ASYMPTOTIC SYMMETRIES OF DE SITTER SPACE-TIME

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The general form of the metric of an axially-symmetrical asymptotically de Sitter space-time fulfilling a radiation condition was found. Using the Bondi-Metzner method, the group of asymptotic symmetries of de Sitter space-time was found. The results obtained in this work agree only partially with Penrose's theory.

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1. *Introduction*

Transformations conserving the form of equations of physics are called symmetry transformations. Noether's theorem [1] states, that each s-parameter group of symmetries provides s conservation laws. Einstein's general covariance principle requires the description of physical quantities by covariant tensors, and the independence of physical equations of the choice of coordinates. The Einstein group — the group of coordinate transformations of a four-dimensional space — appears thus to be the symmetry group of each physical theory. The infinite dimensional Einstein group contains an infinite number of one-parameter groups, providing an infinite number of conservation laws [3]. Only a few of them possess a clear physical interpretation. In order to extract the physically meaningful conservation laws, one has to find a criterion for the selection of the appropriate coordinate transformations. One possible approach is to restrict the coordinate transformations to exact isometries. However, only very specifical space-times admit a non-trivial isometry group, therefore such a requirement is too strong. One can weaken it by admitting approximate isometries. This is the approach used in the asymptotic symmetries theory. The requirement that coordinate transformations preserve the form of the metric for $r \to \infty$ allows one to select the asymptotic symmetry group (AS group).

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Bondi et al. [2] introduced the AS group as the group of transformations conserving certain boundary conditions in the class of asymptotically flat solutions of Einstein equations (this method of studying AS groups will be referred to as Bondi's method). Bondi et al. [2] assumed axial symmetry of solutions, later Sachs [13] dropped this assumption, and obtained the so-called Bondi-Metzner-Sachs group (BMS group) [4, 13, 14]. The same result has been obtained independently by Newman and Unti [9].

The BMS group appeared also in Penrose's conformal method of studying global properties of solutions of Einstein equations [10], as the symmetry group of the boundary of Minkowski space-time [6, 10] (or a space-time with a boundary homeomorphic and conformally equivalent to the boundary of the Minkowski space-time). In the class of asymptotically flat space-times (AF space-times) all solutions have the same boundary as the Minkowski space-time. Thus it appeared that the problem of finding the AS group of an AF space-time is equivalent to the problem of finding the symmetry group of the boundary of the Minkowski space-time.

From the astrophysical point of view, "asymptotically cosmological" space-times seem to be more relevant than AF space-times. The de Sitter space-time, or the Robertson--Walker space-times, admit a boundary at temporal infinity [7] while AF space-times have a boundary at null infinity [6, 10]. Penrose's theory states, that the symmetry group of the boundary of a space-time admitting a boundary at temporal infinity is the group of conformal transformations of the boundary. Such groups are finite-dimensional. It seemed interesting to see whether the AS group is isomorphic to the symmetry group of the boundary, for spaces with boundary at temporal infinity. This work is an attempt to give an answer to this question in the case of the simplest cosmological model — the de Sitter space-time. The comparison of the results obtained by Bondi et al. [2] and by Sachs [13] suggests, that the most important property of the BMS group — the infinite dimensionality — appears already in the case of axial symmetry. For the sake of simplicity the analysis has been limited to axial symmetry.

The results of Section 3 are obtained under the assumption, that the function γ appearing in the metric (3) has the expansion

$$\gamma = \overset{\circ}{\gamma}(u, \theta)+c(u, \theta)/r+O^*(1/r^2), \tag{1}$$

where $O^*(r^p)$ means "asymptotic smoothness":

$$O^*(r^p) = O(r^p) \qquad \partial O^*(r^p)/\partial r = O^*(r^{p-1}) \qquad \partial O^*(r^p)/\partial u = O^*(r^p),$$

$$\partial O^*(r^p)/\partial \theta = O^*(r^p).$$

In Section 3, the analycity in u and θ of the functions $\overset{\circ}{\gamma}$, c and the integration functions is required for the existence theorem used. The AS group obtained in Section 4 does not depend upon the analycity of $\overset{\circ}{\gamma}$, and the coordinate transformations need not to be analytical but to have an expansion similar to (1). In Sections 4.2, 4.3 and 4.4 an expansion similar to (1) for γ is required up to the third order in $1/r$.

1.1. Penrose's theory for the de Sitter space-time

Let $(\tilde{M}, \tilde{g})$ be a space-time $\tilde{M}$ with a Lorentz metric $\tilde{g}$. An asymptote [6, 10] of $\tilde{M}$ is the triplet (M, Ψ, Ω), where M is a manifold with boundary I, with a Lorentz metric g, Ψ is a diffeomorphism

$$\Psi : \tilde{M} \to M \backslash I$$

and Ω a smooth function on M satisfying the following conditions:

1. $g|_{M\backslash I} = \Omega^2 \Psi^* \tilde{g}$,
2. $\Omega|_I = 0 \quad \nabla\Omega|_I \neq 0$.

I will be called the boundary of the space-time $\tilde{M}$. When $g^{ab}\nabla_a\Omega\nabla_b\Omega|_I = 0$ I will be called the boundary at null infinity, when $g^{ab}\nabla_a\Omega\nabla_b\Omega|_I > 0$ (signature $+ - - -$) I will be called the boundary at temporal infinity.

The model for the de Sitter space-time is a hyperboloid in a five-dimensional space [7]:

$$S = \{(v, w, x, y, z) \in R^5 : -v^2 + w^2 + x^2 + y^2 + z^2 = \alpha^2\}.$$

The metric of the de Sitter space-time fulfills the vacuum Einstein equations with a cosmological constant $\lambda = \frac{R}{4} = \frac{3}{\alpha^2}$, or the Einstein equations with a zero cosmological constant and a stress-energy tensor of a perfect fluid with a constant density

$$\varrho = R/(32\pi) = 3/(8\pi\alpha^2), \qquad (c = G = 1)$$

and a constant negative pressure $p = -\varrho$.

On the hyperboloid one can fix the coordinates t, χ, θ, ϕ:

$$v = \alpha \sinh(t/\alpha), \qquad w = \alpha \cosh(t/\alpha) \cos\chi, \qquad x = \alpha \cosh(t/\alpha) \sin\chi \cos\theta,$$

$$y = \alpha \cosh(t/\alpha) \sin\chi \sin\theta \cos\phi, \qquad z = \alpha \cosh(t/\alpha) \sin\chi \sin\theta \sin\phi.$$

In these coordinates the line element takes the form

$$ds^2 = dt^2 - \alpha^2 \cosh^2(t/\alpha)\{d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2)\}.$$

The asymptote of de Sitter space-time is the manifold

$$A = [-\pi/2, \pi/2] \times S^3$$

with the boundary

$$I = I^+ U I^-, \qquad I^\pm = \{\pm\pi/2\} \times S^3 = S^3.$$

Embedding A in a five-dimensional space one can fix natural coordinates $t', \chi', \theta', \phi'$:

$$A = \{(t', w, x, y, z) : t' \in [-\pi/2, \pi/2], \qquad w = \cos\chi',$$

$$x = \sin\chi' \cos\theta', \qquad y = \sin\chi' \sin\theta' \cos\phi', \qquad z = \sin\chi' \sin\theta' \sin\phi'\}.$$

In these coordinates the line-element takes the form

$$ds^2 = dt'^2 - \{d\chi'^2 + \sin^2\chi'(d\theta'^2 + \sin^2\theta' d\phi'^2)\}.$$

The diffeomorphism Ψ is

$$t' = 2 \arctan(\exp t/\alpha) - \pi/2, \tag{2}$$

$$\chi' = \chi \quad \theta' = \theta \quad \phi' = \phi,$$

and the function Ω

$$\Omega = \sin(t' + \pi/2)/\alpha.$$

$\nabla\Omega$ is a time-like vector on I, thus I is the boundary at temporal infinity of the de Sitter space-time. I^+ and I^- are three dimensional spheres, thus the symmetry group of the boundary is the group of conformal transformations of a three-dimensional sphere, which is isomorphic to the O (1,4) group.

