SUPERENERGY TENSORS IN THE EINSTEIN-CARTAN THEORY OF GRAVITATION*

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In this paper we study systematically a generalization of the notion of "superenergy tensors" which has been introduced previously in the framework of the General Theory of Relativity on the Einstein-Cartan Theory of Gravitation. It is shown, by means of expansion in the normal coordinate system that the generalization is analytically simple only for the Einstein formulation of conservation laws.

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1. *Introduction*

It is known that in the framework of General Relativity Theory (GRT), the gravitational field which is identified with the Riemannian connexion $\tilde{\omega}^{\mu\cdot}_{\cdot\nu} = \{^{\mu}_{\nu\alpha}\}dx^{\alpha}$ does not have any energy-momentum tensor. It has only so called canonical Einstein's energy-momentum pseudotensor, $_Et$, with the following components, $_Et_{\mu\cdot}^{\cdot\nu}$, in holonomic frames[1]

$$_{E}t_{\mu\cdot}^{\cdot\nu} = \frac{c^4}{16\pi G}\eta^{\nu\alpha\beta\gamma}_{\cdot\cdot\cdot\cdot}\left(\eta_{\mu\varrho\cdot\alpha}^{\cdot\cdot\delta\cdot}\begin{Bmatrix}\sigma\\ \delta\beta\end{Bmatrix} - \eta_{\delta\varrho\cdot\alpha}^{\cdot\cdot\sigma\cdot}\begin{Bmatrix}\delta\\ \mu\beta\end{Bmatrix}\right)\begin{Bmatrix}\varrho\\ \sigma\gamma\end{Bmatrix}. \tag{1}$$

In the normal coordinate system, NCS(P), the pseudotensor $_Et$, if we assume its analyticity, is equivalent to the set of "the generalized normal tensors" [1–3]. It is an equivalence in the sense that the generalized normal tensors are the coefficients of the Taylor series of the pseudotensor with the centre at the point P. Analogously, an analytic tensor field is equivalent, in NCS(P) and in above mentioned sense, to the set of the tensors which are called, following Veblen [1], "extensions" of this tensor.

In the papers [4, 5] the second generalized normal tensor belonging to the pseudotensor $_Et$ was used to construct the superenergy tensor $_gS$ of the gravitational field $\tilde{\omega}^{\mu\cdot}_{\cdot\nu}$ and the second extension belonging to the metric energy-momentum tensor of matter $_mT$ was

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[1] We use only holonomic frames.

used to construct the superenergy tensor of matter ${}_{\mathrm{m}}S$. The components of these tensors were calculated, in NCS(P), according to the following, general definition

$$S_{\mu\cdot}^{\cdot\nu}(P) := \lim_{a\to 0} \frac{\iiiint\limits_{-a}^{a} (T_{\mu\cdot}^{\cdot\nu} - \overset{\circ}{T}{}_{\mu\cdot}^{\cdot\nu}) dy^0 dy^1 dy^2 dy^3}{\frac{8}{3} a^6}$$

$$= (2\overset{\circ}{v}{}^{\alpha}\overset{\circ}{v}{}^{\beta} - \overset{\circ}{g}{}^{\alpha\beta}) \overset{\circ}{T}{}_{\mu\cdot,\alpha\beta}^{\cdot\nu} \underset{\mathrm{NCS}(P)}{\overset{*}{=}} \overset{\circ}{T}{}_{\mu\cdot,00}^{\cdot\nu} + \overset{\circ}{T}{}_{\mu\cdot,11}^{\cdot\nu} + \overset{\circ}{T}{}_{\mu\cdot,22}^{\cdot\nu} + \overset{\circ}{T}{}_{\mu\cdot,33}^{\cdot\nu}. \tag{2}$$

for the field T of the class C^r, $r \geqslant 3$. In the above expression P denotes the origin of the normal coordinate system NCS(P) and $\overset{\circ}{v} \underset{\mathrm{NCS}(P)}{\overset{*}{=}} \delta^{\alpha}_{0}$; $T_{\mu\cdot}^{\cdot\nu}$ are the components of the energy-momentum tensor or pseudotensor and a comma "," denotes partial differentiation. The sign "o" above tensors or other fields denotes the value of the considered field at the point P.

$\overset{\circ}{T}{}_{\mu\cdot,\alpha\beta}^{\cdot\nu}$ mean the partial derivatives of the second order of the tensor or pseudotensor with the components $T_{\mu\cdot}^{\cdot\nu}$ calculated at the point P. These derivatives form a true tensor [1–3].

