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RELATIVISTIC SOLUTION FOR ONE SPIN-1/2 AND ONE SPIN-0 PARTICLE BOUND BY COULOMB POTENTIAL

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A relativistic solution is given for a quantum-dynamical system which, like $e^-\alpha$ or $e^-\pi^+$, consists of one Dirac particle and one Klein-Gordon particle bound by Coulomb potential. A new fine-structure formula follows displaying explicitly the mass dependence of energy spectrum in the relativistic two-body problem.

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So far, in quantum mechanics we know only a few relativistic solutions for particles bound by Coulomb potential. In particular, such a solution for two Dirac particles is still lacking because the relevant relativistic wave equations [1, 2] give rather involved systems of radial equations [3]. In this note we find such a solution for a dynamical system which, like helium ion $e^-\alpha$ or electron-pion atom $e^-\pi^+$, consists of one Dirac particle and one Klein-Gordon particle.

First, we recall the respective relativistic wave equation [4]. Denoting

$$D_1 = \vec{\alpha}\cdot\vec{p}_1+\beta m_1, \qquad K_2 = \sqrt{\vec{p}_2^{\,2}+m_2^2} \tag{1}$$

we can write such an equation in the free case as follows

$$(E-D_1-K_2)(E-D_1+K_2)\psi_0(\vec{r}_1,\vec{r}_2) = 0 \tag{2}$$

or explicitly

$$[(E-\vec{\alpha}\cdot\vec{p}_1-\beta m_1)^2-\vec{p}_2^{\,2}-m_2^2]\psi_0(\vec{r}_1,\vec{r}_2) = 0. \tag{3}$$

In the Coulomb case we substitute $E\to E-V$ in Eq. (3), obtaining

$$\{(E-V)^2-2(E-V)(\vec{\alpha}\cdot\vec{p}_1+\beta m_1)+\vec{\alpha}\cdot[\vec{p}_1, V]+\vec{p}_1^{\,2}-\vec{p}_2^{\,2}+m_1^2-m_2^2\}\psi(\vec{r}_1,\vec{r}_2) = 0. \tag{4}$$

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In the centre-of-mass frame where $\vec{p}_1 = -\vec{p}_2 \equiv \vec{p}$ and $\vec{r}_1 - \vec{r}_2 \equiv \vec{r}$, we get multiplying Eq. (4) by $(E-V)^{-1/2}$:

$$\left[E-V-2(\vec{\alpha}\cdot\vec{p}+\beta m_1)+\frac{m_1^2-m_2^2}{E-V}\right]\sqrt{E-V}\;\psi(\vec{r}) = 0. \tag{5}$$

In the case of equal masses $m_1 = m_2 \equiv m$, Eq. (5) can be reduced to the Dirac-like equation (but for the internal motion)

$$[E-V-2(\vec{\alpha}\cdot\vec{p}+\beta m)]\sqrt{E-V}\;\psi(\vec{r}) = 0 \tag{6}$$

giving for $V = -\alpha/r$ the Sommerfeld-like formula [4] as its exact solution

$$E = 2m\left[1+\left(\frac{\alpha/2}{n_r+\gamma}\right)^2\right]^{-1/2}, \quad \gamma = \left[(j+\tfrac{1}{2})^2-\left(\frac{\alpha}{2}\right)^2\right]^{1/2}, \tag{7}$$

where $n_r = 0, 1, 2, \ldots$ and $j = 1/2, 3/2, \ldots$. In the opposite case of the one-body limit when $m_1/m_2 \to 0$ *and* $V/m_2 \to 0$, Eq. (5) transits into the usual Dirac equation with energy $\varepsilon_1 = E-m_2$, implying for $V = -\alpha/r$ the usual Sommerfeld formula. Note that for finite masses $m_1 \ll m_2$ the Dirac equation with $V = \mp\alpha/r$ follows (in some approximation) from Eq. (5) only at $\alpha/r \ll m_2$. It is the reason why for finite masses Eq. (5) cannot have the same behaviour at $r \to 0$ as the Dirac equation, unless V/E is neglected *before* $r \to 0$ is discussed.

Now, we go over to the general case of different masses $m_1 \neq m_2$. Since the Coulomb potential $V = \mp\alpha/r$ is a physically reliable static interaction up to the first order in α, we expand the effective interaction appearing in Eq. (5) into powers of α, retaining the first-order terms only

$$E-V+\frac{m_1^2-m_2^2}{E-V} = \frac{E^2+m_1^2-m_2^2}{E} - 2V_{\text{eff}}+O(\alpha^2)\,(m_1-m_2), \tag{8}$$

where

$$V_{\text{eff}} = V\frac{m_2}{m_1+m_2} = \mp\frac{\alpha_{\text{eff}}}{r}, \quad \alpha_{\text{eff}} = \alpha\frac{m_2}{m_1+m_2} \tag{9}$$

because of $E = m_1+m_2+O(\alpha^2)$. Obviously, in the case of $m_1 = m_2$ we get Eq. (6) exactly, while in the case of $m_1/m_2 \to 0$ the Dirac equation with energy $\varepsilon_1 = E-m_2$ follows. Note that in the physically required approximation given in Eq. (8) we neglect V/E *before* the limit of $r \to 0$ is applied in order to fix the behaviour of Eq. (5) at $r \to 0$.

