ON THE POMERON EQUATION FOR LARGE $k_\perp^2$

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Solutions of the Pomeron equation are investigated for large gluon virtual mass $k_\perp^2$. Effects of the running coupling constant for this equation are studied; it is also shown that large $k_\perp^2$ solutions of the Pomeron equation agree with the small x limit of the DDT equation.

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Much progress has been recently achieved in understanding the high energy behaviour of nonabelian gauge theories (NAGT) — specifically in QCD. One can distinguish two regimes in which the theory has been investigated: (*a*) hard scattering regime in which all kinematical invariants are large — like deep inelastic scattering (DIS); for a review see Ref. [1] and references therein, (*b*) Regge limit, where the invariant squared energy s is large but other invariants (like t) remain fixed [2–7].

In the hard scattering region one calculates amplitudes in a leading logarithmic [1, 8] approximation summing a series of the form

$$A = \sum_{n=0}^{\infty} a_n^{\mathrm{H}} (g^2 \ln Q^2)^n, \tag{1}$$

where Q^2 is a large, positive energy scale in a given process.

In the Regge limit [3, 4], on the other hand, the series for the amplitude has the following form:

$$A = \sum_{n=0}^{\infty} a_n^{\mathrm{R}} (g^2 \ln s)^n, \tag{2}$$

where large logarithms of s (in DIS $\log s$ is equivalent to $\log \frac{1}{x}$ [2, 8]) are summed up with some coefficient functions containing full information on all other kinematical variables (like Q^2) [3, 4, 7]. Since $s \to \infty$, this limit corresponds, in DIS language to $x \to 0$.

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In this paper we show that the results for suitably defined parton probabilities calculated in both regimes, by means of two different calculation techniques, coincide in a region where s, $Q^2 \to \infty$ but x is small. First we consider an approximation in which the running coupling constant effects are neglected in both regions. In the Regge limit the Pomeron equation [6, 7] describing colour singlet exchange can be diagonalized by the Mellin transform, and its solution for large $k_\perp^2$ can be obtained by approximating the transform of the kernel by the pole term.

Then we ask how the running coupling changes this integral equation and its solutions. The replacement $g^2 \to \alpha_S(Q^2)$ corresponds to the inclusion of some next-to-leading log s terms important in the large $k_\perp^2$ limit. We find that the Pomeron equation "improved" in that way can be written as a differential equation whose solutions are governed exactly by the pole term of the Mellin transform of the kernel.

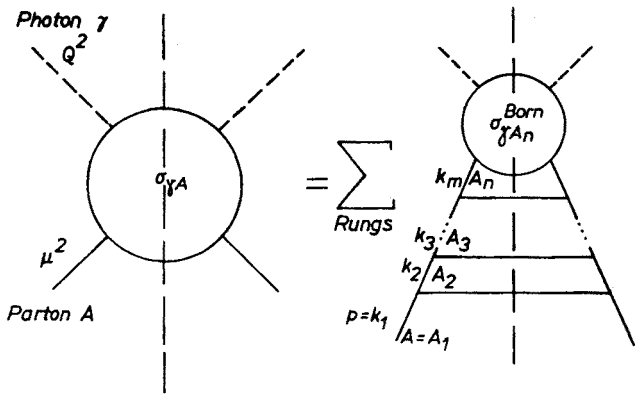

Fig. 1. Full photon-parton cross-section in DIS as a sum of ladder diagrams convoluted with the Born photon-parton cross-section

In DIS one extracts a leading log Q^2 contribution (1) by summing ladder [1, 8] diagrams (see Fig. 1) in the axial gauge, in a kinematical region

$$\mu^2 < |k_{1\perp}^2| < |k_{2\perp}^2| < \ldots |k_{n\perp}^2| < Q^2, \tag{3}$$

$$x \sim \beta_n < \beta_{n-1} < \ldots \beta_1 < 1, \tag{4}$$

where we have introduced Sudakov parametrization for integration momenta (Fig. 1):

$$k_n = \alpha_n q' + \beta_n p' + k_{n\perp},$$

$$q' = q + xp,$$

$$p' = p + \left(\frac{\mu^2}{s}\right) q.$$

The Q^2 behaviour of this process does not depend on the photon-parton Born cross-section $\sigma^{BORN}_{\gamma A_n}$ and therefore one defines the universal Q^2 dependent parton distributions