From the superenergy tensor of gravitation, ${}_{\mathrm{g}}S$, and from the superenergy tensor of matter, ${}_{\mathrm{m}}S$, we constructed, in the above mentioned papers, so called "total superenergy tensor" in GRT ${}^{\mathrm{t}}S$,

$${}^{\mathrm{t}}S := {}_{\mathrm{g}}S + {}_{\mathrm{m}}S. \tag{3}$$

In the papers [6, 7] the global superenergetic quantities of a closed system, the physical meaning of the superenergy and the superenergy tensor of the Einstein–Rosen gravitational wave were examined (all essentially in the framework of GRT). In papers [4–7] we also gave some remarks about the superenergy tensors of matter and gravitation in the framework of the Einstein-Cartan Theory (ECT).

The paper presented here is devoted to a systematic analysis of the superenergy tensors in the framework of ECT.

2. *The normal coordinate systems in ECT and the superenergy tensors in ECT*

In the framework of ECT, the mathematical model of space-time is a differentiable manifold U_4 [2]. The metric and connection of this manifold are determined by the Einstein–Cartan equations [8, 9]

$$G_{\mu\cdot}^{\cdot\nu} = \frac{8\pi G}{c^4}\, {}_{\mathrm{c}}T_{\mu\cdot}^{\cdot\nu}, \qquad T_{\mu\nu\cdot}^{\cdot\cdot\lambda} = \frac{8\pi G}{c^4}\, {}_{\mathrm{c}}\tau_{\mu\nu\cdot}^{\cdot\cdot\lambda}, \qquad \nabla_\alpha g_{\mu\nu} = 0. \tag{4}$$

In the above equations ${}_{\mathrm{c}}\tau_{\mu\nu\cdot}^{\cdot\cdot\lambda} = -{}_{\mathrm{c}}\tau_{\nu\mu\cdot}^{\cdot\cdot\lambda}$ denote the components of the canonical spin-tensor of matter, ${}_{\mathrm{c}}\tau$; ${}_{\mathrm{c}}T_{\mu\cdot}^{\cdot\nu}$ are the components of the canonical energy-momentum tensor of matter, ${}_{\mathrm{c}}T$, and $G_{\mu\cdot}^{\cdot\nu}$, $T_{\mu\nu\cdot}^{\cdot\cdot\lambda}$, $g_{\mu\nu}$ denote, respectively, the components of the Einstein tensor G of the U_4-geometry, the components of the modified torsion tensor T and the components of the metric tensor g (See, e.g., [9]).

The linear and metric connection $\omega^{\mu\cdot}_{\cdot\nu} = \Gamma^{\mu}_{\nu\alpha} dx^{\alpha}$ of space-time in ECT is physically interpreted as the gravitational field. We will call this gravitational field "Cartan's gravitation" or "Cartan's gravitational field".

The Riemann-Cartan connection $\omega^{\mu\cdot}_{\cdot\nu}$ is not symmetric, i.e.,

$$\Gamma^{\mu}_{[\nu\alpha]} =: S^{\cdot\cdot\mu}_{\nu\alpha\cdot} \neq 0.$$

It is known [1, 2] that the construction of the normal coordinates, NCS(P) involves only symmetric part of a non-symmetric connection. Therefore, in ECT, we can formally construct, at every point P of space-time, two different NCS(P):

1. The normal coordinate system NCS(P; $\tilde{\omega}$) belonging to the Riemannian part $\tilde{\omega}^{\mu\cdot}_{\cdot\nu} = \{^{\mu}_{\nu\alpha}\} dx^{\alpha}$ of the Riemann–Cartan connection $\omega^{\mu\cdot}_{\cdot\nu} = \Gamma^{\mu}_{\nu\alpha} dx^{\alpha}$;

2. The normal coordinate system NCS(P; ω) belonging to the symmetric part $\overset{s}{\omega}{}^{\mu\cdot}_{\cdot\nu} = \Gamma^{\mu}_{(\nu\alpha)} dx^{\alpha}$ of the Riemann–Cartan connection $\omega^{\mu\cdot}_{\cdot\nu}$.

These two coordinate systems are holonomic and can have the same natural basis at the point P.