2. *The field equations*

In order to find the most general form of the metric of an asymptotically de Sitter space-time, a generalisation of the method developed by Bondi et al. [2] will be used. Bondi [2] has shown, that every axially symmetrical metric can be written in the following form:

$$ds^2 = (Vr^{-1}e^{2\beta} - U^2r^2e^{2\gamma})du^2 + 2e^{2\beta}dudr + 2Ur^2e^{2\gamma}dud\theta$$

$$-r^2(e^{2\gamma}d\theta^2 + e^{-2\gamma}\sin^2\theta d\phi^2), \tag{3}$$

(where U, V, β and γ are functions of r, u and θ), under the assumption, that u is a timelike coordinate for all $r > r_0$, for some r_0. In this work a stronger assumption will be needed, namely, if u is timelike for $r_0 < r < r_1$, then the metric can be written in the form (3) for all $r > r_0$.

Coordinates u, r, θ, ϕ in which the metric takes the form (3) will be called Bondi's coordinates. A space-time will be called asymptotically de Sitter if, in Bondi's coordinates

$$R_{\mu\nu} - Rg_{\mu\nu}/2 \to 0 \qquad r \to \infty.$$

Statement 1:

If the stress-energy tensor $T_{\mu\nu}$ in Bondi's coordinates fulfills the following conditions:

1. $T_{\mu\nu}$ fulfills the energy conservation law ($T^{\mu\nu}{}_{;\nu} = 0$),
2. $T_{\alpha\beta,\phi} = 0$,
3. $T_{u\phi} = T_{r\phi} = T_{\theta\phi} = 0$,

then Einstein equations resolve into three groups

$$(\tilde{T}_{\mu\nu} = 8\pi(T_{\mu\nu} - T^{\alpha}_{\alpha}g_{\mu\nu}/2))$$

1. main equations

$$R_{11} = \tilde{T}_{11}, \quad R_{12} = \tilde{T}_{12}, \quad R_{22} = \tilde{T}_{22}, \quad R_{33} = \tilde{T}_{33},$$

2. trivial equation

$$R_{10} = \tilde{T}_{10},$$

3. supplementary conditions

$$(R_{20} - \tilde{T}_{20})|_P = 0, \quad (R_{00} - \tilde{T}_{00})|_P = 0,$$

(where P is any hypersurface $r = \text{const.}$).

Bondi et al. [2] have prooved a similar statement for a vanishing stress-energy tensor. The proof of this statement can be carried on by replacing $R_{\mu\nu}$ by $R_{\mu\nu} - \tilde{T}_{\mu\nu}$ in the proof proposed by Bondi et al. [2].

The Einstein equations with a cosmological constant can be integrated in the same way, since they can be treated as zero cosmological constant equations with an appropriately modified stress-energy tensor. If the stress-energy tensor in the cosmological constant equations fulfills the conditions of statement 1, then the modified stress-energy tensor will also fulfill them.

For the de Sitter space-time, the main equations are[1] (from Bondi et al. [2]):

$$0 = R_{11} \leftrightarrow \beta_1 = r\gamma_1^2/2, \tag{4}$$

$$0 = R_{12} \leftrightarrow [r^4 e^{2(\gamma-\beta)} U_1]_1 = 2r^2\beta_{12} - 2r^2\gamma_{12} + 4r^2\gamma_1\gamma_2 - 4r\beta_2 - 4r^2\gamma_1 \cot 0, \tag{5}$$

$$\begin{aligned} 2\lambda = R/2 = -R_{22}e^{-2\gamma}r^{-2} + R^3{}_3 \leftrightarrow V_1 = -Rr^2e^{2\beta}/4 + r^2U_1 \cot\theta/2 \\ + 2rU\cot\theta + e^{2(\beta-\gamma)}[1 + (3\gamma_2 - \beta_2)\cot\theta + \gamma_{22} - \beta_{22} - \beta_2^2 \\ + 2\gamma_2(\beta_2 - \gamma_2)] - r^4e^{2(\gamma-\beta)}U_1{}^2/4 + r^2U_{12}/2 + 2rU_2, \end{aligned} \tag{6}$$

$$\begin{aligned} 0 = R^3{}_3 + R_{22}e^{-2\gamma}r^{-2} \leftrightarrow 0 = 2r(r\gamma)_{01} - r^4e^{2(\gamma-\beta)}U_1{}^2/4 + r(1+r\gamma_1)U_2 \\ - r^2(\cot\theta/2 - \gamma_2)U_1 + r(2r\gamma_{12} + 2\gamma_2 + r\gamma_1\cot\theta - \cot\theta)U + e^{2(\beta-\gamma)}(\beta_2\cot\theta - \beta_{22} \\ - \beta_2{}^2) - r\gamma_1V_1 - (r\gamma_{11} + \gamma_1)V + r^2U_{12}/2 \end{aligned} \tag{7}$$

Consider the structure of these equations, without, at first, worrying about the functions of integration. Knowing γ at one value of u, Eq. (4) determines β, Eq. (5) determines U, Eq. (6) V, and one can compute the u derivative of γ from (7). Thus γ may be found at the next instant of u, and one can go again through the whole cycle. One can easily count the functions of integration, which are all independent of r. In Eq. (4), β is determined apart from an additive function $H(u, \theta)$, Eq. (5) determines U apart from two functions, $L(u, \theta)$ and $-6N(u, \theta)$, the first being an addition to u itself and the second an addition to $r^4 \exp\{2(\gamma-\beta)\}U_1$. Eq. (6) determines V apart from a function $-2m(u, \theta)$.

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[1] The addition of a subscript to a function means differentiation with respect to the appropriate coordinate ($u = 0$, $r = 1$, $\theta = 2$), unless specified otherwise.

If one assumes γ of the form

$$\gamma = \overset{\circ}{\gamma}(u, \theta) + c(u, \theta)/r + \dots \tag{8}$$

one obtains

$$\beta = H - c^2/(4r^2) + \dots,$$

$$U = L + 2He^{2(H-\overset{\circ}{\gamma})}/r - e^{2(H-\overset{\circ}{\gamma})}[A(c) + 2cH_2]/r^2 + \dots,$$

$$V = -Re^{2H}r^3/12 + (L_2 + L \cot\theta)r^2 + \dots,$$

$$c = 12\overset{\circ}{\gamma}_0 \exp[-2H]/R, \qquad A(c) = c_2 - 2\overset{\circ}{\gamma}_2 c + 2c \cot\theta. \tag{9}$$

If one adds additional terms in expansion of γ, Eqs. (4)–(6) determine expansion functions of U, V, and β through the expansion functions of γ and the integration functions H, L, m and N. Higher order expansion functions of γ may then be determined through lower order expansion functions of γ and the functions H, L, m and N from Eq. (7). The exceedingly complicated supplementary conditions (expressions for R_{20} and R_{00} may be found in appendix to the paper of Bondi et al. [2]) are then constraint equations for the functions $\overset{\circ}{\gamma}$, H, L, m and N.

