SPHERICALLY SYMMETRIC COSMOLOGICAL SOLUTIONS OF THE LYTTLETON-BONDI UNIVERSE

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We investigate the dynamics of a spherically symmetric distribution of matter, in the circumstance that we allow creation of matter in the Lyttleton-Bondi sense, on the basis of Einstein's field equations for the adiabatic case with non-vanishing internal pressure gradient. It has been shown that if the density is uniform throughout the body at each instant (and so is a function of time only) and the motion is shear free, then either the sphere is static and has a definite charge distribution or it is non-static and continually expands.

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1. *Introduction*

Lyttleton and Bondi (1959) have developed a cosmological model based on the possibility of a general excess of charge in the Universe. This excess of charge may arise from the difference in magnitude of the charge of the proton and that of the electron, or from the difference in the number of protons as compared to the number of electrons. The imbalance is of such order that it would not render the body appreciably conducting.

Further, if the charge excess exceeds a certain critical value expansion will result. If this expansion alone occurred, the space density of material would steadily diminish and the acceleration of expansion would decrease with it. To off-set the decrease in density implied by the expansion, Lyttleton and Bondi postulated the creation of matter, and also necessarily charge, everywhere in space. This creation process will then keep the density invariable. Since charge is not then conserved in the strict sense some modification of the Maxwell equations must be made to permit the resulting breach of conservation of charge.

Lyttleton and Bondi (1959) have investigated the nature of the fields in both Newtonian and general relativistic frameworks in the circumstance that the electromagnetic field is assumed to vanish. Burman (1971) has studied the static spherically symmetric exterior

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solution and has shown that the field has no detectable departure from the predictions of the Schwarzschild solution. Rao and Panda (1975) studied the exterior solutions in the case of the cylindrically symmetric Einstein-Rosen metric and obtained exact solutions, one of which is static and the other time-dependent.

As far as I know, nobody has investigated the nature of the interior fields yet. The purpose of the present work is to study the interior fields of a spherically symmetrical perfect matter fluid. In our work we shall relax the condition that the matter density is invariant, in that we shall require it to be a function of the time only. This step is motivated by recent developments in cosmology. It is now generally believed that the universe has evolved to its present state from an initially dense configuration. It must be pointed out that allowing the density to vary with time does not invalidate the basic assumptions of the theory — creation of matter will still occur but the density will not necessarily remain the same.

In order to obtain exact solutions of our field equations we have imposed the following conditions on the system: the components of the metric tensor satisfy the condition $\dot{\gamma} = \dot{\mu}$, which is the requirement for shear-free motion. Of course, besides this the usual regularity condition at the centre and the boundary conditions at the interface between the region occupied by the matter and the surrounding empty space are assumed to hold. We shall show that the implied creation of matter and the expansion of the universe may not continue indefinitely. Specifically, we establish that if the charge distribution throughout our model sphere attains a certain critical value the creation rate vanishes and the sphere becomes static, otherwise the sphere expands continually. The new thing here is that the assumptions $\dot{\gamma} = \dot{\mu}$ and $\varrho = \varrho(t)$ do not necessarily lead to a static solution. This must be contrasted with the situation which obtains if matter creation is not allowed. In fact, it has been shown (Misra and Srivastava 1973) that for a charged fluid sphere embedded in empty space the conditions $\varrho = \varrho(t)$ and $\dot{\gamma} = \dot{\mu}$ are inconsistent in the sense that either the solutions are static or matter is uncharged. Thus, allowing matter creation to occur presents us with an entirely new picture of the interior fields of the spherical matter.

2. *The field equations*

We choose the line element in the form

$$ds^2 = e^{\nu}dt^2 - e^{\gamma}dr^2 - e^{\mu}(d\theta^2 + \sin^2\theta d\varphi^2), \tag{2.1}$$

where (t, r, θ, φ) are comoving co-ordinates. Here the functions ν, γ and μ are functions of r and t only. Throughout this paper units are chosen so that G and c are each unity.