The natural existence of the two above mentioned kinds of the normal coordinates in ECT follows from the two natural decompositions of the Riemann–Cartan connection $\omega^{\mu\cdot}_{\cdot\nu}$. In terms of components these two decompositions have the following form

$$\Gamma^{\alpha}_{\beta\gamma} = \begin{Bmatrix} \alpha \\ \beta\gamma \end{Bmatrix} + K^{\cdot\cdot\alpha}_{\beta\gamma\cdot}, \tag{5}$$

$$\Gamma^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{(\beta\gamma)} + \Gamma^{\alpha}_{[\beta\gamma]} =: \overset{s}{\Gamma}{}^{\alpha}_{\beta\gamma} + S^{\cdot\cdot\alpha}_{\beta\gamma\cdot}. \tag{6}$$

In the decompositions (5) and (6) $\{^{\alpha}_{\beta\gamma}\}$ and $\overset{s}{\Gamma}{}^{\alpha}_{\beta\gamma}$ are the components of the symmetric linear connections $\tilde{\omega}^{\mu\cdot}_{\cdot\nu}$ and $\overset{s}{\omega}{}^{\mu\cdot}_{\cdot\nu}$ respectively. From these two connections the connection $\tilde{\omega}^{\mu\cdot}_{\cdot\nu} = \{^{\eta}_{\nu\alpha}\} dx^{\alpha}$ is metric.

We may interpret physically the decomposition 1° as a splitting of Cartan's gravitational field $\omega^{\mu\cdot}_{\cdot\nu}$ into Einsteinian gravitation $\tilde{\omega}^{\mu\cdot}_{\cdot\nu}$ reducible by a suitable choice of the coordinates and tensorial part, K, with components $K^{\cdot\cdot\alpha}_{\beta\gamma\cdot}$ $(-)$ $K^{\cdot\cdot\alpha}_{\beta\gamma\cdot}$ are the components of the contortion tensor [9]).

From the fact that in ECT the connection between torsion of space-time and canonical spin is purely algebraic, it follows that we can write the ECT equations in the two equivalent forms:

1. In the form (4). In this formulation of ECT equations explicitly occur Cartan's gravitation $\omega^{\mu\cdot}_{\cdot\nu}$ with the components $\Gamma^{\mu}_{\nu\alpha}$ and the canonical energy-momentum tensor of matter, ${}_{c}T$, with the components ${}_{c}T^{\cdot\nu}_{\mu\cdot}$.

2. In Einstein's form [9, 10]

$$G^{\cdot\nu}_{\mu\cdot}(\tilde{\omega}) = \frac{8\pi G}{c^4}\, {}_{\text{com}}T^{\cdot\nu}_{\mu\cdot}. \tag{7}$$

$G^{\cdot\nu}_{\mu\cdot}(\tilde{\omega})$ denote here the components of the Einstein tensor G constructed from the Riemannian part $\tilde{\omega}^{\mu\cdot}_{\cdot\nu}$ of the Riemann–Cartan connection $\omega^{\mu\cdot}_{\cdot\nu}$ and ${}_{\text{com}}T^{\cdot\nu}_{\mu\cdot}$ are the components of the so-called combined, symmetric energy-momentum tensor of matter ${}_{\text{com}}T$ [9, 11].

In this formulation of ECT equations explicitly occur Einstein's gravitation $\tilde{\omega}^{\mu\cdot}_{\cdot\nu}$ with the components $\{^{\mu}_{\nu\alpha}\}$ and the combined energy-momentum tensor of matter ${}_{\rm com}T$ with the components ${}_{\rm com}T^{\cdot\nu}_{\mu\cdot}$. The tensorial part K of the Cartan gravitational field $\omega^{\mu\cdot}_{\cdot\nu}$ is, in this formulation of the ECT equations, formally eliminated from the considerations.

In each of the two above mentioned formulations of ECT equations the gravitation involved does not have any energy-momentum tensor but only a pseudotensor and in each of the two above formulations of ECT equations there exist a specific energy-momentum pseudotensor of appearing gravitation and a specific energy-momentum tensor of matter. For example, in the formulation 1 of ECT equations, the energy-momentum tensor of matter is the canonical one, ${}_{c}T$, with the components

$$ {}_{c}T^{\cdot\nu}_{\mu\cdot} = L\delta^{\nu}_{\mu} - \frac{\partial L}{\partial(\Psi_{,\nu})}\nabla_{\mu}\Psi. \tag{8}$$