$D_{A\to B}(x, \mu^2, Q^2)$ [1, 8]. The well known DDT equation for parton distributions [1] can be solved by performing the Laplace-Mellin transform of $D_{A\to B}(x, \mu^2, Q^2)$ in x and in the "time" parameter ξ defined as

$$\xi(Q^2) \equiv \int_{\mu^2}^{Q^2} \frac{dk^2}{k^2} \frac{\alpha_S(k^2)}{4\pi}.$$

The Laplace-Mellin transform converts the integral DDT equation into an algebraical one, which can be easily solved in the $x \to 0$ limit where one can neglect quark-gluon mixing terms. If we neglect the running coupling constant effects parameter ξ becomes [8]:

$$\xi = \tilde{g}^2 \ln \frac{Q^2}{\mu^2},$$

$$\tilde{g}^2 \equiv \frac{Ng^2}{(2\pi)^2}$$

and the solution for $D_{G\to G}(x, \mu^2, Q^2)$ reads:

$$D_{G\to G}(x, \mu^2, Q^2) = \frac{1}{x}\left(\frac{\ln Q^2/\mu^2}{\ln 1/x}\right)^{1/2} I_1\left(2\sqrt{\tilde{g}^2 \ln \frac{Q^2}{\mu^2} \ln \frac{1}{x}}\right),$$

where N is the number of colours, $I_1(z)$ is the modified Bessel function of the first kind [9].

In the Regge limit one is concerned mainly with hadron-hadron collisions where the forward amplitude for, say, proton-proton scattering can be represented (Fig. 2) as [6]:

$$\text{Im } A(s, t = 0) = \int d^2k_{1\perp} d^2k_{2\perp} \frac{F_1(k_{1\perp}) f(s, k_{1\perp}^2, k_{2\perp}^2) F_2(k_{2\perp})}{k_{1\perp}^2 \cdot k_{1\perp}^2}, \tag{5}$$

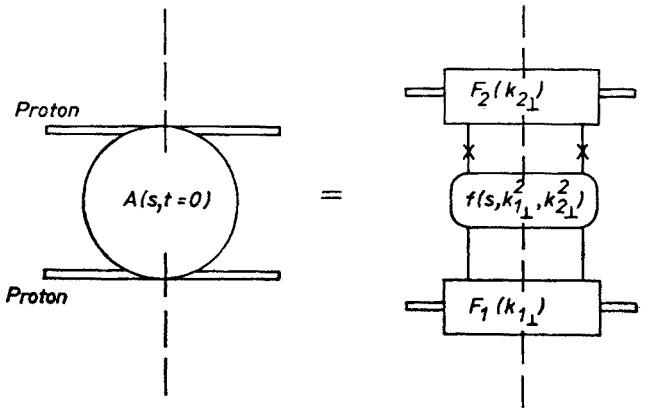

Fig. 2. Graphical illustration of the Eq. (5)

where $F(k_\perp)$ is the proton wave function and $f(s, k_{1\perp}^2, k_{2\perp}^2)$ describes the propagation of gluons in a singlet colour state. Equation (5) can be treated as a definition of $f(s, k_{1\perp}^2, k_{2\perp}^2)$ whose properties have been investigated in Refs. [4, 6, 7].

As we see from Eq. (5) the function $f(s, k_{1\perp}^2, k_{2\perp}^2)$ can be treated as a probability of finding a gluon (in fact a reggeized [2, 7, 10] gluon) with transverse virtualness $k_{2\perp}^2$ in a dressed gluon with virtualness $k_{1\perp}^2$ in a process with an external energy scale $s \to \infty$. The function $f(s, k_{1\perp}^2, k_{2\perp}^2)$ as it stands in Eq. (5) cannot be compared with the gluon distribution $D_{G\to G}(x, k_{1\perp}^2, k_{2\perp}^2)$ since all integrations over the longitudinal degrees of freedom have been carried out, whereas in the parton distributions $D_{G\to G}$ longitudinal momenta are fixed [1, 8]. These remaining integrations over dx or $d^2k_{2\perp}$ are performed when $D_{G\to G}(x)$ or $f(k_{2\perp}^2)$ are convoluted with a photon-gluon cross-section $\sigma_{\gamma G}^{\text{BORN}}(x)$ or proton wave function $F_2(k_{2\perp})$ to obtain the total cross-section. So in order to compare the quantities calculated in both regions one has to integrate $D_{G\to G}(x', \mu^2, Q^2)$ over dx' up to $x = \dfrac{Q^2}{s}$ and $f(s, k_{1\perp}^2, k_{2\perp}^2)$ over $dk_{2\perp}^2$ up to Q^2. Quantities obtained in such a way represent the full gluon-gluon transition functions integrated over the last cell (see Fig. 1, 2) over the transverse and longitudinal degree of freedom, and should coincide in a small x region.

