ANALYTICITY AND THE RENORMALIZATION GROUP IN φ^4 THEORY

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It is shown, under weak analyticity assumptions, that high energy bounds on the imaginary and on the real part of a scattering amplitude are correlated. If in a theory the Froissart bound on the imaginary part is substituted by a stronger one, then the bound on the real part is improved as well. Applications of the result to various situations are discussed and the case of the φ^4 theory is explicitly analyzed. The resulting bounds on the real part and on the phase are sensitive to the original deviation from the Froissart bound.

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1. Introduction

Some applications in elementary particle physics lead to scattering amplitudes for which the Froissart bound is modified. For instance, the possibility of a zero mass exchange in the t-channel invalidates the proof of this bound. Another example is the fixed-t non--forward scattering amplitude $F(E, t)$ in strong interactions. Axiomatic field theory proves that $F(E, t)$ is asymptotically bounded from above by $E \ln^{3/2} E$ and by $E^{2-\varepsilon}$ for $t \leqslant -\varepsilon < 0$ and for $0 < t \leqslant 4$, respectively [1, 2]. Constraints on the phase induced by these bounds were obtained in Ref. [3].

The case of φ^4 theory can serve as a third example. A strong upper bound on $\sigma(E)$ was obtained by Khuri [4] by using the renormalization group method. He showed that, at sufficiently high energies, the imaginary part of the forward scattering amplitude $F(E)$ obeys the following inequality

$$\operatorname{Im} F(E) \leqslant \text{const } E^{1-2\gamma} \ln^2 E, \tag{1}$$

where γ is the anomalous dimension of the φ field at the ultraviolet-stable fixed point. It is a non-negative number, but its exact value is not known. For $\gamma = 0$, (1) coincides with the Froissart-Martin bound on the imaginary part of $F(E)$. If γ is positive, (1) represents a stronger restriction.

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Relation (1) was obtained [4] by combining the renormalization group method with rigorous inequalities due to Singh [5] and Jin and Martin [2]. Because of optical theorem, (1) implies a bound on the total cross section

$$\sigma(E) \leqslant \text{const}\, E^{-2\gamma} \ln^2 E. \tag{2}$$

This approach, on the other hand, does not yield information on other quantities like the real part or the phase.

We show in the present paper that a systematic use of the general principles (analyticity, polynomial boundedness, crossing symmetry, unitarity, etc.) together with the renormalization group approach does lead to constraints on the phase as well as on the real part of the scattering amplitude at high energies. For example, we show that if the Froissart-Martin bound on $F(E)$,

$$|F(E)| \leqslant \text{const}\, E \ln^2 E, \tag{3}$$

is supplemented with the stronger bound (1) on $\text{Im}\, F(E)$, then also the real part of the amplitude must obey a constraint which is more restrictive than the original inequality (3). Details are given in Theorem 2.

We use the approach developed in Ref. [3], but introduce several new points. Firstly, the method used in Ref. [3] to investigate the phase is extended to the investigation of the real part as well. Secondly, in addition to high-energy correlations between the real part, the imaginary part and the phase, we obtain a tight correlation between the rigorous bounds on them.

The paper is organized as follows. We briefly review in Section 2 the renormalization group approach which was used in Ref. [4] to obtain the bound (1). Then, in Section 3, we list the properties of a scattering amplitude which follow from locality, crossing symmetry, unitarity condition, polynomial boundedness and other general principles of quantum field theory. Theorem 2 (Section 4) establishes the asymptotic constraints which follow for $\text{Re}\, F(E)$ from (1) and from the general principles mentioned above. Similarly, Theorem 3 (Section 5) contains analogous bounds on the phase of the scattering amplitude. Section 6 contains concluding remarks. Proofs of the theorems are deferred to the Appendices A and B.

2. Total cross section in φ^4 theory and the renormalization group

When it was shown that the renormalization group approach can be applied on the mass shell [6, 7], a new means appeared for investigating the high-energy behaviour of scattering amplitudes. One could also expect that the general requirements of analyticity, crossing symmetry, unitarity, etc., when supplemented with the renormalization group method, would be able to give more detailed information on high-energy scattering.