Here Ψ describes the field of matter and $L = L(\Psi, \nabla\Psi)$ is the invariant Lagrangian density of matter. The energy-momentum pseudotensor of Cartan's gravitation $\omega^{\mu\cdot}_{\cdot\nu}$, which occurs in this formulation of ECT equations, is the pseudotensor ${}_{\omega}t$ with the components [5, 6]

$$ {}_{\omega}t^{\cdot\nu}_{\mu\cdot} = \frac{c^4}{16\pi G}\eta^{\nu\alpha\beta\gamma}_{\cdot\cdot\cdot\cdot}(\eta_{\mu\varrho\cdot\alpha}^{\cdot\cdot\delta\cdot}\Gamma^{\sigma}_{\delta\beta}\Gamma^{\varrho}_{\sigma\gamma} - \eta_{\delta\varrho\cdot\alpha}^{\cdot\cdot\sigma\cdot}\Gamma^{\delta}_{\mu\beta}\Gamma^{\varrho}_{\sigma\gamma} + \tfrac{1}{2}\eta_{\mu\kappa\cdot\lambda}^{\cdot\cdot\sigma\cdot}S_{\alpha\beta\cdot}^{\cdot\cdot\lambda}\Gamma^{\kappa}_{\sigma\gamma}). \tag{9}$$

This pseudotensor may be decomposed into three parts

$$ {}_{\omega}t = {}_{\tilde{\omega}}t + {}_{g}T + O. \tag{10}$$

In the decomposition (10) ${}_{\tilde{\omega}}t$ denotes the energy-momentum pseudotensor of the Einstein gravitation $\tilde{\omega}^{\mu\cdot}_{\cdot\nu}$ having the same components as the pseudotensor given by (1); ${}_{g}T$ is the energy-momentum tensor of the tensorial part, K, of the Cartan gravitation $\omega^{\mu\cdot}_{\cdot\nu}$ having the components ${}_{g}T^{\cdot\nu}_{\mu\cdot}$,

$$ {}_{g}T^{\cdot\nu}_{\mu\cdot} = \frac{c^4}{16\pi G}\eta^{\nu\alpha\beta\gamma}_{\cdot\cdot\cdot\cdot}(\eta_{\mu\varrho\cdot\alpha}^{\cdot\cdot\delta\cdot}K_{\delta\beta\cdot}^{\cdot\cdot\sigma}K_{\sigma\gamma\cdot}^{\cdot\cdot\varrho} - \eta_{\delta\varrho\cdot\alpha}^{\cdot\cdot\sigma\cdot}K_{\mu\beta\cdot}^{\cdot\cdot\delta}K_{\sigma\gamma\cdot}^{\cdot\cdot\varrho} + \tfrac{1}{2}\eta_{\mu\kappa\cdot\lambda}^{\cdot\cdot\sigma\cdot}S_{\alpha\beta\cdot}^{\cdot\cdot\lambda}K_{\sigma\gamma\cdot}^{\cdot\cdot\kappa}), \tag{11}$$

and O is the energy-momentum pseudotensor which involves energy of interaction of the fields K and $\tilde{\omega}^{\mu\cdot}_{\cdot\nu}$. The pseudotensor O has the following components

$$ O^{\cdot\nu}_{\mu\cdot} = \frac{c^4}{16\pi G}\eta^{\nu\alpha\beta\gamma}_{\cdot\cdot\cdot\cdot}\Bigg[\eta_{\mu\varrho\cdot\alpha}^{\cdot\cdot\delta\cdot}\left(\begin{Bmatrix}\sigma\\ \delta\beta\end{Bmatrix}K_{\sigma\gamma\cdot}^{\cdot\cdot\varrho} + \begin{Bmatrix}\varrho\\ \sigma\gamma\end{Bmatrix}K_{\delta\beta\cdot}^{\cdot\cdot\sigma}\right) - \eta_{\delta\varrho\cdot\alpha}^{\cdot\cdot\sigma\cdot}\left(\begin{Bmatrix}\delta\\ \mu\beta\end{Bmatrix}K_{\sigma\gamma\cdot}^{\cdot\cdot\varrho} + \begin{Bmatrix}\varrho\\ \sigma\gamma\end{Bmatrix}K_{\mu\beta\cdot}^{\cdot\cdot\delta}\right) + \tfrac{1}{2}\eta_{\mu\kappa\cdot\lambda}^{\cdot\cdot\sigma\cdot}S_{\alpha\beta\cdot}^{\cdot\cdot\lambda}\begin{Bmatrix}\kappa\\ \sigma\gamma\end{Bmatrix}\Bigg]. \tag{12}$$