In the Regge limit the kinematics is a little different than in the hard scattering regime; relation (4) still holds, whereas an integration over $d^2k_{i\perp}$ extends over the whole kinematic range [4, 6] (although the dominant contribution comes from the region where all $k_{i\perp}^2 \sim M^2$, M — proton mass [6]). Since we want to compare the asymptotic behaviour of the functions $D_{G\to G}(x, \mu^2, Q^2)$ and $f(s, k_{1\perp}^2, k_{2\perp}^2)$ in a region where both approaches are applicable, we have to impose relation (3) on the function $f(s, k_{1\perp}^2, k_{2\perp}^2)$ pushing $|k_{1\perp}^2| \to \mu^2$ and $|k_{2\perp}^2| \to Q^2$. Graphically this would correspond to the replacement of the upper blob in Fig. 2 representing the proton wave function $F_2(k_{2\perp})$ (see Eq. (5)) by photon-gluon cross-section $\sigma_{\gamma G}^{\text{BORN}}$ from Fig. 1.

A few words should be said about the equation for the function $f(s, k_{1\perp}^2, k_{2\perp}^2)$ which can be derived assuming reggeization of the vector meson in NAGT [4, 6] (for another derivation see Ref. [7]):

$$(\omega - 2\alpha(k_{1\perp}^2)) f_\omega(k_{1\perp}^2, k_{2\perp}^2) = \pi k_{2\perp}^2 \delta(k_{1\perp}^2 - k_{2\perp}^2) + \frac{\tilde{g}^2}{2\pi} \int \frac{d^2k'_\perp}{(k'^2_\perp)^2} K(k_{1\perp}, k'_\perp) f_\omega(k'^2_\perp, k_{2\perp}^2), \tag{6}$$

where the gluon trajectory has the form

$$\alpha(k_\perp^2) = -\frac{\tilde{g}^2}{4\pi} k_\perp^2 \int \frac{d^2k'_\perp}{k'^2_\perp (k_\perp - k'_\perp)^2}, \tag{7}$$

and the kernel is given by

$$K(k_{1\perp}, k'_\perp) = 2 \frac{k_{1\perp}^2 k'^2_\perp}{(k_{1\perp} - k'_\perp)^2}. \tag{8}$$

Here by $f_\omega(k_{1\perp}^2, k_{2\perp}^2)$ we have denoted the Mellin transform of $f(s, k_{1\perp}^2, k_{2\perp}^2)$:

$$f(s, k_{1\perp}^2, k_{2\perp}^2) = \int \frac{d\omega}{2\pi i} \left(\frac{s}{Q^2}\right)^\omega f_\omega(k_{1\perp}^2, k_{2\perp}^2). \tag{9}$$

Equation (6) describes the perturbative Pomeron obtained in pure, spontaneously broken SU(N) theory, in a leading $\log s$ approximation (but within this approximation it is exact in the whole range of k_1^2). In what follows we shall suppress the "transverse" indices for the integration momenta.

Since for the colour singlet exchange in the t-channel all amplitudes are infrared safe [6, 11], we have put all gluon masses in Eq. (6–8) equal to 0 (this is why there is nothing wrong about the infrared divergence of the trajectory (7)). In other channels, however, these divergences do not cancel each other [3, 4, 12], but their exponentiation is crucial for the infrared safety of the equations with singlet exchange.

One of the most important features of the effective kernel

$$K_{\text{eff}}(k_1, k') \equiv \frac{1}{k'^2} K(k_1, k') + \frac{4\pi}{\tilde{g}^2} \alpha(k_1^2) k'^2 \delta^{(2)}(k' - k_1),$$

of the Pomeron equation is scale invariance

$$\int \frac{d^2k'}{k'^2} K_{\text{eff}}(k_1, k') = \int \frac{d^2k'}{k'^2} K_{\text{eff}}(ak_1, ak'). \tag{10}$$

Due to (10) one can diagonalize Eq. (6) by the Mellin transform

$$f_\omega(k_1^2, k_2^2) = \frac{k_1^2}{(k_2^2)^2} \int \frac{dv}{2\pi i} (k_1^2)^{-v} \varphi_\omega(v; k_2^2), \tag{11}$$

converting the integral Pomeron equation into the algebraic one. The solution for $\varphi_\omega(v)$ reads

$$\varphi_\omega(v) = \frac{(k_2^2)^v}{\omega - \tilde{g}^2 K(v)}, \tag{12}$$

$$K(v) = -2\gamma_E - \psi(1-v) - \psi(1+v) + \frac{1}{v},$$

where $\gamma_E = 0.577\ldots$ is the Euler constant and $\psi(z)$ a digamma function [9].