The field equations are (Lyttleton and Bondi 1959):

$$G^i{}_j = -8\pi T^i{}_j, \tag{2.2}$$

$$F_{ij} = A_{i,j} - A_{j,i}, \tag{2.3}$$

$$F^{ij}{}_{;j} = 4\pi \mathscr{J}^i, \tag{2.4}$$

$$J^i{}_{;i} = q. \tag{2.5}$$

Here $T^i{}_j$ is the energy-momentum tensor for the system and we express it in the form

$$T^i{}_j = t^i{}_j + E^i{}_j, \tag{2.6}$$

where $t^i{}_j$ is the energy-momentum tensor of perfect fluid

$$t^i{}_j = (\varrho + p)u^i u_j - p\delta^i{}_j \tag{2.7}$$

and $E^i{}_j$ is the electromagnetic stress-energy tensor

$$4\pi E^i{}_j = -F^{ik}F_{jk} + \tfrac{1}{4}\,\delta^i{}_j F^{hk}F_{hk} + \lambda(A^i A_j - \tfrac{1}{2}\,\delta^i{}_j A_k A^k). \tag{2.8}$$

The quantity q is the rate of creation of charge per unit volume and $\mathscr{J}^i$ is defined by the equation

$$\mathscr{J}^i = J^i - (\lambda/4\pi)A^i. \tag{2.9}$$

The four velocity of matter is the unit timelike vector

$$u^i = e^{-\nu/2}\delta^i{}_0. \tag{2.10}$$

In the case of spherical symmetry, the only non-zero component of F^{ik} is $F^{01} = -F^{10}$, which depends on r and t.

$i = 1$ in Eq. (2.4) gives

$$F^{10} = Q(r)e^{-\mu-(\gamma+\nu)/2}, \tag{2.11}$$

where $Q(r)$ is an arbitrary function of r, while $i = 0$ implies that

$$Q' = -4\pi\mathscr{J}^0 e^{\mu+(\gamma+\nu)/2}. \tag{2.12}$$

If we define $A_0 = \phi$, then by Eq. (2.3) we have

$$\phi' = Q(r)e^{(\gamma+\nu)/2-\mu}, \tag{2.13}$$

where a prime denotes differentiation with respect to r.

Eq. (2.11) is an explicit solution for the field $F^{10}(r, t)$ in terms of the yet unknown functions ν and γ. If we write R for $e^{\mu/2}$ we note, from the boundary condition that the geometry is Euclidean at infinity so that ν and γ approach zero as $R \to \infty$, that the solution (2.11) has the usual classical form, at least for large R. Thus the function $Q(r)$ can be identified as the charge up to the co-ordinate radius r.

The non-trivial field equations for the metric (2.1) are (Eq. (2.2)):

$$-8\pi T^0{}_0 = e^{-\gamma}(\mu'' + \tfrac{3}{4}\,\mu'^2 - \tfrac{1}{2}\,\mu'\gamma') - e^{-\nu}(\tfrac{1}{2}\,\dot{\gamma}\dot{\mu} + \tfrac{1}{4}\,\dot{\mu}^2) - e^{-\mu}, \tag{2.14}$$

$$-8\pi T^1{}_1 = e^{-\gamma}(\tfrac{1}{2}\,\mu'\nu' + \tfrac{1}{4}\,\mu'^2) + e^{-\nu}(\tfrac{1}{2}\,\dot{\mu}\dot{\nu} - \ddot{\mu} - \tfrac{3}{4}\,\dot{\mu}^2) - e^{-\mu}, \tag{2.15}$$

$$\begin{aligned}-8\pi T^2{}_2 = {} & e^{-\gamma}(\tfrac{1}{2}\,\mu'' + \tfrac{1}{4}\,\mu'^2 + \tfrac{1}{2}\,\nu'' + \tfrac{1}{4}\,\nu'^2 - \tfrac{1}{4}\,\mu'\gamma' + \tfrac{1}{4}\,\mu'\nu' - \tfrac{1}{4}\,\nu'\gamma') \\ & + e^{-\nu}(-\tfrac{1}{2}\,\ddot{\gamma} - \tfrac{1}{4}\,\dot{\gamma}^2 - \tfrac{1}{2}\,\ddot{\mu} - \tfrac{1}{4}\,\dot{\mu}^2 + \tfrac{1}{4}\,\dot{\gamma}\dot{\nu} + \tfrac{1}{4}\,\dot{\nu}\dot{\mu} - \tfrac{1}{4}\,\dot{\gamma}\dot{\mu}),\end{aligned} \tag{2.16}$$