The choice of γ under the form (8) does probably not give the most general form of the metric of a space-time fulfilling the "asymptotically de Sitter" condition, as stated at the beginning of this section. However, such a choice of γ in Bondi et al. [2] lead to a metric which was, "at infinity", the Minkowski metric + gravitational radiation. One would be interested in similar solutions in the "asymptotically de Sitter" case. The existence of terms increasing with r in γ would cause drastic changes in the $d\theta^2$ and $d\phi^2$ terms at infinity. It is not probable that gravitational radiation could cause such strong effects. Therefore the form (8) of the expansion of γ may be interpreted as a "radiation condition". It will be seen in Section 4.2 that even such a condition gives rise to a change in the structure of the boundary.

3. *Boundary conditions*

The metrics of asymptotically de Sitter space-times contain at least three arbitrary functions of variables u and θ. In order to know whether different metrics correspond to different space-times, one has to know whether it is possible to set to zero, by a coordinate transformation, some of the functions appearing in the metric, for each metric.

One can get rid of the function L by performing a transformation

$$\bar{\theta} = \alpha(u, \theta), \qquad \bar{r} = r(\alpha_\theta \sin\alpha/\sin\theta)^{1/2},$$

where the function α is a solution of the equation $\alpha_u = L(u, \alpha)$ (this equation has always a solution when L is suitably regular [8]). If the functions $\overset{\circ}{\gamma}$ and H are analytical it may be shown, using Cauchy-Kowalewska theorem [12], that one can always find a coordinate transformation of the form

$$
\begin{aligned}
u &= \alpha(\bar{u}, \bar{\theta}) + \overset{1}{\alpha}(\bar{u}, \bar{\theta})/\bar{r} + O(1/\bar{r}^2), \\
r &= K(\bar{u}, \bar{\theta})\bar{r} + \varrho(\bar{u}, \bar{\theta}) + O(1/\bar{r}), \\
\theta &= g(\bar{u}, \bar{\theta}) + \overset{1}{g}(\bar{u}, \bar{\theta})/\bar{r} + O(1/\bar{r}^2),
\end{aligned}
\tag{10}
$$

which leads to a metric with $\bar{H} = f(\bar{\mathring{\gamma}})$ (f being an arbitrary function, in particular f can be equal to zero), or to a metric with $\bar{\mathring{\gamma}} = 0$, $\bar{H}$ — arbitrary. For practical reasons, the coordinates for which $\bar{\mathring{\gamma}} = 2\bar{H}$ will be used.

One can ask whether there exist coordinates in which $\mathring{\gamma}$, L and H disappear simultaneously. Under the basic assumption of this paper, that all coordinate transformations preserving the form of the metric for $r \to \infty$ are of the form (10), it can be shown that such coordinates do exist only for very specifical $\mathring{\gamma}$.

Assuming the coordinate transformation $\bar{u}\bar{r}\bar{\theta}\bar{\phi} \to ur\theta\phi$ of the form (10) ($ur\theta\phi$ are so chosen, that $H = \mathring{\gamma}/2$) and demanding that in coordinates $\bar{u}\bar{r}\bar{\theta}\bar{\phi}$ $\bar{\mathring{\gamma}} = \bar{H} = 0$, one obtains, in the highest order in $1/\bar{r}$, the following set of equations for the expansion functions:

$$
\begin{aligned}
&\text{(a)}\ K^2 e^{2\mathring{\gamma}}\{R\alpha_{\bar{u}}^2/12 + g_{\bar{u}}^2\} = R/12, \\
&\text{(b)}\ K^2 e^{2\mathring{\gamma}}\{R\alpha_{\bar{\theta}}^2/12 + g_{\bar{\theta}}^2\} = 1, \\
&\text{(c)}\ \sin^2\bar{\theta} = K^2 e^{-2\mathring{\gamma}} \sin^2 g, \\
&\text{(d)}\ RK\tilde{\alpha}^2/12 + 2\tilde{\alpha} + K\tilde{g}^2 = 0, \qquad \tilde{\alpha} = e^{\mathring{\gamma}}\overset{1}{\alpha}, \qquad \tilde{g} = e^{\mathring{\gamma}}\overset{1}{g}, \\
&\text{(e)}\ Ke^{\mathring{\gamma}}\{(1 + RK\tilde{\alpha}/12)\alpha_{\bar{u}} + K\tilde{g}g_{\bar{u}}\} = 1, \\
&\text{(f)}\ \frac{R}{12}\alpha_{\bar{u}}\alpha_{\bar{\theta}} + g_{\bar{u}}g_{\bar{\theta}} = 0, \\
&\text{(g)}\ (RK\tilde{\alpha}/12 + 1)\alpha_{\bar{\theta}} + K\tilde{g}g_{\bar{\theta}} = 0.
\end{aligned}
\tag{11}
$$

From (11g), (11f), (11b) and (11e) one obtains

$$
\begin{aligned}
&\text{(g}'\text{)}\ \alpha_{\bar{\theta}} = -\overset{1}{g}, \\
&\text{(f}'\text{)}\ \alpha_{\bar{u}} = \varepsilon(e^{-2\mathring{\gamma}}K^{-2} - R\overset{1}{g}{}^2/12)^{1/2}, \qquad \varepsilon = \pm 1, \\
&\text{(b}'\text{)}\ g_{\bar{u}} = \alpha_{\bar{u}}, \\
&\text{(e}'\text{)}\ g_{\bar{u}} = R\overset{1}{g}/12.
\end{aligned}
\tag{11}
$$

It appears, that the Eq. (11a) is automatically fulfilled when (11e′) and (11f′) are fulfilled. From (11b′), (11f′), (11g′), (11a′) and (11c) one obtains

$$g_{\bar{u}}^2 + Rg_{\bar{\theta}}^2/12 = R\sin^2 g \sin^{-2}\bar{\theta}\exp\{-4\mathring{\gamma}(\alpha, g)\}/12,$$

$$g_{\bar{u}} = -R\alpha_{\bar{\theta}}/12, \qquad g_{\bar{\theta}} = \alpha_{\bar{u}}. \tag{12}$$

Let

$$\bar{x} = (R/12)^{1/2}\bar{u}, \qquad \mathring{\alpha} = (R/12)^{1/2}\alpha.$$

Now Eq. (12) reads

$$\text{(a)}\quad g_{\bar{x}}^2 + g_{\bar{\theta}}^2 = \sin^2 g \sin^{-2}\bar{\theta}\exp\{-4\mathring{\gamma}(\alpha, g)\},$$

$$\text{(b)}\quad g_{\bar{x}} = -\mathring{\alpha}_{\bar{\theta}},$$

$$\text{(c)}\quad g_{\bar{\theta}} = \mathring{\alpha}_{\bar{x}}. \tag{13}$$

Eqs. (13b) and (13c) show, that the function

$$h = \mathring{\alpha} + ig$$

is a holomorphic function of the variable $\bar{x}+i\bar{\theta}$. Clearly, $K \neq 0$ for any $\bar{u}$ and $\bar{\theta}$, thus Eq. (11c) shows that $g(\bar{u}, \bar{\theta} = 0) = 0$. Then $h|_{\bar{\theta}=0} = \mathring{\alpha}$, so the expansion coefficients of h are real. It follows that $(h(x+i\theta))^* = h(x-i\theta)$ (* denotes complex conjugacy).