The energy-momentum complex of matter and gravitation in the formulation 1° of ECT is the complex K [6] having the components $\overline{K}^{\cdot\nu}_{\mu\cdot} = \sqrt{|g|}\,({}_{\omega}t^{\cdot\nu}_{\mu\cdot} + {}_{c}T^{\cdot\nu}_{\mu\cdot})$. In the Einstein formulation 2° of ECT equations, the energy-momentum tensor of matter is the so-called combined energy-momentum tensor ${}_{\mathrm{com}}T$ with the components [9]

$$
{}_{\mathrm{com}}T^{\mu\nu}_{\cdot\cdot} = {}_{\mathrm{com}}T^{\nu\mu}_{\cdot\cdot} = {}_{\mathrm{m}}T^{\mu\nu}_{\cdot\cdot} + \frac{8\pi G}{c^4}\Big[-4\,{}_{\mathrm{c}}\tau^{\mu\alpha\cdot}_{\cdot\cdot[\beta}\,{}_{\mathrm{c}}\tau^{\nu\beta\cdot}_{\cdot\cdot\alpha]} - 2\,{}_{\mathrm{c}}\tau^{\mu\alpha\beta}_{\cdot\cdot\cdot}\,{}_{\mathrm{c}}\tau^{\nu\cdot\cdot}_{\cdot\alpha\beta}
$$

$$
+\,{}_{\mathrm{c}}\tau^{\alpha\beta\mu}_{\cdot\cdot\cdot}\,{}_{\mathrm{c}}\tau^{\cdot\cdot\nu}_{\alpha\beta\cdot} + \tfrac{1}{2}\,g^{\mu\nu}\big(4\,{}_{\mathrm{c}}\tau^{\cdot\alpha\cdot}_{\gamma\cdot[\beta}\,{}_{\mathrm{c}}\tau^{\gamma\beta}_{\cdot\cdot\alpha]} + {}_{\mathrm{c}}\tau^{\gamma\alpha\beta}_{\cdot\cdot\cdot}\,{}_{\mathrm{c}}\tau^{\cdot\cdot\cdot}_{\gamma\alpha\beta}\big)\Big], \tag{13}
$$

where $\sqrt{|g|}\,{}_{\mathrm{m}}T^{\mu\nu}_{\cdot\cdot} := 2\dfrac{\delta L}{\delta g_{\mu\nu}}$ (see [13], [15]), and the energy-momentum pseudotensor of the Einstein gravitation $\tilde{\omega}^{\mu}_{\cdot\nu}$ appearing in this formulation of ECT equations is the pseudotensor ${}_{\tilde{\omega}}t$ which has formally the same analytic form as the Einstein canonical pseudotensor ${}_{\mathrm{E}}t$ given by (1).

The energy-momentum complex of matter and gravitation appearing in this formulation of ECT is the complex ${}_{\mathrm{E}}\bar{K}$ [6] with the components

$$
{}_{\mathrm{E}}\bar{K}^{\cdot\nu}_{\mu\cdot} = \sqrt{|g|}\,\big({}_{\tilde{\omega}}t^{\cdot\nu}_{\mu\cdot} + {}_{\mathrm{com}}T^{\cdot\nu}_{\mu\cdot}\big).
$$

It follows from the above mentioned facts that in the framework of ECT we can formally consider four different superenergy tensors of gravitation and four different superenergy tensors of matter:

1. The superenergy tensor ${}_{\omega}S(P;\tilde{\omega};\vec{v})$ of the Cartan gravitation $\omega^{\mu}_{\cdot\nu}$ constructed from the ${}_{\omega}t$ in the normal coordinate system NCS$(P;\tilde{\omega})$ and the superenergy tensor ${}^{\mathrm{c}}_{\mathrm{m}}S(P;\tilde{\omega};\vec{v})$ of matter constructed from ${}_{\mathrm{c}}T$ in the NCS$(P;\tilde{\omega})$.

2. The superenergy tensor ${}_{\omega}S(P;\omega;\vec{v})$ of the Cartan gravitation $\omega^{\mu}_{\cdot\nu}$ constructed from the ${}_{\omega}t$ in the normal coordinate system NCS$(P;\omega)$ and the superenergy tensor ${}^{\mathrm{c}}_{\mathrm{m}}S(P;\omega;\vec{v})$ of matter constructed from ${}_{\mathrm{c}}T$ in the NCS$(P;\omega)$.