$$-8\pi T^1{}_0 = -\dot{\mu}' - \tfrac{1}{2}\,\dot{\mu}\mu' + \tfrac{1}{2}\,\dot{\gamma}\mu' + \tfrac{1}{2}\,\dot{\mu}\nu' = 0, \tag{2.17}$$

where

$$T^0{}_0 = \varrho + \frac{Q^2}{8\pi}e^{-2\mu} + \frac{\lambda\phi^2}{8\pi}e^{-\nu}, \tag{2.18a}$$

$$T^1{}_1 = -p + \frac{Q^2}{8\pi}e^{-2\mu} - \frac{\lambda\phi^2}{8\pi}e^{-\nu}, \tag{2.18b}$$

$$T^2{}_2 = T^3{}_3 = -p - \frac{Q^2}{8\pi}e^{-2\mu} - \frac{\lambda\phi^2}{8\pi}e^{-\nu} \tag{2.18c}$$

and the dot denotes differentiation with respect to time.

We shall now establish the implication of the requirements that $\varrho = \varrho(t)$ and $\dot{\gamma} = \dot{\mu}$. We note that the condition to be established also applies, of course, to the special case when ϱ is a constant, independent of r and t. Subtracting Eq. (2.15) from Eq. (2.16) and substituting for ν' from Eq. (2.17) we obtain

$$4\{(\mu' e^{-\gamma/2})' e^{\mu}\}^{\cdot} + (1-\dot{\gamma}/\dot{\mu})(\mu'^2 e^{-\gamma})^{\cdot} e^{\mu+\gamma/2} + 4\dot{\mu}e^{\gamma/2}$$
$$+2\dot{\mu}e^{-\nu/2}\{(\dot{\mu}-\dot{\gamma})e^{\mu+(\gamma-\nu)/2}\}^{\cdot} - 8\dot{\mu}Q^2 e^{-\mu+\gamma/2} = 0. \tag{2.19}$$

On inserting the condition $\dot{\gamma} = \dot{\mu}$ in Eq. (2.17), we obtain

$$e^{\nu} = \dot{\mu}^2 A(t), \tag{2.20}$$

where $A(t)$ is an arbitrary function of time.

Also on inserting $\dot{\gamma} = \dot{\mu}$ and hence $\gamma = \mu - a(r)$ into Eq. (2.19) we have

$$\{(\mu' e^{-\gamma/2})' e^{\mu}\}^{\cdot} + \dot{\mu}e^{(\mu-a)/2} - 2\dot{\mu}Q^2 e^{-(\mu+a)/2} = 0.$$

Integration of this equation with respect to time yields

$$2\mu'' - \mu'^2 + \mu' a' + 4e^{-a} + 8Q^2 e^{-(\mu+a)} = b(r)e^{-\mu/2}, \tag{2.21}$$

where $b(r)$ is an arbitrary function of integration.

Eq. (2.21) will now be used to show that the functions ϕ, Q and $b(r)$ are connected by a single subsidiary condition which must be imposed on the solution of our problem; so that the choice of the function $b(r)$ is not completely arbitrary. On inserting $\dot{\gamma} = \dot{\mu}$ into Eq. (2.14) and evaluating the derivative with respect to r of the resulting equation, we obtain

$$(8\pi T^0{}_0)' = -\mu' e^{-\mu} + \tfrac{3}{4}\dot{\mu}e^{-\nu}(2\dot{\mu}' - \dot{\mu}\nu')$$
$$+e^{-\gamma}(\mu''\gamma' + \tfrac{3}{4}\gamma'\mu'^2 - \tfrac{1}{2}\mu'\gamma'^2 - \mu''' - \tfrac{3}{2}\mu'\mu'' + \tfrac{1}{2}\mu''\gamma' + \tfrac{1}{2}\mu'\gamma'').$$

The second term on the right-hand side of this equation vanishes on account of Eq. (2.17). We next put $\gamma = \mu - a$ and obtain μ''' from Eq. (2.21) to write the last equation in the form

$$(8\pi T^0{}_0)' = e^{-\mu}\{(4Q^2)' e^{-\mu} - 2Q^2\mu' e^{-\mu} - \tfrac{1}{2}(be^{a})' e^{-\mu/2} + 2Q^2\mu' e^{-\mu}\}.$$