Expressing g by h in Eq. (13a) one obtains

$$h'(\bar{x}+i\bar{\theta})h'(\bar{x}-i\bar{\theta}) = \sin^2\{[h(\bar{x}+i\bar{\theta})-h(\bar{x}-i\bar{\theta})]/2i\}$$

$$\times\sin^{-2}\bar{\theta}\exp\{-4\mathring{\gamma}(\mathring{\alpha}, g)\}. \tag{14}$$

Using

$$\lim_{\bar{\theta}\to 0}\sin\{[h(\bar{x}+i\bar{\theta})-h(\bar{x}-i\bar{\theta})]/2i\}\sin^{-1}\bar{\theta} = h'(\bar{x}) \tag{15}$$

and the regularity of $\mathring{\gamma}$ for $\bar{\theta} = 0$, one obtains from (14)

$$h'(\bar{x})^2 = h'(\bar{x})^2\exp\{-4\mathring{\gamma}(2h, 0)\}. \tag{16}$$

A necessary condition for the existence of solutions of Eq. (14) is thus

$$\exp\{\mathring{\gamma}(u, 0)\} = 1.$$

This shows that if we do not impose any supplementary conditions on the form of asymptotically de Sitter metrics, then, in general, one cannot find coordinates in which L, H and $\mathring{\gamma}$ disappear simultaneously.

4. AS transformations

4.1. AS transformations for general asymptotically de Sitter space-times

Assuming the transformation $ur\theta\phi \to \bar{u}\bar{r}\bar{\theta}\bar{\phi}$ in the form (10), and demanding $\overset{\circ}{\gamma} = 2H$, $\bar{\overset{\circ}{\gamma}} = 2\bar{H}$, in coordinates $ur\theta\phi$, $\bar{u}\bar{r}\bar{\theta}\bar{\phi}$ respectively, one obtains the following set of equations for the transformation functions:

$$\begin{aligned}
&\text{(a) } K^2 e^{2\overset{\circ}{\gamma}-2\bar{\overset{\circ}{\gamma}}}[R\alpha_{\bar{u}}^2/12+g_{\bar{u}}^2] = R/12,\\
&\text{(b) } K^2 e^{2\overset{\circ}{\gamma}-2\bar{\overset{\circ}{\gamma}}}[R\alpha_{\bar{\theta}}^2/12+g_{\bar{\theta}}^2] = 1,\\
&\text{(c) } \sin^2\bar{\theta} = K^2 e^{-2(\overset{\circ}{\gamma}-\bar{\overset{\circ}{\gamma}})}\sin^2 g,\\
&\text{(d) } RK\tilde{\alpha}^2/12+2\tilde{\alpha}+K\tilde{g}^2 = 0 \qquad \tilde{\alpha} = e^{\overset{\circ}{\gamma}}\overset{1}{\alpha} \qquad \tilde{g} = e^{\overset{\circ}{\gamma}}\overset{1}{g},\\
&\text{(e) } Ke^{\overset{\circ}{\gamma}-\bar{\overset{\circ}{\gamma}}}[(1+RK\tilde{\alpha}/12)\alpha_{\bar{u}}+K\tilde{g}g_{\bar{u}}] = 1,\\
&\text{(f) } R\alpha_{\bar{u}}\alpha_{\bar{\theta}}/12+g_{\bar{u}}g_{\bar{\theta}} = 0,\\
&\text{(g) } (RK\tilde{\alpha}/12+1)\alpha_{\bar{\theta}}+K\tilde{g}g_{\bar{\theta}} = 0.
\end{aligned} \tag{17}$$

These equations are Eqs. (11) with $\overset{\circ}{\gamma}(\alpha, g)$ replaced by $\overset{\circ}{\gamma}(\alpha, g)-\bar{\overset{\circ}{\gamma}}(\bar{u}, \bar{\theta})$. Thus the analysis can be carried out exactly in the same way as for Eqs. (11), and one obtains

$$e^{4\bar{\overset{\circ}{\gamma}}(\bar{u},\bar{\theta})} = (g_{\bar{x}}^2+g_{\bar{\theta}}^2)e^{4\overset{\circ}{\gamma}(\alpha,g)}\sin^2\bar{\theta}\sin^{-2} g, \tag{18}$$

$$g_{\bar{x}} = -\overset{\circ}{\alpha}_{\bar{\theta}}, \qquad g_{\bar{\theta}} = \overset{\circ}{\alpha}_{\bar{x}}. \tag{19}$$

Eqs. (19) show, that there exists a harmonic function $\phi(\bar{x}, \bar{\theta})$ such that

$$g = \phi_{\bar{x}}, \qquad \overset{\circ}{\alpha} = \phi_{\bar{\theta}}, \qquad (\phi_{\bar{x}\bar{x}}+\phi_{\bar{\theta}\bar{\theta}} = 0), \tag{20}$$

$\overset{1}{g}$ can be calculated from

$$\overset{1}{g} = (12/R)^{1/2}\phi_{\bar{\theta}\bar{\theta}}$$

then K is obtained from

$$K = \sin\bar{\theta}\sin^{-1} g\exp\{\overset{\circ}{\gamma}(\alpha, g)-\bar{\overset{\circ}{\gamma}}(\bar{\theta}, \bar{x})\}$$

and Eq. (17d) determines $\overset{1}{\alpha}$. Lower order equations determine lower order expansion coefficients, and one obtains a realisation of the AS group of the de Sitter space-time. There are simpler transformation groups isomorphic to this group. Composing two AS transformations, one obtains

$$\begin{aligned}
u &= \alpha(\bar{u}, \bar{\theta})+\overset{1}{\alpha}(\bar{u}, \bar{\theta})/\bar{r}+ \ldots = \alpha(\bar{\alpha}(\bar{\bar{u}}, \bar{\bar{\theta}})+\overset{1}{\bar{\alpha}}(\bar{\bar{u}}, \bar{\bar{\theta}})/\bar{\bar{r}}+ \ldots,\\
&\bar{g}(\bar{\bar{u}}, \bar{\bar{\theta}})+\overset{1}{\bar{g}}(\bar{\bar{u}}, \bar{\bar{\theta}})/\bar{\bar{r}}+ \ldots)+\overset{1}{\alpha}(\bar{\alpha}+\overset{1}{\bar{\alpha}}/\bar{\bar{r}}+ \ldots, \bar{g}+\overset{1}{\bar{g}}/\bar{\bar{r}}+ \ldots)/(\bar{K}\bar{\bar{r}}+\bar{\varrho}+ \ldots)\\
&= \alpha(\bar{\alpha}, \bar{g})+\mu(\bar{\bar{u}}, \bar{\bar{\theta}})/\bar{\bar{r}}+ \ldots,
\end{aligned}$$

$$\theta = g(\bar{u}, \bar{\theta}) + \overset{1}{g}(\bar{u}, \bar{\theta})/\bar{r} + \ldots = g(\bar{\alpha}, \bar{g}) + \varepsilon(\bar{\bar{u}}, \bar{\bar{\theta}})/\bar{\bar{r}} + \ldots,$$

$$r = K(\bar{u}, \bar{\theta})\bar{r} + \varrho + \ldots = K(\bar{\alpha}, \bar{g})\bar{K}(\bar{\bar{u}}, \bar{\bar{\theta}})\bar{\bar{r}} + \nu(\bar{\bar{u}}, \bar{\bar{\theta}}) + \ldots, \tag{21}$$

where μ, ν and ε are functions of $\overset{1}{\alpha}$, $\overset{\bar{1}}{\alpha}$, $\overset{1}{g}$, $\overset{\bar{1}}{g}$, ϱ, K and of the derivatives of α, g and K. Eqs. (21) show, that the group of transformations

$$u = \alpha(\bar{u}, \bar{\theta}), \quad \theta = g(\bar{u}, \bar{\theta}), \quad r = K(\bar{u}, \bar{\theta})\bar{r} \tag{22}$$

is isomorphic to the AS group (one assumes that the freedom in lower order coefficients, if there is any, is irrelevant from the physical point of view). When α and g determine K univocally, one obtains another group of transformations isomorphic to the AS group:

$$\bar{u} = \alpha(u, \theta), \quad \bar{\theta} = g(u, \theta). \tag{23}$$

The AS group is usually identified with its realisation (23). For the boundary conditions

$$\lim_{r\to\infty} \gamma = \mathring{\gamma}, \quad \lim_{r\to\infty} \beta = \mathring{\gamma}/2, \quad \lim_{r\to\infty} U = 0 \tag{24}$$

one obtains, for the AS group, the group of transformations

$$\bar{x} = \phi_\theta(x, \theta), \quad \bar{\theta} = \phi_x(x, \theta), \quad \phi_{xx} + \phi_{\theta\theta} = 0 \tag{25}$$

or, since holomorphic functions are complex gradients of harmonic functions

$$\bar{z} = f(z), \quad z = x + i\theta, \tag{26}$$

f — any holomorphic function.