However, the difficulty of a simultaneous use of these two apporaches is that they apply to two disjoint high-energy regions [8]. The analyticity approach is suitable at fixed momentum transfer, but cannot be used at fixed scattering angle, in which case the negative values of energy are not related to the physical region of a crossed reaction. On the other hand, the renormalization group method applies to non-forward fixed angle scattering, but cannot be used at fixed momentum transfer, The reason is that, at sufficiently high energies, a fixed t amounts to the forward scattering angle, at which the four-particle vertex function has a singularity.

The main idea which led to relation (1) was that this singularity might be not very strong. If we use as a guide perturbation theory and its order-by-order properties in φ^4 theory, we can assume that the singularity is integrable in $\cos\theta$. Then, after some technical assumptions, the following integral asymptotic relation for the scattering amplitude can be obtained [4]

$$\int_{-\frac{\delta E}{2m}}^{0} dt F(E, t) \sim \text{const}\, E^{1-2\gamma}. \tag{4}$$

Here, t is momentum transfer squared and $\delta > 0$ is arbitrary.

This relation, combined with the asymptotic inequality due to Singh [5]

$$\frac{\operatorname{Im} F(E, t)}{\operatorname{Im} F(E, 0)} \geqslant 1 + \frac{t}{16m^2}\left[\ln \frac{E}{E_0^2 \sigma(E)}\right]^2, \quad t \leqslant 0 \tag{5}$$

and with the Jin-Martin lower bound [2]

$$\sigma(E) \geqslant \text{const}\, E^{-6}, \tag{6}$$

gives the high energy bounds (1) and (2) for the total cross section and for the imaginary part of the scattering amplitude, respectively. The value of γ is not exactly known, but some estimates can be used. From the positivity of the propagator one knows that $\gamma \geqslant 0$; on the other hand, combining the inequality (1) with (6) one gets $\gamma \leqslant 3$.

3. *Analyticity properties of the scattering amplitude*

The amplitude of the forward scattering of two particles with nonvanishing masses is a function of one complex variable, for which the laboratory energy E can be chosen. For complex (unphysical) values of E, we shall use the symbol z.

We shall use the following assumptions on the forward scattering amplitude $F(z)$:

(F1) $F(z)$ is analytic in the upper half of the complex z-plane excluding the semicircular disk of radius r_0 (r_0 is a positive constant). This domain will be denoted $\mathscr{D}$. $F(z)$ is continuous in the closure of $\mathscr{D}$. except possibly the point $E = \infty$.

(F2) $F(z)$ is crossing symmetric, i.e.

$$F(z) = F^*(-z^*)$$

for every z from $\mathscr{D}$.

(F3) $F(z)$ is polynomially bounded in $\mathscr{D}$ for $|z| \to \infty$.

(F4) $F(E)$ satisfies the Froissart bound

$$|F(E)| \leqslant \text{const}\, E \ln^2 E$$

for sufficiently high E. ($F(E)$ is defined as $\lim_{\varepsilon \to 0_+} F(E + i\varepsilon)$.)

(F5) The imaginary part of the amplitude Im $F(E)$ is nonnegative for all sufficiently high E. The optical theorem is taken in the form

$$\operatorname{Im} F(E) = a\sqrt{E^2-b}\,\sigma(E)$$

where a and b are positive constants.

For a closer discussion of the properties (F1)–(F5) see Ref. [9].

We recall now the basic theorem of Ref. [9], which will be systematically exploited in the sequel.

Theorem 1. Suppose that $F(z)$ fulfills the conditions (F1)–(F5). Let E_s be a positive constant such that

$$\lim_{E\to\infty} F(E)E^{2n+a-2} = 0, \tag{7}$$

$$\text{the function } g(E) \equiv \cos\frac{\pi a}{2}\operatorname{Im} F(E) - \sin\frac{\pi a}{2}\operatorname{Re} F(E)$$

$$\text{does not change sign for every } E > E_s, \tag{8}$$

and

$$\left|\int_{E_s}^{\infty} g(E)E^{2n+a-1}dE\right| = \infty; \tag{9}$$

here, n is some integer and a is real, $-1 \leqslant a \leqslant 1$. Denote $\mu(E)$ and $\nu(E)$ two functions which are integrable on the interval (E_s, E) for every $E > E_s$ and fulfill the constraints

$$-\tfrac{1}{2}-\frac{a}{2} \leqslant \mu(E) \leqslant \nu(E) \leqslant \tfrac{1}{2}-\frac{a}{2}. \tag{10}$$