On substituting for $T^0{}_0$ from Eq. (2.18a) and putting $\varrho' = 0$ we obtain

$$(3Q^2)'e^{-2\mu}-(\lambda\phi^2 e^{-\nu})'-\tfrac{1}{2}(be^{\alpha})'e^{-3\mu/2} = 0;$$

which is equivalent to the condition

$$\frac{\lambda}{6}(\nu'\phi^2-2\phi\phi')e^{\frac{3\mu}{2}-\nu}+QQ'e^{-\mu/2}-\tfrac{1}{2}(be^{\alpha})' = 0. \tag{2.22}$$

Eq. (2.22) is the condition we seek. This condition is then seen to arise as a consequence of the assumption that the density is uniform and that $\dot{\gamma} = \dot{\mu}$. However, before discussing it further we derive it again in a more general way. The general approach is indeed necessary because cf the wealth of information we extract from it. Writing down the other as yet hidden implications of Eq. (2.22), without explicitly showing how they arise, will make the whole procedure look artificial.

Use can be made of the Bianchi identities to establish the following relationships for any energy-momentum tensor:

$$(T^1{}_1)' = \tfrac{1}{2}\nu'(T^0{}_0-T^1{}_1)+\mu'(T^2{}_2-T^1{}_1), \tag{2.23}$$

$$(\dot{T}^0{}_0) = (\dot{\mu}+\tfrac{1}{2}\dot{\gamma})(T^1{}_1-T^0{}_0)+\dot{\mu}(T^2{}_2-T^1{}_1). \tag{2.24}$$

Eqs. (2.23) and (2.24) may be interpreted physically as representing the conservation of linear momentum and energy, respectively.

We introduce a new variable $m(r, t)$ defined by the following equation (Thompson and Whitrow, 1967):

$$8m = \dot{\mu}^2 e^{\frac{3\mu}{2}-\nu}-\mu'^2 e^{\frac{3\mu}{2}-\gamma}+4e^{\mu/2}. \tag{2.25}$$

Eqs. (2.15) and (2.14) with the help of (2.17) then yield

$$\dot{m} = 2\pi\dot{\mu}e^{3\mu/2}T^1{}_1 \tag{2.26}$$

and

$$m' = 2\pi\mu' e^{3\mu/2}T^0{}_0, \tag{2.27}$$

respectively. Eqs. (2.23)–(2.27) are completely equivalent to the field equations (2.14)–(2.17).

The set of Eqs. (2.23) and (2.24) on account of Eqs. (2.18a) and (2.18b) yield

$$p' = -\tfrac{1}{2}\nu'(\varrho+p)+\frac{QQ'}{4\pi}e^{-2\mu}-\frac{\lambda}{4\pi}\phi\phi' e^{-\nu}, \tag{2.28}$$

$$\dot{\varrho} = -(\dot{\mu}+\tfrac{1}{2}\dot{\gamma})\left(\varrho+p+\frac{\lambda\phi^2}{4\pi}e^{-\nu}\right)+\frac{\lambda e^{-\nu}}{8\pi}(\dot{\nu}\phi^2-2\phi\dot{\phi}), \tag{2.29}$$

while Eqs. (2.26) and (2.27) yield

$$\dot{m} = -2\pi\dot{\mu}pe^{\frac{3\mu}{2}} + \frac{\dot{\mu}Q^2}{4}e^{-\mu/2} - \frac{\lambda\dot{\mu}}{4}\phi^2 e^{\frac{3\mu}{2}-\nu}, \tag{2.30}$$

$$m' = 2\pi\mu'\varrho e^{3\mu/2} + \frac{\mu' Q^2}{4}e^{-\mu/2} + \frac{\lambda\mu'}{4}\phi^2 e^{\frac{3\mu}{2}-\nu}. \tag{2.31}$$

In the case $\dot{\mu} = 0$, the model is static, as is easily seen from Eqs. (2.30) and (2.31).