3. The superenergy tensor ${}_{\tilde{\omega}}S(P;\tilde{\omega};\vec{v})$ of the Einstein gravitation $\tilde{\omega}^{\mu}_{\cdot\nu}$ constructed from the ${}_{\tilde{\omega}}t$ in the normal coordinate system NCS$(P;\tilde{\omega})$ and the superenergy tensor ${}^{\mathrm{com}}_{\mathrm{m}}S(P;\tilde{\omega};\vec{v})$ of matter constructed from the ${}_{\mathrm{com}}T$ in the NCS$(P;\tilde{\omega})$.

4. The superenergy tensor ${}_{\tilde{\omega}}S(P;\omega;\vec{v})$ of the Einstein gravitation $\tilde{\omega}^{\mu}_{\cdot\nu}$ constructed from the ${}_{\tilde{\omega}}t$ in the normal coordinate system NCS$(P;\omega)$ and the superenergy tensor ${}^{\mathrm{com}}_{\mathrm{m}}S(P;\omega;\vec{v})$ of matter constructed from the ${}_{\mathrm{com}}T$ in the NCS$(P;\omega)$.

All the above superenergy tensors can be constructed in NCS$(P;\tilde{\omega})$ or in NCS$(P;\omega)$ in the same way as described in Section 1 (See also [4, 5]).

From considerations presented up to now it follows immediately that, in the framework of ECT, we can also formally introduce four different superenergy complexes:

1. The superenergy complex ${}_{1}S := \sqrt{|g|}\,[{}_{\tilde{\omega}}S(P;\tilde{\omega};\vec{v}) + {}^{\mathrm{com}}_{\mathrm{m}}S(P;\tilde{\omega};\vec{v})]$ of the Einstein gravitation $\tilde{\omega}^{\mu}_{\cdot\nu}$ and matter described by ${}_{\mathrm{com}}T$, constructed in the normal coordinate system NCS$(P;\tilde{\omega})$.

2. The superenergy complex ${}_{2}S := \sqrt{|g|}\,[{}_{\omega}S(P;\tilde{\omega};\vec{v}) + {}^{\mathrm{c}}_{\mathrm{m}}S(P;\tilde{\omega};\vec{v})]$ of the Cartan gravitation $\omega^{\mu}_{\cdot\nu}$ and matter described by ${}_{\mathrm{c}}T$, constructed in the normal coordinate system NCS$(P;\tilde{\omega})$.

3. The superenergy complex ${}_{3}S := \sqrt{|g|}\,[{}_{\tilde{\omega}}S(P;\omega;\vec{v}) + {}^{\mathrm{com}}_{\mathrm{m}}S(P;\omega;\vec{v})]$ of the Einstein gravitation $\tilde{\omega}^{\mu}_{\cdot\nu}$ and matter described by ${}_{\mathrm{com}}T$, constructed in the normal coordinate system NCS$(P;\omega)$.

4. The superenergy complex ${}_4S := \sqrt{|g|}\,[{}_{\omega}S(P;\omega;\vec{v}) + {}^{\mathrm{c}}_{\mathrm{m}}S(P;\omega;\vec{v})]$ of the Cartan gravitation $\omega^{\mu\cdot}_{\cdot\nu}$ and matter described by ${}_{\mathrm{c}}T$, constructed in the normal coordinate system NCS$(P;\omega)$

From all the considered up to now superenergy tensors of gravitation the simplest analytic form has the tensor ${}_{\tilde{\omega}}S(P;\tilde{\omega};\vec{v})$ and, in consequence, the simplest analytic form has the superenergy complex ${}_1S$.

This complex in terms of components, has the following form:

$$
\begin{aligned}
{}_1S_{\mu\cdot}^{\cdot\nu} &= \sqrt{|g|}\,[{}_{\tilde{\omega}}S_{\mu\cdot}^{\cdot\nu}(P;\tilde{\omega};\vec{v}) + {}^{\mathrm{com}}_{\mathrm{m}}S_{\mu\cdot}^{\cdot\nu}(P;\tilde{\omega};\vec{v})] = \sqrt{|g|}(2v^{\alpha}v^{\beta} - g^{\alpha\beta})\,\{\tfrac{2}{9}\,\kappa[4K^{\cdot\nu(\lambda\sigma)\cdot}_{\cdot\cdot\cdot\cdot(\alpha|}K_{\mu\lambda\sigma|\beta)}^{\cdot\cdot\cdot} \\
&\quad - 2\delta^{\nu}_{\mu}K^{\gamma\delta\varrho\cdot}_{\cdot\cdot\cdot\beta}K_{\gamma(\delta\varrho)\alpha}^{\cdot\cdot\cdot\cdot} + 2\delta^{\nu}_{\mu}K_{(\alpha|\cdot}^{\cdot\gamma}K_{\gamma|\beta)}^{\cdot} - 3K_{\mu(\alpha|}K^{\nu}_{\cdot|\beta)} \\
&\quad - 2K^{(\gamma\cdot\nu)\cdot}_{\cdot\mu\cdot(\alpha|}K_{\gamma|\beta)}^{\cdot\cdot}] + \overset{*}{\nabla}_{(\alpha}\overset{*}{\nabla}_{\beta)}\,{}_{\mathrm{com}}T_{\mu\cdot}^{\cdot\nu} + \tfrac{1}{3}\,K_{(\alpha|\mu|\beta)\cdot}^{\cdot\cdot\cdot\cdot\varrho}\,{}_{\mathrm{com}}T_{\varrho\cdot}^{\cdot\nu} - \tfrac{1}{3}\,K_{(\alpha|\varrho|\beta)\cdot}^{\cdot\cdot\cdot\cdot\nu}\,{}_{\mathrm{com}}T_{\mu\cdot}^{\cdot\varrho}\}. \qquad (14)
\end{aligned}
$$

In the above expression $\overset{*}{\nabla}$ denotes the covariant differentiation with respect to the connection $\tilde{\omega}^{\mu\cdot}_{\cdot\nu} = \{{}^{\mu}_{\nu\alpha}\}dx^{\alpha}$ and $K_{\mu\nu\alpha\cdot}^{\cdot\cdot\cdot\beta}$ mean the components of the curvature tensor of the connection $\tilde{\omega}^{\mu\cdot}_{\cdot\nu}$.

The analytic forms of the other superenergy tensors of gravitation and of the superenergy complexes considered in this paper are too complicated to be given here. The very complicated forms of these objects (about 20 times more complicated than the forms of ${}_{\tilde{\omega}}S(P;\tilde{\omega};\vec{v})$ and ${}_1S$) are caused by the non-vanishing torsion of space-time in the domains occupied by matter.

3. Discussion of the results and conclusions

From the mathematical point of view all the superenergy tensors considered in the previous section, especially the four superenergy complexes described there, are equally good. However, the physical and, especially, practical arguments distinguish the superenergy tensor ${}_{\tilde{\omega}}S(P;\tilde{\omega};\vec{v})$ and the superenergy complex ${}_1S(P;\tilde{\omega};\vec{v})$ as having the most useful properties.

The physical argumentation in favour of the tensor ${}_{\tilde{\omega}}S(P;\tilde{\omega};\vec{v})$ and the complex ${}_1S(P;\tilde{\omega};\vec{v})$ goes as follows:

1. The test particles which establish a local, inertial frame of reference in ECT (spinless test particles and photons) move along the geodesics of the Riemannian part $\tilde{\omega}^{\mu\cdot}_{\cdot\nu}$ of the Riemann-Cartan connection $\omega^{\mu\cdot}_{\cdot\nu}$.

In consequence, the coordinate system NCS$(P;\tilde{\omega})$ is the local, inertial frame of reference in ECT with the origin at the point P while the coordinate system NCS$(P;\omega)$ does not have any physical meaning.

Thus, from the physical point of view, one should expand the tensors and other fields and construct superenergy tensors in the coordinate system NCS$(P;\tilde{\omega})$. Therefore, the construction method distinguishes the superenergy tensors ${}_{\omega}S(P;\tilde{\omega};\vec{v})$, ${}^{\mathrm{c}}_{\mathrm{m}}S(P;\tilde{\omega};\vec{v})$, ${}_{\tilde{\omega}}S(P;\tilde{\omega};\vec{v})$, ${}^{\mathrm{com}}_{\mathrm{m}}S(P;\tilde{\omega};\vec{v})$ and the superenergy complexes ${}_1S(P;\tilde{\omega};\vec{v})$ and ${}_2S(P;\tilde{\omega};\vec{v})$, all obtained in NCS$(P;\tilde{\omega};\vec{v})$.