Define the quantity $\varrho(E) = \operatorname{Re} F(E)/\operatorname{Im} F(E)$. If the inequalities

$$\operatorname{tg}[\pi\mu(E)] \leqslant \varrho(E) \leqslant \operatorname{tg}[\pi\nu(E)] \tag{11}$$

hold for every $E > E_s$, then positive constants c_1 and c_2 and an infinite sequence of energies $\{E_k\}$, $E_k \to \infty$ with $k \to \infty$, exist such that the inequalities

$$c_1E_k^{-2n+1}\exp\left\{2\int^{E_k}\mu(E)\frac{dE}{E}\right\} \leqslant |F(E_k)| \leqslant c_2E_k^{-2n+1}\exp\left\{2\int^{E_k}\nu(E)\frac{dE}{E}\right\} \tag{12}$$

hold for every $k = 1, 2, 3, \ldots$ (We shall suppress the lower, fixed, integration limits in such expressions.)

For proof of the theorem see Theorem 2 and Appendix B of Ref. [9].

4. *Asymptotic constraints on the real part of the amplitude*

We shall mostly be concerned with the case $\gamma > 0$, when the relation (1) does not coincide with — and is more restrictive than — the Froissart-Martin bound (3). The class of functions possessing the properties (F1) to (F5) and satisfying the high-energy bound (1) with some γ, $0 < \gamma \leqslant 3$, will be denoted $\mathscr{F}_\gamma$.

Let $F(z)$ belong to $\mathscr{F}_\gamma$. Then, applying Theorem 1, we can find that the real part $\operatorname{Re} F(E)$ obeys a bound, which is more restrictive than (3) and depends on the value of γ. Details are contained in the following Theorem 2. Certain smoothness properties at infinity are assumed, which are discussed in Remark 1.

Theorem 2. Let $F(z)$ be an element of the class $\mathscr{F}_\gamma$ and let the following conditions be satisfied:

(*i*) the sign of $\operatorname{Re} F(E)$ remains unchanged above some energy:

(*ii*) the limits (finite or infinite) $\lim_{E\to\infty} \operatorname{Re} F(E)$, $\lim_{E\to\infty} \operatorname{Re} F(E)E^{-1}$, and $\lim_{E\to\infty} \varrho(E)E^{-\varepsilon}$ for all $\varepsilon \in (0, \varepsilon_0)$ and some positive ε_0 exist.

Then either

$$\int^{\infty} |\operatorname{Re} F(E)| dE \text{ converges,} \tag{13a}$$

or $\operatorname{Re} F(E)$ is bounded by one of the following two inequalities

$$c_1 \leqslant |\operatorname{Re} F(E)| \leqslant c_2 \tag{13b}$$

$$|\operatorname{Re} F(E)| \leqslant c_3 E^{1-2\gamma+\varepsilon} \ln^2 E \tag{13c}$$

for all energies higher than a certain value and for every $\varepsilon \in (0, \varepsilon_0)$. Here, c_1, c_2 and c_3 are some unknown positive constants. If γ lies between 0 and 1/2, c_1 can be zero; besides, (13b) holds only if $\operatorname{Re} F(E)$ is asymptotically positive and (13c) holds only if $\operatorname{Re} F(E)$ is asymptotically negative.

Proof: See Appendix A.

Theorem 2 yields the result which was announced in the Introduction: the inequalities (13a), (13b) and (13c) amount to a single, γ-dependent, high-energy bound on $\operatorname{Re} F(E)$ which is analogous to the bound (1) on $\operatorname{Im} F(E)$. Indeed, to apply it, one needs no information on $\operatorname{Im} F(E)$ except the value of γ. This is a merit with respect to Theorem 1, which correlates two quantities, $\varrho(E)$ and $|F(E)|$, which contain both the real and the imaginary part of the amplitude (see (11) and (12)).