Under the approximation (8), when eliminating from Eq. (5) angular coordinates in the standard way [5], we obtain in the representation where

$$\alpha_r \equiv \frac{\vec{r}}{r}\cdot\vec{\alpha} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \psi(r) = \begin{pmatrix} \psi^+(r) \\ \psi^-(r) \end{pmatrix} \tag{10}$$

the following system of two radial equations:

$$\left(\frac{d}{dr}\mp\frac{k}{r}\right)f^{\pm}-\left(\frac{1}{a^{\pm}}\mp V_{\text{eff}}\right)f^{\mp} = 0. \tag{11}$$

Here

$$f^{\pm} = r\sqrt{E-V}\,\psi^{\pm}, \qquad \frac{1}{a^{\pm}} = m_1 \pm \frac{E^2+m_1^2-m_2^2}{2E} \tag{12}$$

and $k = \varepsilon(j+1/2)$ with $\varepsilon = \pm 1$ corresponding to the parity $P = (-1)^{j-\varepsilon/2}$. When $V = -\alpha/r$, Eqs. (11) imply the asymptotic behaviour

$$f^{\pm} \underset{r\to 0}{\sim} r^{\gamma}, \qquad f^{\pm} \underset{r\to\infty}{\sim} \exp\left(-\frac{r}{a}\right), \tag{13}$$

the latter for bound states, where

$$\gamma = [(j+\tfrac{1}{2})^2-\alpha_{\text{eff}}^2]^{1/2}, \qquad a = \sqrt{a^+a^-}\,. \tag{14}$$

Thus, substituting in Eqs. (11)

$$f^{\pm} = r^{\gamma}\exp\left(-\frac{r}{a}\right)v^{\pm} \tag{15}$$

we get for Coulomb bound states the equations

$$\left(\frac{d}{dr}+\frac{\gamma\mp k}{r}-\frac{1}{a}\right)v^{\pm}-\left(\frac{1}{a^{\pm}}\pm\frac{\alpha_{\text{eff}}}{r}\right)v^{\mp} = 0, \tag{16}$$

where

$$v^{\pm} = \sum_{\nu=0}^{n_r} c_{\nu}^{\pm} r^{\nu} \tag{17}$$

are polynomials. Inserting the polynomials (17) into Eqs. (16) we determine energy levels E corresponding to quantum numbers $n_r = 0, 1, 2, \ldots$ and $j = 1/2, 3/2, \ldots$ (and degenerate with respect to $P = \pm 1$)

$$\left(\frac{2m_1E}{E^2+m_1^2-m_2^2}\right)^2 = 1+A, \tag{18}$$

where

$$A = \left(\frac{\alpha_{\text{eff}}}{n_r+\gamma}\right)^2 = \left\{\frac{\alpha\dfrac{m_2}{m_1+m_2}}{n_r+\left[(j+\tfrac{1}{2})^2-\left(\alpha\dfrac{m_2}{m_1+m_2}\right)^2\right]^{1/2}}\right\}^2. \tag{19}$$

Thus, the explicit energy-spectrum formula for our dynamical system is

$$E = x\left\{1+\left[1-\left(\frac{m_1^2-m_2^2}{x}\right)^2\right]^{1/2}\right\}^{1/2}, \tag{20}$$

where

$$x = \left[\frac{(1-A)m_1^2+(1+A)m_2^2}{1+A}\right]^{1/2}. \tag{21}$$

Expanding Eq. (20) into powers of α^2 up to α^4 we obtain the new fine-structure formula

$$E = M - \frac{\alpha^2\mu}{2n^2} - \frac{\alpha^4\mu}{2n^4}\left(1-\frac{m_1}{M}\right)^2\left\{\frac{n}{j+\frac{1}{2}} - \frac{3}{4}\left[1-\frac{1}{3}\frac{m_1(m_1-m_2)}{m_2^2}\right]\right\} + O(\alpha^6), \tag{22}$$

where $n \equiv n_r+j+\frac{1}{2} = 1, 2, 3, \ldots$ and

$$M = m_1+m_2, \frac{1}{\mu} = \frac{1}{m_1}+\frac{1}{m_2}. \tag{23}$$

Eq. (22) displays explicitly the mass dependence of the fine-structure terms in the relativistic two-body problem considered in this paper[1].