One should emphasize here that (12) is an *exact* solution for the function $\varphi_\omega(v)$, however in order to invert the Mellin transform (11) one has to make some approximations.

We invert the Mellin transform (11) in a limit where

$$\frac{k_2^2}{k_1^2} \gg 1,$$

so that one can deform the integration contour closing it in a left-hand half-plane. Then $f_\omega(k_1^2, k_2^2)$ is given in terms of the sum over the poles on the negative real axis, whose positions are in general complicated functions of $\frac{\tilde{g}^2}{\omega}$, but the dominant leading $\log k^2$

contribution comes from the most right pole at

$$\nu_0 = \frac{\tilde{g}^2}{\omega} + o\left(\left(\frac{\tilde{g}^2}{\omega}\right)^2\right).$$

The solution of the Pomeron equation in this approximation is given by

$$f_\omega(k_1^2, k_2^2) = \frac{\tilde{g}^2}{\omega^2} \frac{k_1^2}{(k_2^2)^2} \left[\frac{k_2^2}{k_1^2}\right]^{\frac{\tilde{g}^2}{\omega}}. \tag{13}$$

In other words the leading log k^2 formula for $f_\omega(k_1^2, k_2^2)$ — Eq. (13) — was obtained by approximating the Mellin transform of the kernel $K(\nu)$ by its pole term

$$K(\nu) \approx \frac{1}{\nu}.$$

To invert the ω-transform (9) we use the Hankel formula for $1/\Gamma(z)$ which yields:

$$f(s, k_1^2, k_2^2) = \sqrt{\frac{g^2 \ln \frac{s}{Q^2}}{\ln \frac{k_2^2}{k_1^2}}}\, I_1\left(\tfrac{1}{2}\sqrt{\tilde{g}^2 \ln \frac{s}{Q^2} \ln \frac{k_2^2}{k_1^2}}\right).$$

According to our previous remarks in order to compare the gluon distributions calculated in both limits we have to integrate $D_{G\to G}(x')$ over dx' and $f(k_2^2)$ over dk_2^2. For integrated gluon distributions we obtain

$$F(x, Q^2) \equiv \int_x^1 dx' D_{G\to G}(x', \mu^2, Q^2) = \int_{\mu^2}^{Q^2} dk_2^2 f\left(\frac{1}{x}, \mu^2, k_2^2\right),$$

$$F(x, Q^2) = \sum_{n=0}^{\infty} \frac{\left(\tilde{g}^2 \ln \frac{Q^2}{\mu^2} \ln \frac{1}{x}\right)^{n+1}}{(n+1)!(n+1)!}. \tag{14}$$

From (14) we see that, as in fact it should be expected, both quantities have the same asymptotic form for s, Q^2 large and x small.

In the hard scattering region renormalization effects have been taken explicitly into account [1, 8] and therefore the running coupling constant appeared in the DDT equation. In the Regge limit the leading log s approximation does not require renormalization, since renormalization effects are non-leading in this region [3, 4].

Now we come to the question; how the running coupling constant affects the Pomeron equation [6] and its solutions [10, 13]. This we can do by replacing $g^2/4\pi$ in Eq. (6) by $\alpha_S(k^2)$. We should point out here that we have no systematic way to include the next-to-

-leading terms in the $\log s$ expansion (2), however one can argue that this replacement sums up all the leading $\log s$ and $\log Q^2$ terms for the scattering amplitude.

So if we replace

$$\tilde{g}^2 \to \frac{16\pi^2}{(\frac{11}{3} N - \frac{2}{3} n_f) \ln \frac{k_2}{k_1}} \tag{15}$$

(n_f number of flavours) and perform the Mellin transform

$$\varphi_\omega(k_1^2, k_2^2) = \int \frac{dv}{2\pi i} \left(\frac{k_1^2}{k_2^2}\right)^{-v} \tilde{\varphi}_\omega(v) \tag{16}$$

we obtain a differential equation for the function $\tilde{\varphi}_\omega(v)$

$$\frac{d}{dv} \tilde{\varphi}_\omega(v) = -\frac{\bar{g}^2}{\omega} \tilde{\varphi}_\omega(v) K(v),$$

$$\bar{g}^2 = 4N/(\tfrac{11}{3} N - \tfrac{2}{3} n_f). \tag{17}$$

Equation (17) can be exactly solved giving

$$\tilde{\varphi}_\omega(v) = \frac{1}{\omega} \exp\left(-\frac{\bar{g}^2}{\omega} \overline{K}(v)\right),$$

$$\frac{d}{dv} \overline{K}(v) = K(v), \tag{18}$$

where the explicit form for $\overline{K}(v)$ is given by

$$\overline{K}(v) = -2\gamma_E v - \ln\left(\frac{\Gamma(1+v)}{\Gamma(1-v)}\right) + \ln v.$$

The factor $\frac{1}{\omega}$ comes from the inhomogeneous term in the integral equation (6) for $f_\omega(k_1^2, k_2^2)$.