Solutions (40)–(42) are also of the form (25), thus the group (25) contains as a subgroup the group of conformal transformations of a three-dimensional sphere leaving the ϕ-angle unchanged (which will be called the axi-symmetric $O(1,4)$ group).

The choice $\mathring{\gamma} = 2H$ does not seem to be any better nor worse than the choice $\mathring{\gamma} = 0$, H — arbitrary, or $H = 0$, $\mathring{\gamma}$ — arbitrary, or $H = f(\mathring{\gamma})$. Therefore one would be interested in knowing what would be the AS group obtained for another choice of boundary conditions. Unfortunately, all attemps to solve the differential equations for the case $\mathring{\gamma} \neq 2H$ have failed. This is due to the fact that the equivalent of Eq. (19) for cases $\mathring{\gamma} \neq 2H$ have multiplicative terms $e^{\mathring{\gamma}(\alpha, g)}$ and $e^{\bar{\mathring{\gamma}}(\bar{u}, \bar{\theta})}$, which complicate these equations considerably. However one can show, under certain assumptions, that all these groups are isomorphic.

Let Z_f be a coordinate transformation:

$$Z_f(\mathring{\gamma}, \mathring{\gamma}') : ur\theta\phi \to u'r'\theta'\phi',$$

(where $ur\theta\phi$, $u'r'\theta'\phi'$ are so chosen, that $\mathring{\gamma} = 2H$, $H' = f(\mathring{\gamma}')$ respectively). Having one $\mathring{\gamma}'_0$ [2] resulting from such a transformation for a given $\mathring{\gamma}_0$, one can obtain a transformation

[2] The subscript 0 does not mean differentiation here.

leading from $\mathring{\gamma}$ to $\mathring{\gamma}'$ from the formula:

$$Z_f(\mathring{\gamma}, \mathring{\gamma}') = g_f(\mathring{\gamma}'_0, \mathring{\gamma}') \circ Z_f(\mathring{\gamma}_0, \mathring{\gamma}'_0) \circ g_{1/2}(\mathring{\gamma}, \mathring{\gamma}_0),$$

(where $g_f \in G_f$ — the AS group for boundary conditions $H = f(\mathring{\gamma})$, and $g_f(\mathring{\gamma}_1, \mathring{\gamma}_2)$ is a coordinate transformation leading from a metric with a function $\mathring{\gamma}_1$ to a metric with a function $\mathring{\gamma}_2$). This shows, that one can find a Z_f transformation for every $\mathring{\gamma}$ and $\mathring{\gamma}'$, provided that there exist $\mathring{\gamma}_0$ and $\mathring{\gamma}'_0$, such that $g_{1/2}(\mathring{\gamma}, \mathring{\gamma}_0)$ and $g_f(\mathring{\gamma}'_0, \mathring{\gamma}')$ exist.

Let $\{\mathring{\gamma}\}^{1/2}_{\mathring{\gamma}_0}$ be the set of all possible $\mathring{\gamma}$ functions which can be obtained by a $G_{1/2}$ transformation from the metric with $H = \mathring{\gamma}_0/2$, and let $\{\mathring{\gamma}'\}^{f}_{\mathring{\gamma}_0'}$ be the set of all possible $\mathring{\gamma}$ functions which can be obtained by a G_f transformation from the metric with $H' = f(\mathring{\gamma}'_0)$, and $\mathring{\gamma}'_0$ is such, that the metric with $H' = f(\mathring{\gamma}'_0)$ can be obtained from the metric $H = \mathring{\gamma}_0/2$ by a Z_f transformation. If there exists a bijective mapping ϱ

$$\{\mathring{\gamma}\}^{1/2}_{\mathring{\gamma}_0} \xrightarrow{\varrho} \{\mathring{\gamma}'\}^{f}_{\mathring{\gamma}_0'}, \tag{27}$$

the mapping

$$G_{1/2} \ni g_{1/2}(\mathring{\gamma}, \mathring{\gamma}') \xrightarrow{h} g_f(\varrho(\mathring{\gamma}), \varrho(\mathring{\gamma}')) = Z_f(\mathring{\gamma}', \varrho(\mathring{\gamma}'))$$
$$\circ\, g_{1/2}(\mathring{\gamma}, \mathring{\gamma}') \circ Z_f^{-1}(\mathring{\gamma}, \varrho(\mathring{\gamma})) \in G_f$$

is a homomorphism between $G_{1/2}$ and G_f. The mapping

$$G_f \ni g_f(\varrho(\mathring{\gamma}), \varrho(\mathring{\gamma}')) \to g_{1/2}(\mathring{\gamma}, \mathring{\gamma}') = Z_f^{-1}(\mathring{\gamma}', \varrho(\mathring{\gamma}'))$$
$$\circ\, g_f(\varrho(\mathring{\gamma}), \varrho(\mathring{\gamma}')) \circ Z_f(\mathring{\gamma}, \varrho(\mathring{\gamma})) \in G_{1/2}$$

is the homorphism inverse to h, thus $G_{1/2}$ and G_f are isomorphic. Similarly, one can show that the AS group for boundary conditions $\mathring{\gamma} = 0$, H — arbitrary is isomorphic to $G_{1/2}$.

One can provide arguments for the existence of the mapping (27). For any given $\mathring{\gamma}$, the problem of finding $Z_f(\mathring{\gamma}, \mathring{\gamma}')$ can be reduced to one partial differential equation, depending only upon $\mathring{\gamma}$. Since $\mathring{\gamma} \in \{\mathring{\gamma}\}^{1/2}_{\mathring{\gamma}_0}$, for some $\mathring{\gamma}_0$, there exists a solution $Z_f(\mathring{\gamma}, \mathring{\gamma}')$ for every $\mathring{\gamma}' \in \{\mathring{\gamma}'\}^{f}_{\mathring{\gamma}_0'}$. Changing the boundary conditions on any fixed non-characteristic surface of the differential equation for $Z_f(\mathring{\gamma}, \mathring{\gamma}')$, one obtains all possible $\mathring{\gamma}'$. Similarly, for a given $\mathring{\gamma}' \in \{\mathring{\gamma}'\}^{f}_{\mathring{\gamma}_0'}$, the solutions of the differential equation for $Z_f^{-1}(\mathring{\gamma}, \mathring{\gamma}')$, depending only upon $\mathring{\gamma}'$, may be found for all $\mathring{\gamma} \in \{\mathring{\gamma}\}^{1/2}_{\mathring{\gamma}_0}$. This suggests that, $\{\mathring{\gamma}\}^{1/2}_{\mathring{\gamma}_0}$ and $\{\mathring{\gamma}'\}^{f}_{\mathring{\gamma}_0'}$ should have "the same dimension". Now if, for different $\mathring{\gamma}$, we arbitrarily choose to adopt some fixed boundary value for the solutions of the $Z_f(\mathring{\gamma}, \mathring{\gamma}')$ equation on some arbitrarily fixed non-characteristic surface, this should provide the one-to-one correspondence between elements of $\{\mathring{\gamma}\}^{1/2}_{\mathring{\gamma}_0}$ and of $\{\mathring{\gamma}'\}^{f}_{\mathring{\gamma}_0'}$.