Alternatively, Eqs. (2.28) and (2.29) are derivable from the conservation equations

$$T^i_{k;i} = 0.$$

Evaluating $T^i_{k;i}$ with the help of Eqs. (2.6), (2.7) and (2.8) and comparing the results with Eqs. (2.28) and (2.29) we deduce that

$$J_0 = \frac{\lambda\phi\phi'}{Q(r)}e^{\mu-(\gamma+\nu)/2} - Q' e^{(\nu-\gamma)/2-\mu} \tag{2.32}$$

and

$$q = \frac{\lambda e^{-\nu}}{2}\{2\dot{\phi} + \phi(\dot{\gamma} + 2\dot{\mu} - \dot{\nu})\}. \tag{2.33}$$

It is seen from Eq. (2.33) that $q \equiv 0$ if the model is static; so that a sufficient condition for the creation rate to vanish, under our assumptions, is that the model should become static.

Eqs. (2.28)–(2.31) will now be used to derive the condition (2.22). Let us introduce the function $\bar{\varrho}(r, t)$ defined by the equation

$$m = \frac{4\pi}{3}\bar{\varrho}e^{3\mu/2}. \tag{2.34}$$

The function $\bar{\varrho}$ may be interpreted as the mean density of the matter distribution within the point (r, θ, Q, t), if $e^{\mu/2}$ is interpreted as the distance of the particle from the centre and m the mass of the fluid. Differentiating Eq. (2.34) with respect to time and using Eq. (2.30) we obtain

$$p = -\frac{2}{3}\frac{\dot{\bar{\varrho}}}{\dot{\mu}} - \bar{\varrho} + \frac{Q^2}{8\pi}e^{-2\mu} - \frac{\lambda\phi^2}{8\pi}e^{-\nu}.$$

Multiplying this equation by $\dot{\mu} + \frac{1}{2}\dot{\gamma}$, we obtain

$$-(\dot{\mu} + \tfrac{1}{2}\dot{\gamma})p = \frac{2\dot{\bar{\varrho}}}{3\dot{\mu}}(\dot{\mu} + \tfrac{1}{2}\dot{\gamma}) + \bar{\varrho}(\dot{\mu} + \tfrac{1}{2}\,,) + \frac{(\dot{\mu} + \frac{1}{2}\dot{\gamma})}{8\pi}(\phi^2\lambda e^{-\nu} - Q^2 e^{-2\mu}).$$

Rewriting Eq. (2.29) in the form

$$-(\dot{\mu} + \tfrac{1}{2}\dot{\gamma})p = \dot{\varrho} + (\dot{\mu} + \tfrac{1}{2}\dot{\gamma})\left(\varrho + \frac{\lambda\phi^2}{4\pi}e^{-\nu}\right) - \frac{\lambda e^{-\nu}}{8\pi}(\dot{\nu}\phi^2 - 2\phi\dot{\phi})$$

and comparing the last two equations we find that

$$\left\{\dot{\bar{\varrho}}-\dot{\varrho}+\frac{\lambda e^{-\nu}}{8\pi}(\dot{\nu}\phi^2-2\phi\dot{\phi})\right\}+(\dot{\mu}+\tfrac{1}{2}\dot{\gamma})\left(\bar{\varrho}-\varrho-\frac{\lambda\phi^2}{8\pi}e^{-\nu}\right)-\frac{(\dot{\mu}+\frac{1}{2}\dot{\gamma})}{8\pi}Q^2e^{-2\mu}$$

$$=\frac{\dot{\bar{\varrho}}}{3}(1-\dot{\gamma}/\dot{\mu}).$$

Multiply by $e^{\mu+\frac{1}{2}\gamma}$ and obtain this equation in the form

$$\left\{\left(\bar{\varrho}-\varrho-\frac{\lambda\phi^2}{8\pi}e^{-\nu}-\frac{Q^2e^{-2\mu}}{8\pi}\right)e^{\mu+\frac{1}{2}\gamma}\right\}^{\cdot}=\frac{\dot{\bar{\varrho}}}{3}(1-\dot{\gamma}/\dot{\mu})e^{\mu+\frac{1}{2}\gamma}+\frac{\dot{\mu}Q^2}{4\pi}e^{-\mu+\frac{1}{2}\gamma}. \tag{2.35}$$

Eq. (2.35) may be deduced from the equation

$$\{(\bar{\varrho}-T^0{}_0)e^{\mu+\gamma/2}\}^{\cdot}=\frac{\dot{\bar{\varrho}}}{3}(1-\dot{\gamma}/\dot{\mu})e^{\mu+\frac{1}{2}\gamma}+\dot{\mu}(T^1{}_1-T^2{}_2)e^{\mu+\frac{1}{2}\gamma}, \tag{2.36}$$

which holds for any energy-momentum tensor.