Although $\tilde{\varphi}_\omega(v)$, given by (18), is an exact solution of the Eq. (17), it is valid only for large $\frac{k_2^2}{k_1^2}$, because the equation itself is valid only for large k_2^2, since we have approximated $\alpha_S(k_2^2)$ in Eq. (6) by its asymptotic form (15).

The function $\tilde{\varphi}_\omega(v)$ has a pole at $v = 0$ and it can be seen that the leading $\log k_2^2$ behaviour of $f_\omega(k_1^2, k_2^2)$ is governed not approximately, like in the case of the "fixed" coupling constant (see Eq. (12)), but *exactly* by $\frac{1}{v}$ pole term of the kernel $K(v)$.

If we look at Eq. (18) we realize that $\varphi_\omega(v)$ is singular exactly in $\omega = 0$, whereas for the "fixed" coupling constant the leading singularity [4, 6] was a branch point in $\omega > 0$.

This change in position and character of the singularity is caused by the logarithmic term introduced into the effective kernel by $\alpha_S(k^2)$ (so in fact it is due to our approximation valid only for large k_2^2).

Now we can invert the Mellin transform (16):

$$f\omega(k_1^2, k_2^2) = \frac{k_1^2}{(k_2^2)^2} \frac{\bar{g}^2}{\omega} \frac{\exp\left(\frac{\bar{g}^2}{\omega} \ln \ln \frac{k_2^2}{k_1^2}\right)}{\Gamma\left(\frac{\bar{g}^2}{\omega}\right)}.$$

Finally we get the result for the integrated gluon distributions defined in Eq. (14):

$$F(x, Q^2) = \sum_{n=0}^{\infty} \frac{\left(\bar{g}^2 \ln \frac{1}{x} \ln \ln \frac{Q^2}{\mu^2}\right)^{n+1}}{(n+1)!(n+1)!}.$$

To sum up: we have shown that two different calculation schemes in the Regge and hard scattering regions give the same results for the gluon distribution functions in a limit where s, Q^2 are large but s/Q^2 remains small. In the "fixed" coupling constant case solutions of Eq. (6) for large $k_{2\perp}^2$ can be obtained by approximating the Mellin transform of the kernel $K(\nu)$ by its pole term, whereas in the case of the running coupling constant the "improved" Pomeron equation (valid only for large k_2^2) solutions are given exactly by the pole term of $K(\nu)$.

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REFERENCES

[1] Yu. L. Dokshitzer, D. I. D'yakonov, S. I. Troyan, *Phys. Rep.* **58** (1980).
[2] For review see: J. Bartels, *Acta Phys. Pol.* **B11**, 281 (1980) and references therein.
[3] L. N. Lipatov, *Yad. Fiz.* **23**, 642 (1976).
[4] E. A. Kuraev, L. N. Lipatov, V. S. Fadin, *Zh. Eksp. Theor. Fiz.* **71**, 840 (1976).
[5] L. Łukaszuk, L. Szymanowski, *Nucl. Phys.* **B151**, 293 (1979).
[6] Ya. Ya. Balitzkij, L. N. Lipatov, *Yad. Fiz.* **28**, 1597 (1978).
[7] T. Jaroszewicz, *Acta Phys. Pol.* **B11**, 93 (1980).
[8] Yu. L. Dokshitzer, *Zh. Eksp. Teor. Fiz.* **73**, 1216 (1977).
[9] M. Abramowitz, I. A. Stegun, *Handbook of Mathematical Functions*, Dover Pub. Inc., New York. 1970.
[10] J. B. Bronzen, R. L. Sugar, *Phys. Rev.* **D17**, 585 (1978).
[11] J. Kwieciński, M. Praszałowicz, *Phys. Lett.* **94B**, 413 (1980).
[12] J. Kwieciński, M. Praszałowicz, *Acta Phys. Pol.* **B12**, 103 (1981).
[13] D. Heckatorn, *Phys. Rev.* **D18**, 1286 (1978).