It is interesting to understand why the results obtained above differ from the predictions of Penrose's theory. In coordinates where $L = 0$, the metric of an asymptotically de Sitter space-time takes the form

$$ds^2 = [-Re^{4H}r^2/12+O(1)]du^2+[2e^{2H}+O(1/r^2)]dudr$$
$$+O(r)dud\theta-r^2\{[r^{2\mathring{\gamma}}+O(1/r)]d\theta^2+\sin^2\theta[e^{-2\mathring{\gamma}}+O(1/r)]d\phi^2\}. \tag{28}$$

Performing a coordinate transformation

$$\sinh(t/\alpha) = \sinh(u/\alpha)+\bar{r}\cosh(u/\alpha)/\alpha,$$
$$\sin\chi = \{(1+\bar{r}^{-2}\alpha^2)\cosh^2(u/\alpha)+2\alpha\cosh(u/\alpha)\sinh(u/\alpha)/\bar{r}\}^{-1/2},$$
$$\bar{r} = e^{2H}r \tag{29}$$

one obtains, for large t

$$ds^2 = [1+O(e^{-t/\alpha})]dt^2+O(1)d\chi dt-\alpha^2e^{2t/\alpha}\{[1+O(e^{-t/\alpha})]d\chi^2+O(e^{-2t/\alpha})$$
$$\times d\theta dt+O(e^{-t/\alpha})d\theta d\chi+\sin^2\chi e^{-4H}[(e^{2\mathring{\gamma}}+O(e^{-t/\alpha}))d\theta^2$$
$$+(e^{-2\mathring{\gamma}}+O(e^{-t/\alpha}))\sin^2\theta d\phi^2]\}. \tag{30}$$

Performing the coordinate transformation (2) one obtains the metric of the boundary by passing to the limit $t \to \pm\pi/2$:

$$ds^2 = d\chi^2+\sin^2\chi e^{-4\tilde{H}}\{e^{2\tilde{\gamma}}d\theta^2+e^{-2\tilde{\gamma}}\sin^2\theta d\phi^2\}, \tag{31}$$

where

$$\tilde{H} = H(\alpha\ln\{(\cos\chi+1)/\sin\chi\}, \theta),$$
$$\tilde{\gamma} = \mathring{\gamma}(\alpha\ln\{(\cos\chi+1)/\sin\chi\}, \theta).$$

It may be shown, using Cauchy-Kowalewska theorem, that, for all analytical $\mathring{\gamma}$, $H=f(\mathring{\gamma})$ (or for all analytical H, $\mathring{\gamma} = 0$), each of the metrics (31) is conformally equivalent to the metric of some three-dimensional hypersurface embedded in a four-dimensional space, given by the equation $r = r(\chi, \theta)$ ($r\chi\theta\phi$ — standard four-dimensional spherical coordinates). Such hypersurfaces are not conformally equivalent, thus the boundaries of space-times of the family of asymptotically de Sitter space-times are different for different spaces. If one wants to compare the results of this method with Penrose's results, one has to limit the analysis to those of asymptotically de Sitter space-times, boundaries of which are three dimensional spheres, that is to spaces, for which there exist coordinates, in which $\mathring{\gamma} = H = L = 0$.

4.2. Analysis of solutions in the case $\mathring{\gamma} = H = L = 0$

In the case $\mathring{\gamma} = H = L = 0$, the function c from the expansion of γ disappears ($c = (12/R)\mathring{\gamma}_0$), thus one obtains

$$\gamma = \overset{2}{c}/r^2+\ldots.$$

Now in U appears a term $A_0(\overset{2}{c})\ln r/r^3 (A_0(\overset{2}{c}) = \overset{2}{c}_2+2\overset{2}{c}\cot\theta)$, and V has $\ln^2 r/r^5$ terms. In Eq. (7) one obtains a term $A_0^2(\overset{2}{c})\ln^2 r/r^4$ from the $r^4 \exp\{2(\gamma-\beta)\}U_1^2/4$ term; other terms containing $\ln^2 r$ appear in the form $\ln^2 r/r^5$, or with higher powers of $1/r$, which shows that $A_0(\overset{2}{c}) = 0$. The regularity conditions for $\overset{2}{c}$ show then that $\overset{2}{c} = 0$. Subsequently γ is of the form

$$\gamma = \overset{3}{c}/r^3+\overset{4}{c}/r^4+O^*(r^{-5}). \tag{32}$$

From the main equations (4)–(7), one obtains

$$\begin{aligned}
&\text{(a) } \beta = -3\overset{3}{c}{}^2/(4r^6)-12\overset{3}{c}\overset{4}{c}/(7r^7)+O^*(1/r^8),\\
&\text{(b) } U = 2N/r^3+3A_0(\overset{3}{c})/(2r^4)+4A_0(\overset{4}{c})/(5r^5)+O^*(1/r^6),\\
&\text{(c) } V = -Rr^3/12+r-2m-(N_2+N\cot\theta)/r\\
&\qquad -(3\overset{3}{c}\cot\theta+\overset{3}{c}_{22}-2\overset{3}{c})/(2r^2)+\overset{3}{V}(\overset{3}{c},\overset{4}{c})/r^3+O^*(1/r^4)\,^3,\\
&\text{(d) } R\overset{4}{c}/3 = 4\overset{3}{c}_0+N_2-N\cot\theta
\end{aligned} \tag{33}$$

and from the supplementary conditions

$$\begin{aligned}
&\text{(a) } RA_0(\overset{3}{c})/2 = m_2+3N_0 \Leftrightarrow (\overset{3}{c}\sin\theta)_\theta = 2\sin^2\theta(m_2+3N_0)/R,\\
&\text{(b) } 4m_0+R(N_2+N\cot\theta) = 0 \Leftrightarrow R(\sin\theta N)_\theta = -4\sin\theta m_0.
\end{aligned} \tag{34}$$

With any given $m = m(u,\theta)$, Eq. (34b) determines N univocally (the integration function $N_0 = f(u)/\sin\theta$ must be equal to zero from regularity conditions). Then Eq. (34a) determines $\overset{3}{c}$ univocally, from (33d) one can determine $\overset{4}{c}$. If one expands γ in $O(r^{-(n+1)})$, Eq. (4) determines β in $O(r^{-(n+4)})$, Eq. (5) determines U in $O(r^{-(n+2)})$, and Eq. (6) determines V in $O(r^{-n})$. The $\overset{n}{c}$ function of the expansion of γ appears in β in $r^{-(n+3)}$ and lower terms, in U in $r^{-(n+1)}$ and lower terms, in V in $r^{-(n-1)}$ and lower terms. Eq. (7) determines $\overset{n}{c}$ algebraically through m and $\overset{i}{c}$, for $i = 3, \ldots, n-1$. In this way, for any given suitably regular m, one can determine the functions γ, β, U and V univocally. Unfortunately, one cannot state anything about the convergence of such series.

In the case $m = \text{const.}$ ($\rightarrow \gamma = \beta = U = 0$) one obtains an exact solution of the field equations with a cosmological constant:

$$ds^2 = (1-Rr^2/12-2m/r)du^2+2du\,dr+r^2(d\theta^2+\sin^2\theta\, d\phi^2)$$

which is the Weyl-Trefftz [5] metric. One interprets the $2m/r$ term as the gravitational field of a point mass in a space with a "background" de Sitter metric [5]. In the general case one can then interpret $m = m(u,\theta)$ as Bondi's "mass aspect" [2].

