at g_0

$$\begin{bmatrix} A_0^1 \\ A_0^2 \\ A_0^3 \\ \vec{A}_1 \\ \vec{A}_2 \\ \vec{A}_3 \end{bmatrix} = \begin{bmatrix} \varphi \\ 0 \\ 0 \\ \vec{0} \\ \vec{0} \\ \vec{0} \end{bmatrix} \pm \sqrt{\frac{-L_{11}\lambda}{L_{30}}} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vec{0} \\ \overrightarrow{\delta A} \\ \overrightarrow{\delta A} \end{bmatrix} + o(\sqrt{\lambda}) \tag{17}$$

where $\frac{o(x)}{x} \to 0$ as $x \to 0$, φ is the known abelian potential (see formula (12b) in [8]), $\overrightarrow{\delta A}$ — the solution of the linearized equations (12), (13), L_{11}, L_{30} are constants proportional to the numerator or the denumerator correspondingly on the right hand side of (23).

Let us return to the second and third isospin components of Eqs (5a, b)

$$-2g\delta A_i^3 \partial_i \varphi = 0, \quad 2g\delta A_i^2 \partial_i \varphi = 0. \tag{18}$$

It is evident that nonzero solutions $\delta A_i^{2,3}$ exist only under some symmetry assumptions about φ (e.g. a cylindrically symmetric φ induces a form of $\delta A_i^{2,3}$ as in (8)). It seems to be interesting whether bifurcation is always related to symmetry (spatial or gauge) breaking. Our examples (see below) suggest the affirmative answer, but in general this problem is unsolved (not only in the Yang-Mills theory — see [14]).

Remark 1. It is possible to obtain subsequent terms of the expansion (17) using the Lyapunov-Schmidt method [11], but the computations are then much more involved.

Remark 2. For a spherically symmetric φ, inserting $\delta A_i = \sum_J f_J(r) \varepsilon_{ikl} x_k \partial_l Y_{J0}(v)$ into (12), one gets

$$\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - J(J+1)\frac{1}{r^2} + g^2\varphi^2\right) f_J(r) = 0. \tag{19}$$

(19) may have a nontrivial solution $f_J(r) \neq 0$ only when

$$\int_\Omega \frac{J(J+1)}{r^2} f_J^2(r) dV < g^2 \int_\Omega f_J^2(r) \varphi^2(r) dV. \tag{20}$$

It follows that g must be large enough.

We now present two examples of the simple bifurcation, assuming that the domain is a ball bounded by a sphere of a radius R, and dealing only with the smallest possible critical value g_0^2.

1. The potential is constant, $\varphi = \varphi_0$. Then

$$\delta A_i = \frac{\varepsilon_{i3l} x_l \operatorname{const} J_{3/2}(g_0 \varphi_0 r)}{r^{3/2}}, \quad \delta A = 0. \tag{21}$$

The expression for A_μ^a is as in (17), with δA_i from (21). The critical value g_0 is given by min $(g\colon \tan(g\varphi_0 R) = g\varphi_0 R)$.

2. The ball $r \leqslant R$ is uniformly charged with a charge density σ. Then

$$\delta A_i = \varepsilon_{i3l} x_l \,\text{const}\, J_{1/2}(|g_0|\,|\sigma| r^3 \tfrac{1}{18}) \frac{1}{r^{1/2}}, \qquad \delta A = 0. \tag{22}$$

g_0 is given by min $(g\colon J_{1/2}(|g||\sigma|R^3\frac{1}{18}) = 0)$, $J_{1/2}$ — the Bessel function, A_μ^a are given by (17) where δA_i is as in (22).

Remark 3. The Coulomb potential $\varphi = q/r$ on R^3 (except possibly a small ball near the origin, in order to ensure the smoothness).

The function spaces of Nirenberg-Walker-Cantor [15] which are correct in this non-compact case (in this sense, that they ensure that all coefficients $L_{i\mu}$ in (17) are finite) do not contain nonzero solutions of the linearized equations (since the decreasing condition at infinity is too strong). So the Coulomb potential does not bifurcate.

4. *Stability*

Supposing simple bifurcation one can prove that the bifurcating solutions exist above the critical point g_0^2 (an over critical bifurcation) for some charge distributions.