Eq. (2.35) cannot be integrated in general, but under the condition $\dot{\gamma}=\dot{\mu}$ it becomes integrable with respect to time. Thus, after inserting $\dot{\gamma}=\dot{\mu}$, $\gamma=\mu-a(r)$ into Eq. (2.35) we have

$$\bar{\varrho}-\varrho-\frac{\lambda\phi^2}{8\pi}e^{-\nu}+\frac{3Q^2}{8\pi}e^{-\mu}=H(r)e^{-3\mu/2},$$

where $H(r)$ is an arbitrary function of integration.

Use can now be made of Eq. (2.34) to eliminate $\bar{\varrho}$, we obtain

$$m-\frac{4\pi}{3}\varrho e^{3\mu/2}-\frac{\lambda\phi^2}{6}e^{3\mu/2-\nu}+\frac{Q^2}{2}e^{-\mu/2}=h(r), \tag{2.37}$$

here $h(r)=\dfrac{4\pi}{3}H(r)$. Now differentiate Eq. (2.37) with respect to r and use Eq. (2.31) to obtain the final equation in the form

$$\frac{\lambda}{6}(\nu'\phi^2-2\phi\phi')e^{3\mu/2-\nu}+(QQ'e^{-\mu/2}-h')=0, \tag{2.38}$$

which is exactly the same as Eq. (2.22) if we make the identification

$$h=\tfrac{1}{2}be^{a}. \tag{2.39}$$

Now if $\nu'\phi^2-2\phi\phi'=0$, we conclude from Eq. (2.38) that $e^{\mu/2}$ becomes a function of r only. In other words, the model becomes static. The condition $\nu'\phi^2-2\phi\phi'=0$ implies that $\phi=\Gamma_0e^{\nu/2}$, where Γ_0 is a constant of integration. It then follows from Eq. (2.13) that the charge distribution $Q(r)$ is given by

$$Q(r)=\Gamma_0(\nu'/2)e^{\mu-\gamma/2}. \tag{2.40}$$

As was already remarked these circumstances lead to the conclusion that the matter creation rate q vanishes. This conclusion, of course, comes from Eq. (2.33). Thus for a Lyttleton--Bondi sphere embedded in empty space the conditions $\varrho = \varrho(t)$, $\dot{\gamma} = \dot{\mu}$ and $\phi = \Gamma_0 e^{\nu/2}$ imply that the sphere is static and has the electric charge distribution (2.40).

3. *The model solution*

We now consider a particular solution of Eq. (2.21) for which the condition $\nu'\phi^2 - 2\phi\phi' = 0$ is not satisfied. Under this circumstance e^{μ} is a function of time so that the model is non-static. In order to extract a solvable model from the variety of models obtainable from Eq. (2.21) we use Eq. (2.38). This constraint equation offers us some freedom in our choice of the function $h = \frac{1}{2}b\dot{e}^{a}$. Further we assume that the creation rate is such that the quantity $Qe^{-\mu/2}$, which we interpret as the potential at the point $R = e^{\mu/2}$ due to $Q(r)$ — the charge up to the co-ordinate radius r, is a constant independent of time. With this we choose be^{a} in Eq. (2.38) such that the condition

$$8\lambda(\phi^2 e^{-\nu})' e^{\mu} + 24Q^2(e^{-\mu})' - 3(be^{a})(e^{-\mu/2})' + (be^{a})' e^{-\mu/2} = 0 \tag{3.1}$$

is satisfied. With this choice Eq. (2.21) becomes

$$2\mu'' - \mu'^2 + \mu' a' + 4e^{-a} = 0. \tag{3.2}$$

It should be pointed out that Thompson and Whitrow (1967) derived an equation similar to Eq. (2.21) and reduced it to Eq. (3.2) under entirely different circumstances. Thus our discussion of Eq. (3.2) will be similar to theirs.