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[3] The exact form of $\overset{3}{V}$ is irrelevant for our purposes.

4.3. AS transformations in Bondi's coordinates, in the case $\overset{\circ}{\gamma} = H = L = 0$

Assuming the AS transformations for the boundary conditions

$$\text{(a) } \lim_{r\to\infty} r^3\gamma = \overset{3}{c}(u, \theta),$$

$$\text{(b) } \lim_{r\to\infty} r^6\beta = -3\overset{3}{c}{}^2/4,$$

$$\text{(c) } \lim_{r\to\infty} r^3 U = 2N(u, \theta),$$

$$\text{(d) } \lim_{r\to\infty} V r^{-3} = -R/12 \tag{35}$$

of the form (10), one obtains the system of Eqs. (17) with $\overset{\circ}{\gamma}$ and $\overset{\circ}{\gamma}$ set to zero.

These equations can be solved as follows: from (17b), (17e), (17g) and (17f) one obtains

$$\text{(b}'\text{) } g_{\bar{u}} = R\overset{1}{g}/12,$$

$$\text{(e}'\text{) } \alpha_{\bar{u}} = \varepsilon(1 - RK^2\overset{1}{g}{}^2/12)^{1/2}/K, \quad \varepsilon = \pm 1,$$

$$\text{(f}'\text{) } g_{\bar{\theta}} = \alpha_{\bar{u}},$$

$$\text{(g}'\text{) } \alpha_{\bar{\theta}} = -\overset{1}{g}. \tag{36}$$

Differentiating (17c) with respect to $\bar{u}$ and using (36b′), one obtains

$$\text{(c}'\text{) } R\overset{1}{g}/12 = -\delta K_{\bar{u}} \sin\theta (K^2 - \sin^2\theta)^{-1/2} K^{-1}, \quad \delta = \operatorname{sgn}(\sin g). \tag{36}$$

Differentiating (17c) with respect to $\bar{\theta}$, using (36c′) and (36f′), one obtains

$$\text{(a) } K_{\bar{u}}^2 + RK_{\bar{\theta}}^2/12 = R(K^2 + 2KK_{\bar{\theta}} \cot\bar{\theta} - 1)/12. \tag{37}$$

Substituting (36c′) in (36e′) and (36g′), using (36f′) one obtains

$$\text{(g}''\text{) } \alpha_{\bar{\theta}} = \frac{12\delta K_{\bar{u}} \sin\bar{\theta}}{RK(K^2 - \sin^2\bar{\theta})^{1/2}},$$

$$\text{(e}''\text{) } \alpha_{\bar{u}} = \frac{\delta(K\cot\bar{\theta} - K_{\bar{\theta}})\sin\bar{\theta}}{K(K^2 - \sin^2\bar{\theta})^{1/2}}. \tag{36}$$

Integrability conditions for α show that

$$\text{(b) } (\ln K)_{\bar{u}\bar{u}} + \frac{R}{12}(\ln K)_{\bar{\theta}\bar{\theta}} = 0. \tag{37}$$

A change of variables $\bar{u} = (R/12)^{1/2}\bar{x}$ leads to

$$\text{(a}'\text{) } K_{\bar{x}}^2 + K_{\bar{\theta}}^2 = K^2 + 2KK_{\bar{\theta}} \cot\bar{\theta} - 1,$$

$$\text{(b}'\text{) } (\ln K)_{\bar{x}\bar{x}} + (\ln K)_{\bar{\theta}\bar{\theta}} = 0. \tag{37}$$

Every analytical solution of Eq. (37b′) may be written in the form

$$\ln K = f(\bar{x}+i\bar{\theta})+f(\bar{x}-i\bar{\theta}), \tag{38}$$

where f is an arbitrary real analytical function of a real variable. Substituting (38) into (37a′) one obtains

$$4f'(\bar{x}+i\ )f'(\bar{x}-i\bar{\theta}) = 1+2i(f'(\bar{x}+i\bar{\theta})-f'(\bar{x}-i\bar{\theta}))\cot\bar{\theta}$$
$$-\exp\{-2[f(\bar{x}+i\bar{\theta})+f(\bar{x}-i\bar{\theta})]\}. \tag{39}$$

Since

$$\lim_{\theta\to 0}\{(f'(x+i\theta)-f'(x-i\theta)\cot\theta\} = 2if''(x)$$

Eq. (39) on the hypersurface $\bar{\theta} = 0$ reads

$$4f''+4f'^2 = 1-\exp\{-4f\}.$$

This differential equation can be reduced to a first order equation by change of variables

$$p = df/dx = f', \quad f'' = p\,dp/df.$$

Solving for f, one obtains 3 independent solutions of equations (37a′–b′)

$$K = (1-2\tau e^{\bar{x}}\cos\bar{\theta}+\tau^2 e^{2\bar{x}})^{1/2},$$
$$\sqrt{R/12}\,\alpha = \bar{x}-\ln(1-2\tau e^{\bar{x}}\cos\bar{\theta}+\tau^2 e^{2\bar{x}})/2, \tag{40}$$
$$g = \arcsin(\sin\bar{\theta}/K);$$

$$K = (1-2\beta e^{-\bar{x}}\cos\bar{\theta}+\beta^2 e^{-2\bar{x}})^{1/2},$$
$$\sqrt{R/12}\,\alpha = \bar{x}+\ln(1-2\beta e^{-\bar{x}}\cos\bar{\theta}+\beta^2 e^{-2\bar{x}})/2, \tag{41}$$
$$g = \arcsin(\sin\bar{\theta}/K);$$

$$K = 1, \quad (R\ 12)^{1/2}\alpha = \bar{x}+\gamma, \quad g = \bar{\theta}. \tag{42}$$

The AS transformations form a three parameter group generated by transformations (40)–(42). The Lie algebra of the realisation (23) of the AS group is spanned on three vectors

$$\zeta_1 = \partial/\partial\tau|_{\tau=0} = -e^{\bar{x}}[\cos\bar{\theta}\,\partial/\partial\bar{x}+\sin\bar{\theta}\,\partial/\partial\bar{\theta}],$$
$$\zeta_2 = \partial/\partial\beta|_{\beta=0} = e^{-\bar{x}}[-\cos\bar{\theta}\,\partial/\partial\bar{x}+\sin\bar{\theta}\,\partial/\partial\bar{\theta}],$$
$$\zeta_3 = \partial/\partial\gamma|_{\gamma=0} = \partial/\partial\bar{x}. \tag{43}$$

Their commutation relations are

$$[\zeta_1,\zeta_2] = -2\zeta_3, \quad [\zeta_1,\zeta_3] = -\zeta_1, \quad [\zeta_2,\zeta_3] = \zeta_2. \tag{44}$$

The Lie algebra of the axi-symmetric $O(1,4)$ is spanned on three vectors

$$\begin{aligned}\eta_1 &= -\cos\theta\,\partial/\partial\chi+\mathrm{ctg}\,\chi\sin\theta\,\partial/\partial\theta,\\ \eta_2 &= -\sin\chi\cos\theta\,\partial/\partial t-\mathrm{th}\,t\cos\theta\cos\chi\,\partial/\partial\chi+\mathrm{th}\,t\sin\theta/\sin\chi\,\partial/\partial\theta,\\ \eta_3 &= \cos\chi\,\partial/\partial t-\mathrm{th}\,t\sin\chi\,\partial/\partial\chi\end{aligned} \tag{45}$$

with the following commutation relations

$$[\eta_1,\eta_2]=\eta_3,\quad [\eta_2,\eta_3]=-\eta_1,\quad [\eta_1,\eta_3]=-\eta_2. \tag{46}$$

The mapping

$$f(\zeta_1)=\eta_1+\eta_2,\quad f(\zeta_2)=\eta_1-\eta_2,\quad f(\zeta_3)=\eta_3$$

is an isomorphism between the Lie algebras (43) and (45), hence the group of transformations (40)–(42) is locally isomorphic to the axi-symmetric $O(1,4)$ group.