2. From these tensors, the tensors ${}_{\tilde{\omega}}S(P;\tilde{\omega};\vec{v})$ and ${}^{\mathrm{m}}_{\mathrm{com}}S(P;\tilde{\omega};\vec{v})$ and, therefore, the

complex ${}_1S(P; \tilde{\omega}; \vec{v})$ are additionally distinguished by the fact that their components can be always easily measured (indirectly) from observations of spinless test particles. This is easily seen from the equations of motion of spinless test particles in ECT and from geodesic deviation of the world lines of such particles [13–15]. The components of the superenergy tensors ${}_{\omega}S(P; \tilde{\omega}; \vec{v})$ and ${}^{\mathrm{c}}_{\mathrm{m}}S(P; \tilde{\omega}; \vec{v})$ are more difficult to measure (indirectly) from observations of test particles and they not always may be measured from such kind of observations because, in ECT, not all components of torsion can be measured from observations of test particles [16].

3. Moreower, from the superenergy tensors of gravitation considered in this paper, only the superenergy tensor ${}_{\tilde{\omega}}S(P; \tilde{\omega}; \vec{v})$ with the components ${}_{\tilde{\omega}}S_{\mu}^{\cdot\nu}(P; \tilde{\omega}; \vec{v})$ admits a simple and interesting physical interpretation as a "relative energy-momentum" (≡ superenergy) tensor of the field of tidal forces described by the Riemann curvature tensor with components $K_{\mu\nu\lambda}^{\cdot\cdot\cdot\kappa}$ [15].

Such physical interpretation of the superenergy tensor ${}_{\tilde{\omega}}S(P; \tilde{\omega}; \vec{v})$ is possible because the components of the tensor ${}_{\tilde{\omega}}S(P; \tilde{\omega}; \vec{v})$ are quadratic in $K_{\mu\nu\lambda}^{\cdot\cdot\cdot\kappa}$. The superenergy tensor ${}_{\omega}S(P; \tilde{\omega}; \vec{v})$ contains, among other things, the covariant derivatives of Riemannian curvature, the torsion and contortion tensors and their covariant derivatives. In consequence, this tensor does not admit as simple and interesting physical interpretation as the tensor ${}_{\tilde{\omega}}S(P; \tilde{\omega}; \vec{v})$. It is interesting that the tensor ${}_{\tilde{\omega}}S(P; \tilde{\omega}; \vec{v})$ mainly consists of the Maxwellian "relative energy-momentum" tensor of the Riemann tensor [4].

The practical argumentation in favour of the superenergy tensor ${}_{\tilde{\omega}}S(P; \tilde{\omega}; \vec{v})$ and the superenergy complex ${}_1S(P; \tilde{\omega}; \vec{v})$ is obvious: only the tensor ${}_{\tilde{\omega}}S(P; \tilde{\omega}; \vec{v})$ and the complex ${}_1S(P; \tilde{\omega}; \vec{v})$ have a simple analytic form.

On the other hand, it seems that the remaining gravitational superenergy tensors and the remaining complexes considered in this paper, especially the tensor ${}_{\omega}S(P; \tilde{\omega}; \vec{v})$ and the complex ${}_2S$, are too complicated for practical calculations.

Resuming one can say that in the framework of ECT there exist two different energy-momentum complexes[2]:

1. The energy-momentum complex $\bar{K} := \sqrt{|g|}\,({}_{\omega}t + {}_{\mathrm{c}}T)$ of the Cartan gravitation and matter,

2. The energy-momentum complex ${}_{\mathrm{E}}\bar{K} := \sqrt{|g|}\,({}_{\tilde{\omega}}t + {}_{\mathrm{com}}T)$ of the Einstein gravitation and matter,

and two distinguished, by the construction method superenergy complexes:

1. The superenergy complex ${}_1S(P; \tilde{\omega}; \vec{v})$ of the Einstein gravitation and matter,
2. The superenergy complex ${}_2S(P; \tilde{\omega}; \vec{v})$ of the Cartan gravitation and matter.

It seems that from these two superenergy complexes only the superenergy complex ${}_1S(P; \tilde{\omega}; \vec{v})$ has unquestionable physical meaning and it may be used in practice as the superenergy complex of matter and gravitation in ECT.

The superenergy complex ${}_2S(P; \tilde{\omega}; \vec{v})$ does not have such satisfactory properties as the complex ${}_1S(P; \tilde{\omega}, \vec{v})$ and it seems to be too complicated for any practical calculations.

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[2] Both of them give the same results for global quantities of a closed system [5].

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