Indeed, one can show that [8]

$$\frac{-L_{11}}{L_{30}} = \frac{-\int\limits_\Omega d\vec{x}(\delta\vec{A}(\vec{x}))^2\varphi^2(\vec{x})}{2\int\limits_\Omega\int\limits_\Omega d\vec{x}d\vec{y}(\delta\vec{A}(\vec{x}))^2(\delta\vec{A}(\vec{y}))^2\varphi(\vec{x})G(\vec{x},\vec{y})\varphi(\vec{y})}, \tag{23}$$

where $G(\vec{x}, \vec{y})$ is the Green function for the laplacian Δ vanishing on the boundary. Expression (23) is positive e.g. for an everywhere positive (or negative) charge density. In this case $g^2 - g_0^2 > 0$, since solutions must be real. That fact is important for stability, as will be proven below.

Theorem 2. The bifurcating cylindrically symmetric solutions are linearization stable for sources such that L_{30} is negative, while the Maxwell potential is stable below g_0^2 and unstable above g_0^2 (where g_0^2 is the smallest critical point).

Proof. At first note that the bilinear form $\int\limits_\Omega (\delta A, \delta\vec{A}) F\begin{pmatrix}\delta A\\ \delta\vec{A}\end{pmatrix} dV$ (see Eqs (12)) is negative for $g^2 < 0$, so the eigenvalues λ of the problem $F\begin{pmatrix}X\\ Y\end{pmatrix} = \lambda\begin{pmatrix}X\\ Y\end{pmatrix}$ are all negative. This means that the Maxwell solution is stable [6, 7]. The operator F depends analytically on g^2, and when g^2 increases, the largest eigenvalue $\lambda_{\max}$ increases also. As g^2 crosses the critical value g_0^2, $\lambda_{\max}$ crosses 0 and above g_0^2 is positive, which corresponds to an instability of the abelian solution φ. (It was assumed that the eigen-problem mentioned above is correctly stated, i.e. $\lambda_{\max}$ is an isolated eigenvalue). The first assertion of our theorem follows now directly from the Th. 3.1. ([6], p. 39) which states that bifurcating solutions are stable when they appear above the critical value which is precisely our case, since $L_{30} < 0$.

5. *Conclusions*

The results of this paper are indicative of a "phase transition" at certain values of the coupling constant. The potentials then change from a Maxwell solution to a composition of the Maxwell and nonabelian part; so the holonomy group increases. Note that the field strength tensor has nonzero magnetic components in addition to electric components.

The bifurcation phenomena seem to be related to symmetry breaking.

The cylindrically symmetric solutions are both stable in the case of simple bifurcation (for some charge distributions). This is in contradiction to the Jackiw-Rossi [4] statement about the stability of the bifurcating solutions. They conclude that one of the bifurcating branches is stable while the second must be unstable. Their conclusion is incorrect (see [5–7] for a general analysis of the stability questions in simple bifurcation), but in any case stability in the Yang-Mills theory with sources demands further explanations.

APPENDIX

Suppose that our nonlinear problem has the following form

$$F(U = 0, \lambda)U + \lambda T(U, \lambda) = 0, \tag{A1}$$

where U belongs to a suitable Banach space, say Hölder $C^{k+\mu}(\Omega)$, F is the Frechet derivative at $U = 0$, T is nonlinear and satisfies the following

$$T(U = 0, \lambda_0) = \frac{\partial}{\partial U} T(U, \lambda_0)_{U=0} = 0. \tag{A2}$$

The kernels of F and F^* are finite when the symbols of F and F^* are injective (this is the case, e.g., when F, F^* are elliptic operators). Suppose that at $\lambda = \lambda_0$ $\ker F = \ker F^* \neq 0$, and for λ sufficiently near λ_0, $\dim\ker F = \dim\ker F^* = 0$. Denote by f_k the elements of $\ker F$ ($k = 1, 2, \ldots \dim\ker F$), multiply (A1) by them and integrate on Ω. Then we obtain a system of equations

$$\int_\Omega f_k[F(0, \lambda) - F(0, \lambda_0)]U dV + \lambda \int_\Omega f_k T(U, \lambda) dV = 0 \tag{A3}$$

which gives itself a restriction upon the space of solutions of (A1).

Inserting into (A3) the expansion analogous to (8) in [8] instead of U, one can easily conclude that Eqs (A3) constitute an algebraic system for $n = \dim\ker F$ unknown functions $\xi_k(\lambda)$, $k = 1, \ldots, n$. For details see [11], Chapters 21–25.

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