Remark 1. There are additional conditions, (*i*) and (*ii*), which do not follow from the general properties (F1) to (F5). Among them, only the requirement of the existence of $\lim_{E\to\infty} \operatorname{Re} F(E)$ is unavoidable. If the other ones are removed, then relations (13b) and (13c) are still valid on some infinite sequence of energies E_k tending to ∞ with $k \to \infty$.

Remark 2. For γ between 0 and 1/2, it is interesting to compare the Theorem 2 with the result obtained by Cornille [10] by means of dispersion relations. The statement of Theorem 2,

$$-c_3 E^{1-2\gamma+\varepsilon} \ln^2 E \leqslant \operatorname{Re} F(E) \leqslant c_2$$

can be improved, by a slight modification of the proof in Appendix A, so as to give

$$-c_3 E^{1-2\gamma} \ln^{3+\varepsilon} E \leqslant \operatorname{Re} F(E) \leqslant c_2.$$

Cornille's result, on the other hand, is that the integral

$$\int^{\infty} \frac{\operatorname{Re} F(E) - \operatorname{Re} F(0)}{E^{2-2\gamma} \ln^{3+\varepsilon} E} dE$$

converges. This implies that the following inequality

$$|\operatorname{Re} F(E) - \operatorname{Re} F(0)| \leqslant \text{const } E^{1-2\gamma} \ln^{2+\varepsilon} E$$

holds on some energy sequence tending to infinity.

5. *Asymptotic constraints on the phase*

We shall now investigate the asymptotic behaviour of the quantity $\varrho(E) = \operatorname{Re} F(E)/\operatorname{Im} F(E)$, which is closely related to the phase. For $F(z)$ belonging to $\mathscr{F}_\gamma$, the asymptotic constraints on $\varrho(E)$ are given in the following theorem:

Theorem 3. Let $F(z)$ be a function of the class $\mathscr{F}_\gamma$. Suppose that the following conditions are fulfilled:

(*i*) Re $F(E)$ does not change sign infinitely many times,

(*ii*) $\int^{\infty} \operatorname{Re} F(E) dE$ diverges,

(*iii*) the limits (finite or infinite) $\lim\limits_{E\to\infty} \operatorname{Re} F(E)E^{-1}$, $\lim\limits_{E\to\infty} \varrho(E)$ and $\lim\limits_{E\to\infty} |\varrho(E)|^{-1}E^{1+2\gamma-\eta}$

for all $\eta \in (0, \eta_0)$ and some η_0 exist.

Then, depending on the value of γ and on the asymptotic sign of the real part, the ratio $\varrho(E) = \operatorname{Re} F(E)/\operatorname{Im} F(E)$ obeys different asymptotic constraints:

1) for Re $F(E)$ asymptotically nonpositive,

$$\text{if } 0 < \gamma \leqslant \tfrac{1}{2}, \text{ then } \lim_{E\to\infty} \varrho(E) \leqslant -\operatorname{tg} \pi\gamma; \tag{14a}$$

$$\text{if } \gamma > \tfrac{1}{2}, \text{ then } \lim_{E\to\infty} \varrho(E) = -\infty \text{ and } \varrho(E) \leqslant -\text{const } E^{2\gamma-1-\eta} \tag{14b}$$

for every $\eta \in (0, \eta_0)$ at least on some sequence of energies tending to ∞.

2) For Re $F(E)$ asymptotically non-negative,

$$\text{if } \tfrac{1}{2} < \gamma \leqslant 1, \text{ then either } \lim_{E\to\infty} \varrho(E) \leqslant \operatorname{tg} \pi(1-\gamma), \text{ or } \lim_{E\to\infty} \varrho(E) = +\infty, \tag{14c}$$

$$\text{if } \gamma > 1, \text{ then } \varrho(E) \geqslant \text{const } E^{2\gamma-1-\eta} \tag{14d}$$

for every $\eta \in (0, \eta_0)$ at least on some sequence of energies tending to ∞.

See Appendix B for proof.

Remark 3. The method gives no constraints on $\lim\limits_{E\to\infty} \varrho(E)$ for Re $F(E) \geqslant 0$ and $0 < \gamma \leqslant 1/2$.

Remark 4. The theorem applies to the case $\gamma = 0$ provided that the additional condition

$$(iv) \lim_{E\to\infty} F(E)E^{-1} = 0$$

is required. The case $\gamma = 0$ was examined in detail in an earlier paper [3]. Some of our present conclusions represent a generalization of the previous results.