4.4. AS transformations in "natural boundary" coordinates, in the case $\overset{\circ}{\gamma} = H = L = 0$

Since the boundary of the de Sitter space-time is a 3-dimensional sphere, the coordinates in which "the boundary looks like a 3-dimensional sphere" seem to be appropriate to study asymptotic symmetries. Using the coordinate transformation (29) one obtains, for large t

$$\begin{aligned}ds^2 &= (1-2me^{-3t/\alpha}/(\alpha\sin^3\chi)+\ldots)dt^2+(4me^{-2t/\alpha}/\sin^3\chi+\ldots)dtd\chi\\ &\quad+(4Ne^{-2t/\alpha}/(\alpha\sin^2\chi)+\ldots)dtd\theta-\alpha^2e^{2t/\alpha}d_3\Omega^2,\\ d_3\Omega^2 &= (1+2me^{-3t/\alpha}/(\alpha\sin^3\chi)+\ldots)d\chi^2+(4Ne^{-3t/\alpha}/(\alpha^2\sin\chi)\\ &\quad+\ldots)d\chi d\theta+\sin^2\chi\{(1+2\overset{3}{c}e^{-3t/\alpha}/\sin^3\chi+\ldots)d\theta^2\\ &\quad+\sin^2\theta(1-2\overset{3}{c}e^{-3t/\alpha}/\sin^3\chi+\ldots)d\phi^2\}.\end{aligned} \tag{47}$$

From the assumption that the metric functions are expandable in powers of $1/r$ in Bondi's coordinates, it follows that the metric functions are expandable in powers of $e^{-t/\alpha}$. Thus one should assume AS transformations of the form

$$\begin{aligned}\chi &= h(\bar\chi,\bar\theta)+\overset{1}{h}(\bar\chi,\bar\theta)e^{-\bar t/\alpha}+\ldots,\\ \theta &= g(\bar\chi,\bar\theta)+\overset{1}{g}(\bar\chi,\bar\theta)e^{-\bar t/\alpha}+\ldots,\\ e^{t/\alpha} &= A(\bar\chi,\bar\theta)e^{\bar t/\alpha}+\overset{1}{A}(\bar\chi,\bar\theta)+\ldots,\\ &\Leftrightarrow\\ t &= \bar t+\alpha\ln A+\alpha\overset{1}{A}e^{-\bar t/\alpha}/A+\ldots.\end{aligned} \tag{48}$$

The $d\bar{\chi}d\bar{t}$ and $d\bar{\theta}d\bar{t}$ equations read

$$2\alpha^2 h\overset{1}{h}e^{\bar{t}/\alpha} = 0(e^{-2\bar{t}/\alpha}), \quad 2\alpha^2 g\overset{1}{g}e^{\bar{t}/\alpha} = 0(e^{-2\bar{t}/\alpha}).$$

Obviously $h \neq 0$, $g \neq 0$, hence

$$\overset{1}{h} = \overset{1}{g} = 0$$

which means that

$$\chi = h + \overset{2}{h}e^{-2\bar{t}/\alpha} + \ldots, \qquad \theta = g + \overset{2}{g}e^{-2\bar{t}/\alpha} + \ldots .$$

The requirement $d_3\Omega^2 \xrightarrow[t\to\infty]{} d_3\Omega_s^2$ (where $d_3\Omega_s^2$ is the metric of a three-dimensional sphere) shows that h and g must be functions of a conformal transformation. Let K be the conformal factor

$$d_3\Omega_s^2 = K^2 d_3\bar{\Omega}_s^2.$$

The $d\bar{\chi}^2$ equation shows that

$$A = K^{-1}.$$

Lower order equations determine lower order expansion functions. The AS group is thus the group of transformations

$$\bar{\chi} = h(\chi, \theta), \qquad \bar{\theta} = g(\chi, \theta),$$

where h and g are functions of a conformal transformation. This group is isomorphic to the axi-symmetric $O(1,4)$ group.

5. *Conclusions*

Penrose's theory results suggested that, in the case of space-times with boundaries at temporal infinity, AS groups were finite dimensional. This result is confirmed in this work (see note added in proof). The AS group for asymptotically de Sitter space-times appears to be the group of conformal symmetries of the three dimensional sphere. This result is apparently in perfect agreement with Penrose's theory, at least at the level of subgroups of AS groups leaving the ϕ-angle unchanged.

The three-dimensional matrics 31 are not conformally equivalent for all $\tilde{\gamma}$, $\tilde{H} = \tilde{\gamma}/2$. It can be shown, using Frobenius-Dieudonné theorem [8], that metrics 31 admit conformal symmetries only for very specific $\tilde{\gamma}$. Penrose's theory provides the AS group for a family of spaces, when the spaces of the family considered admit a common boundary. This is not the case for the asymptotically de Sitter family of spaces, as considered in this paper. To be able to use Penrose's theory one should divide the family of asymptotically de Sitter metrics into classes, two spaces belonging to the same class when their boundaries are conformally equivalent. Using Penrose's theory one would obtain different AS groups for different classes, subject to whether the boundary of the metrics of a given class does or does not admit any conformal symmetries. Therefore the agreement between Penrose's theory and Bondi's method is only apparent in the case of asymptotically de Sitter spaces.

It should be of some interest to explain the differences between both methods which have appeared in this work. The results of Section 4.4 can be generalized to the case without axial symmetry if one considers only space-times with the same boundary as the de Sitter space-time. Using the method of Section 4.4 one obtains, for the AS group, the group of conformal transformations of a three-dimensional sphere.

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Note added in proof

1. At the end of Section 3 it is shown that a necessary condition for the existence of solutions of equation (14) is

$$\exp\{\gamma(u, 0)\} = 1.$$

However this condition must be satisfied by the function $\overset{\circ}{\gamma}$ in order that the metric be regular at the axis $\theta = 0$ [2]. It can be shown, using Frobenius-Dieudonné theorem [8] that the set of equations (13) has a solution if and only if $\overset{\circ}{\gamma}$ satisfies the following condition:

$$\overset{\circ}{\gamma}_{xx}+\overset{\circ}{\gamma}_{\theta\theta} = 0$$

To achieve this, one reduces the set of equations (13) to the Frobenius form by introducing a new dependent variable $p = g_{\bar{x}}$, and the integrability conditions for this set of equations provide the above condition for $\overset{\circ}{\gamma}$.

2. In Section 4.1. it is shown that any harmonic function provides a solution of the set of equations (17) (see equation (20)). This is equivalent to the statement that if h is any holomorphic function of the variable $z = \bar{x}+i\bar{\theta}$ then the functions $\overset{\circ}{\alpha}$ and g defined by

$$g = \operatorname{Im} h \qquad \overset{\circ}{\alpha} = \operatorname{Re} h$$

satisfy the set of equations (17). The condition $\theta, \bar{\theta} \in [0, \pi]$ requires that h transforms bijectively the strip $\operatorname{Im} z \in [0, \pi]$ into itself. This restricts strongly the admissible h functions, and it can be shown that this reduces all the possible transformations to the three-parameter group obtained in Section 4.3.

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