In the case of equal masses when $M = 2m$ and $\mu = m/2$, Eq. (20) reduces to the Sommerfeld-like formula (7) and Eq. (22) gives

$$E = 2m\left[1-\frac{(\alpha/2)^2}{2n^2} - \frac{(\alpha/2)^4}{2n^4}\left(\frac{n}{j+\frac{1}{2}} - \frac{3}{4}\right)\right] + O(\alpha^6). \tag{24}$$

In the one-body limit of $m_1/m_2 \to 0$, Eq. (20) transits into the Sommerfeld formula

$$\varepsilon_1 \equiv E-m_2 = m_1\left[1+\left(\frac{\alpha}{n_r+\gamma}\right)^2\right]^{-1/2}, \quad \gamma = [(j+\tfrac{1}{2})^2-\alpha^2]^{1/2} \tag{25}$$

(to see it cf. Eq. (18)) and Eq. (22) implies

$$\varepsilon_1 \equiv E-m_2 = m_1\left[1-\frac{\alpha^2}{2n^2} - \frac{\alpha^4}{2n^4}\left(\frac{n}{j+\frac{1}{2}} - \frac{3}{4}\right)\right] + O(\alpha^6) \tag{26}$$

which is the familiar fine-structure formula based on the Dirac equation [6].

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[1] Eqs. (20) and (22) are valid if $V_{\mathrm{eff}} = -\alpha_{\mathrm{eff}}/r$ is taken as the effective Coulomb interaction responsible for the proper ladder approximation. Then, corrections to V_{eff} may be treated perturbatively. The next-order corrections following from Eq. (8) have the form

$$\Delta V_{\mathrm{eff}} = -\left(\frac{\alpha^2}{r^2} - \frac{\alpha^3\mu}{n^2 r}\right)\frac{m_1-m_2}{2M^2}$$

and give the shift

$$\Delta E = -\frac{\alpha^4\mu}{2n^4}\frac{\mu(m_1-m_2)}{M^2}\left(\frac{n}{j}-1\right) + O(\alpha^6)\cdot(m_1-m_2)$$

of the level E as obtained in the ladder approximation. Thus, Eq. (22) implies the fine-structure formula

$$E+\Delta E = M - \frac{\alpha^2\mu}{2n^2}$$

$$-\frac{\alpha^4\mu}{2n^4}\left\{\left(1-\frac{m_1}{M}\right)^2\left(\frac{n}{j+{}^1/_2} - \frac{3}{4}\right) + \frac{m_1-m_2}{M}\left[\frac{m_1 m_2}{M^2}\left(\frac{n}{j} - \frac{3}{4}\right) + \frac{1}{4}\frac{m_1^2}{M^2}\right]\right\} + O(\alpha^6).$$

Note that $\Delta E = 0$ both for $m_1 = m_2$ and $m_1/m_2 \to 0$.

In conclusion, we can say that the energy-spectrum formula (20) provides a satisfactory solution to the relativistic wave equation (5) for a dynamical system which, like $e^-\alpha$ or $e^-\pi^+$, consists of one Dirac particle and one Klein-Gordon particle bound by Coulomb potential. This formula follows from Eq. (5) if the physically required approximation given in Eq. (8) is made. When expanded into powers of α^2 up to the second order, Eq. (20) leads to the fine-structure formula (22). In the case of equal masses, the wave equation (5) and its energy-spectrum formula (20) reduce to the Dirac-like equation (6) and the Sommerfeld-like formula (7), respectively. In the one-body limit when the Klein-Gordon particle becomes infinitely heavy, the relativistic two-body wave equation (5) transits into the Dirac equation, while the energy-spectrum formula (20) goes over into the Sommerfeld formula (25). Note that for $n_r = 0$ and $j = 1/2$ the energy spectrum formula (20) gives formally

$$E \to (m_2^2 - m_1^2)^{1/2} \quad \text{if} \quad \alpha_{\text{eff}} \equiv \alpha \frac{m_2}{m_1+m_2} \to 1-0 \tag{27}$$

because then $A \to +\infty$. Thus $\alpha = 1+m_1/m_2$ is the critical value of the Coulombic coupling constant for our dynamical system. We can see that the mass E of the critical Coulombic ground state is zero if $m_1 = m_2$ exactly. If $m_2 - m_1$ is small but positive, the mass E of this state is large for big $m_1 \simeq m_2$, e.g., if $m_2 - m_1 \simeq 5$ MeV, one gets $E \simeq 1.5$ GeV for $m_1 \simeq m_2 \simeq 200$ GeV. Of course, for $\alpha_{\text{eff}} \sim 1$ Eq. (8) cannot be considered as an approximation, unless $m_2 - m_1$ is small.

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