In analogy with Theorem 2, Theorem 3 also gives a γ-dependent high-energy bound, (14a,b,c,d), this time on the phase. We see from the theorem that if γ is sufficiently high, then $\varrho(E)$ cannot be finite at infinite energy and must tend to $+\infty$ or $-\infty$, while the rate of its rise cannot be arbitrary (see relations (14b) and (14d)). Finite values of $\lim_{E\to\infty} \varrho(E)$ are allowed only for γ not too big (see relations (14a) and (14c)).

6. *Summary*

Analyticity and crossing symmetry of a scattering amplitude play a decisive role in obtaining rigorous correlations between its imaginary and its real part, its modulus and its phase, etc. However, they are not sufficient to yield physical predictions, for which reference to some value of a physical quantity is necessary. Having the form of an inequality, the Froissart-Martin bound is a very appropriate supplement to the analyticity property.

As this bound, however, is not common to all field theories, the question arises how the high energy scattering laws change if it is replaced by some other inequality.

We have investigated a special case of such a situation. We have started with $\mathrm{Re}\, F(E)$ obeying the original Froissart-Martin bound (3), but with $\mathrm{Im}\, F(E)$ obeying the stronger bound (1). The concrete high-energy behaviours of these two quantities are, of course, mutually correlated because of analyticity, crossing symmetry, etc. It can therefore be expected that high-energy *bounds* on them must also be correlated; indeed, once $\gamma > 0$ in (1) is given, then $\mathrm{Re}\, F(E)$ could hardly saturate the original bound (3) when $\mathrm{Im}\, F(E)$ is forced to remain small.

We have shown in Theorem 2 that this is indeed the case. Analogous constraints on the phase follow from Theorem 3.

Since the input information (1) on $\mathrm{Im}\, F(E)$ has been obtained within the framework of φ^4 theory, our results can be considered as general features of this theory. Nevertheless, the formalism developed can be applied to every situation satisfying the original requirements, no matter in which context the input information on $\mathrm{Im}\, F(E)$ was obtained.

APPENDIX A

Proof of Theorem 2

Consider Theorem 1 with $n = 0$, $a = 1$. Conditions (7) to (9) read now

$$\lim_{E\to\infty} F(E)/E = 0, \tag{A1}$$

$$\mathrm{Re}\, F(E) \text{ does not change sign above some energy,} \tag{A2}$$

and

$$\int^{\infty} \mathrm{Re}\, F(E)dE \text{ diverges.} \tag{A3}$$

Let us suppose that these conditions as well as conditions (F1) to (F5) of Section 3 are satisfied. In order to fulfill (10), we choose

$$\mu(E) = \nu(E) = \begin{cases} \dfrac{1}{\pi} \operatorname{arc tg} \varrho(E) & \text{for Re } F(E) \text{ asymptotically negative,} \\ \dfrac{1}{\pi} \operatorname{arc tg} \varrho(E) - 1 & \text{for Re } F(E) \text{ asymptotically positive.} \end{cases}$$

All the assumptions of Theorem 1 are now satisfied. Relation (12) reads

$$c_1 \exp \int^{E_k} \left[\frac{2}{\pi} \operatorname{arc tg} \varrho(E) + 1\right] \frac{dE}{E} \leqslant |\operatorname{Re} F(E_k)| \sqrt{1+\varrho^{-2}(E_k)}$$
$$\leqslant c_2 \exp \int^{E_k} \left[\frac{2}{\pi} \operatorname{arc tg} \varrho(E) + 1\right] \frac{dE}{E}, \tag{A4a}$$

$$d_1 \exp \int^{E_k} \left[\frac{1}{\pi} \operatorname{arc tg} \varrho(E) - 1\right] \frac{dE}{E} \leqslant |\operatorname{Re} F(E_k)| \sqrt{1+\varrho^{-2}(E_k)}$$
$$\leqslant d_2 \exp \int^{E_k} \left[\frac{2}{\pi} \operatorname{arc tg} \varrho(E) - 1\right] \frac{dE}{E}, \tag{A4b}$$

for Re $F(E)$ asymptotically nonpositive and nonnegative, respectively, and for every $k = 1, 2, \ldots$ Here, c_1, c_2, d_1 and d_2 are some unknown positive constants.

From (A4a, b) the following statement simply follows:

(*i*) for Re $F(E)$ asymptotically nonpositive,

$$\text{if } \lim_{E\to\infty} \operatorname{Re} F(E) = -\infty, \text{ then } \int^{\infty} \left[\frac{2}{\pi} \operatorname{arc tg} \varrho(E) + 1\right] \frac{dE}{E} = +\infty; \tag{A5}$$

(*ii*) for Re $F(E)$ asymptotically nonnegative, the right hand side of (A4b) is bounded, i.e.,

$$\lim_{E\to\infty} \operatorname{Re} F(E) < \infty, \tag{A6}$$

and thus,

$$\text{if } \lim_{E\to\infty} \operatorname{Re} F(E) = \text{const} \neq 0, \text{ then } \int^{\infty} \left[\frac{2}{\pi} \operatorname{arc tg} \varrho(E) - 1\right] \frac{dE}{E} = \text{const.} \tag{A7}$$

Here, the existence of $\lim_{E\to\infty} \operatorname{Re} F(E)$ has been assumed. If, moreover, $\liminf_{E\to\infty} |\varrho(E)| > 0$, the implications (A5) and (A7) are replaced by equivalences and, in the case (*i*), the limit $\lim_{E\to\infty} \operatorname{Re} F(E)$ is different from zero.

To prove Theorem 2, we use the following two auxiliary statements.

Statement 1. Let $F(z)$ be a function of the class $\mathscr{F}_\gamma$ and let the limit $\lim_{E\to\infty} \operatorname{Re} F(E)/E$ exist. Then

$$\lim_{E\to\infty} \operatorname{Re} F(E)/E = 0. \tag{A8}$$

Proof: If $\operatorname{Re} F(E)$ changes sign above every energy, then (A8) is trivially satisfied, because of the existence of $\lim_{E\to\infty} \operatorname{Re} F(E)$. Thus, we suppose that $\operatorname{Re} F(E)$ has a definite sign for large enough energies.

1) Assume first that $\operatorname{Re} F(E)$ is asymptotically nonnegative. Then

$$\int^{\infty} \frac{\operatorname{Re} F(E)}{E^{1-a}}\, dE \text{ converges for } a \in (-1, 0) \tag{A9}$$

if

$$\int^{\infty} \left(\operatorname{arc\,tg} \varrho(E) - \frac{\pi}{\ln E}\right) \frac{dE}{E} = +\infty. \tag{A10}$$

To prove this, we use Theorem 1 with $n = 0$, $a \in (-1, 0)$. We can put $\mu(E) = \nu(E) = \frac{1}{\pi} \operatorname{arctg} \varrho(E)$. Condition (7) of Theorem 1 reads

$$\lim_{E\to\infty} \frac{F(E)}{E^{2-a}} = 0$$

and is satisfied because of the Froissart-Martin bound (3). The sign of the linear combination

$$g(E) = \cos\frac{\pi a}{2} \operatorname{Im} F(E) - \sin\frac{\pi a}{2} \operatorname{Re} F(E)$$

is definite, as required by condition (8), because a is negative. If now (9) were satisfied, then (12) would hold and (A10) would imply a contradiction with (3). Thus, (9) cannot hold and, consequently, $\int^{\infty} g(E)E^{a-1}dE$ converges. This implies that also (A9) is satisfied. Reversing now this implication, we obtain that if $\int^{\infty} \operatorname{Re} F(E)E^{a-1}dE$ diverges, then for every constant $\tau > 1$ an infinite sequence $\{E_k\}$, $E_k \to \infty$, exists such that

$$\operatorname{Re} F(E_k) \leqslant \pi\tau \operatorname{Im} F(E_k)/\ln E_k \tag{A11}$$

for every $k = 1, 2, 3, \ldots$

Now we can prove the Statement. If $\int^{\infty} \operatorname{Re} F(E)E^{a-1}dE$ diverges, then (A8) follows from (A11) and from (2). If the integral converges, then (A8) follows immediately.

2) If now $\operatorname{Re} F(E)$ is asymptotically nonpositive, we use Theorem 1 with $n = 0$ and $a = -1$. Condition (7) reads now

$$\lim_{E\to\infty} F(E)E^{-3} = 0$$

and is satisfied thanks to (3). The validity of (8) follows from assumption (A2). Condition (9) requires the divergence of the integral $\int^{\infty} \operatorname{Re} F(E)E^{-2}dE$. But if this integral diverged, then (11) would imply a contradiction with the Froissart-Martin bound (3). Therefore, the integral must be convergent and, by this, (A8) is satisfied. The proof of Statement 1 is completed.

Statement 2. Suppose that (A1), (A2), (A3) and (F1) to (F5) are fulfilled, the limits $\lim_{E\to\infty} \operatorname{Re} F(E)$ and $\lim_{E\to\infty} \varrho(E)E^{-\varepsilon}$ for all $\varepsilon \in (0, \varepsilon_0)$ and some positive ε_0 exist, and $\liminf_{E\to\infty} |\varrho(E)| > 0$. Then

$$\text{if } \lim_{E\to\infty} \operatorname{Re} F(E) = -\infty, \text{ then } \operatorname{Re} F(E) \geqslant -C\, E^{\varepsilon} \operatorname{Im} F(E) \tag{A12a}$$

and

$$\text{if } \lim_{E\to\infty} \operatorname{Re} F(E) = 0 \text{ and } \operatorname{Re} F(E) \text{ is asymptotically nonnegative,}$$

$$\text{then } \operatorname{Re} F(E) \leqslant D\, E^{\varepsilon} \operatorname{Im} F(E) \tag{A12b}$$

for every $\varepsilon \in (0, \varepsilon_0)$ and all E above a certain value, C and D being some positive constants.

Proof: The implication (A12a) is proved in the following manner. Because of (A5), the assumption that $\lim_{E\to\infty} \operatorname{Re} F(E) = -\infty$ implies that the inequality

$$|\varrho(E_k)| \leqslant \operatorname{cotg} \frac{c}{E_k^{\varepsilon}} \leqslant \text{const } E_k^{\varepsilon} \tag{A13}$$

holds for every $\varepsilon > 0$ at least on some infinite sequence $\{E_k\}$ tending to infinity with $k \to \infty$. Since $\lim_{E\to\infty} \varrho(E)E^{-\varepsilon}$ is assumed to exist, (A13) holds at all energies above some value and for every $\varepsilon \in (0, \varepsilon_0)$. This immediately leads to (A12a). The implication (A12b) is proved analogously.

We are now in a position to complete the proof of Theorem 2. Let us assume that all the assumptions of the theorem are satisfied. Then the assumptions of both Statement 1 and Statement 2 are fulfilled, too. Further, (A1) is a simple consequence of (1) and of Statement 1. Condition (A2) is exactly the assumption (i) of Theorem 2. In this case, the statement of Theorem 2 follows independently of the validity of condition (A3). Indeed, if $\operatorname{Re} F(E)$ does not obey (A3), we automatically obtain (13a). If, on the other hand, (A3) is satisfied, we can prove that either (13b) or (13c) follow. We shall show this for $\operatorname{Re} F(E)$ asymptotically nonnegative (the proof in the other case is analogous).

Relation (A6) implies that there is a constant c_2 such that

$$\operatorname{Re} F(E) \leqslant c_2 \tag{A14}$$

for sufficiently high E. Using now Statement 2, we arrive at the following conclusion:

$$\text{If } \lim_{E\to\infty} \operatorname{Re} F(E) = 0 \text{ and } \liminf_{E\to\infty} |\varrho(E)| > 0, \text{ then}$$

$$\operatorname{Re} F(E) \leqslant c\, E^{\varepsilon} \operatorname{Im} F(E) \leqslant c_3 E^{1-2\gamma+\varepsilon} (\ln E)^2 \tag{A15}$$

for sufficiently high E and every $\varepsilon \in (0, \varepsilon_0)$, c and c_3 being some unknown positive constants (notice that the latter inequality follows because of (2)).

If now $\lim_{E\to\infty} \operatorname{Re} F(E)$ is nonvanishing, a positive constant c_1 exists such that

$$\operatorname{Re} F(E) \geqslant c_1 \tag{A16}$$

for sufficiently high E. If, on the other hand, $\liminf_{E\to\infty} |\varrho(E)|$ is not positive, then $\lim_{E\to\infty} \varrho(E) = 0$, and for every $\varepsilon > 0$ a constant E_0 exists such that $\varrho(E) < \varepsilon$ for all $E > E_0$, and, consequently,

$$\operatorname{Re} F(E) < \varepsilon \operatorname{Im} F(E) \leqslant \text{const}\, E^{1-2\gamma}(\ln E)^2,$$

where the latter inequality is again a consequence of (2). But this asymptotic constraint on $\operatorname{Re} F(E)$ is already included in (A15). Furthermore, from (A6) we can see that in the case of $\gamma < 1/2$, the absolute upper bound on $\operatorname{Re} F(E)$ is (13b). Combining (A14), (A15) and (A16) we obtain the relations (13b) and (13c).

APPENDIX B

Proof of Theorem 3

Let $\operatorname{Re} F(E)$ be asymptotically nonpositive. The assumptions of Theorem 3 ensure that (A4a) holds. Besides,

$$\lim_{E\to\infty} \operatorname{Im} F(E) E^{2\gamma-1-\eta} = 0 \tag{B1}$$

for all $\eta > 0$. From (A4a) one can see that

$$\frac{\text{const}\, E \exp\left\{\dfrac{2}{\pi}\displaystyle\int^{E} \operatorname{arc\,tg} \varrho(E')\frac{dE'}{E'}\right\}}{\sqrt{1+\varrho^2(E)}} \leqslant \operatorname{Im} F(E). \tag{B2}$$

Combining (B2) with (B1) we obtain that

$$\lim_{E\to\infty}\left\{\int^{E}\left[\frac{2}{\pi}\operatorname{arc\,tg} \varrho(E')+2\gamma-\eta\right]\frac{dE'}{E'} - \tfrac{1}{2}\ln\left(1+\varrho^2(E)\right)\right\} = -\infty \tag{B3}$$

for all $\eta > 0$, provided that this limit exists. For η sufficiently small, the existence of this limit follows from assumption (*iii*) of Theorem 3.

Let us first consider the case of $\lim_{E\to\infty} \varrho(E) > -\infty$. Then the term $\frac{1}{2}\ln(1+\varrho^2(E))$ remains finite and (B3) holds only if

$$\int^{\infty}\left(\frac{2}{\pi}\operatorname{arc\,tg} \varrho(E')+2\gamma-\eta\right)\frac{dE'}{E'} = -\infty. \tag{B3'}$$

This can be satisifed only for $\gamma < 1/2$. Then, (14a) follows directly from (B3').

If $\lim_{E \to \infty} \varrho(E) = -\infty$, then the asymptotic behaviour of the l.h.s. of (B3′) is controlled by the term $\frac{1}{2} \ln (1 + \varrho^2(E))$. Relation (14b) is now a simple consequence of this fact.

Relations (14c) and (14d) are proved in an analogous way.

REFERENCES

[1] A. Martin, *Phys. Rev.* **129**, 1432 (1963).
[2] Y. S. Jin, A. Martin, *Phys. Rev.* **135**, B 1369, B 1375 (1964).
[3] J. Fischer, O. E. Shishanin, *Czech. J. Phys.* **B27**, 1201 (1977).
[4] N. N. Khuri, *Phys. Rev.* **D14**, 2665 (1976).
[5] V. Singh, *Phys. Rev. Lett.* **26**, 530 (1971).
[6] C. G. Callan, Jr., D. J. Gross, *Phys. Rev.* **D11**, 2904 (1975).
[7] M. Creutz, L. L. Wang, *Phys. Rev.* **D10**, 3749 (1974).
[8] K. Huang, E. F. Low, *Zh. Eksp. Teor. Fiz.* **46**, 845 (1964), (*Sov. Phys. JETP* **19**, 579 (1964)).
[9] J. Fischer, P. Kolář, I. Vrkoč, *Phys. Rev.* **D13**, 133 (1976).
[10] H. Cornille, *Nuovo Cimento* **66A**, 217 (1970).