The differential Eq. (3.2) may be solved to obtain

$$e^{-\mu/2} = Bf + C/f, \tag{3.3}$$

where

$$f = \exp\left(-\int e^{-a/2} dr\right) \tag{3.4}$$

and $B = B(t)$, $C = C(t)$ are arbitrary functions of integration. The functions $B(t)$, $C(t)$ are related by an equation derived from the boundary condition $p(r_0) = 0$, where $r = r_0$ at the surface of the sphere. Although, corresponding to a certain degree of freedom in the choice of the equation of state, the functions B, C are not determined uniquely by conditions at the boundary $r = r_0$, there are certain conditions to be imposed on them if a realistic model is sought. In particular, we should expect that the pressure p should decrease as we go from the centre of the sphere to the surface, that is $p' < 0$. We find that this implies that, so long as the sphere expands, $(C/B)' < 0$. The parameters γ, ν, p and ϱ of our problem will now be written in terms of the general solution (3.3) and the quantities Q and ϕ.

We start from Eq. (2.28) which we rewrite in the form

$$(pe^{\nu/2})' = -(\varrho e^{\nu/2})' + \frac{QQ'}{4\pi} e^{-2\mu+\nu/2} - \frac{\lambda\phi\phi'}{4\pi} e^{-\nu}.$$

Integration of this equation yields

$$pe^{\nu/2} = -\varrho e^{\nu/2} + G(r, t) + \Gamma(t),$$

where $\Gamma(t)$ is an arbitrary function of integration. $\Gamma(t)$ is to be evaluated from the boundary condition $p(r_0) = 0$. We then find that

$$p = \varrho\{e^{(\nu_0-\nu)/2} - 1\} + e^{-\nu/2}\{G(r, t) - G(r_0, t)\}, \tag{3.5}$$

where

$$G(r, t) = \int^{r} g(r', t)dr', \qquad \nu_0 = \nu(r_0, t)$$

and

$$g(r, t) = (1/4\pi)\,(QQ'e^{-2\mu+\nu/2} - \lambda\phi\phi' e^{-\nu/2}). \tag{3.6}$$

Similarly, using Eqs. (2.14), (2.18a), (2.20), (3.3) and $\gamma = \mu - a$ we find that

$$8\pi\varrho = \tfrac{3}{4}\,(\dot{\mu}^2 e^{-\nu} + 16BC) - Q^2 e^{-2\mu} - \lambda\dot{\phi}^2 e^{-\nu}. \tag{3.7}$$

In Eq. (2.20) we may normalize the time co-ordinate t so that $e^{\nu(0,t)} = 1$. This is equivalent to choosing $A(t) = 1/\dot{\mu}^2(0, t)$ so that we can write

$$\dot{\mu}^2 e^{-\nu} = \dot{\mu}^2(0, t).$$

But from Eq. (3.3) we find that

$$\dot{\mu} = -2\,\frac{(\dot{B}f^2 + C)}{(Bf^2 + C)}, \tag{3.8}$$

so that we deduce that

$$\dot{\mu}(0, t) = -2\,\frac{(\dot{B}f^2(0) + \dot{C})}{(Bf^2(0) + C)}. \tag{3.9a}$$

Now using the fact that $f = S/r$, where S is a power series in r involving only positive and zero powers of r (Thompson and Whitrow 1967), we find that

$$\dot{\mu}(0, t) = -2\dot{B}/B. \tag{3.9b}$$

It then follows from Eqs. (3.7) and (3.9b) that

$$8\pi\varrho = 3\{(\dot{B}/B)^2 + 4BC\} - Q^2 e^{-2\mu} - \lambda\dot{\phi}^2 e^{-\nu}.$$

By assumption ϱ is not sensitive to variations in r, so that this expression may be evaluated at any convenient value of r. But because we must compare our results with ϱ obtained from Eq. (2.29) it is best to evaluate these r dependent variables at $r = r_0$. We then write for ϱ the expression

$$8\pi\varrho = 3\{(\dot{B}/B)^2 + 4BC\{ - Q_0^2 e^{-2\mu_0} - \lambda\dot{\phi}_0^2 e^{-\nu_0}, \tag{3.10}$$

where $Q_0 = Q(r_0)$, $\quad \mu_0 = \mu(r_0, t)$, $\quad \phi_0 = \phi(r_0, t)$.

If in Eq. (2.29) we put $\dot{\gamma} = \dot{\mu}$ and apply the boundary condition $p(r_0) = 0$, we find that

$$\dot{\varrho} = -\tfrac{3}{2}\dot{\mu}_0\left(\varrho + \frac{\lambda\phi_0^2}{4\pi}e^{-\nu_0}\right) - \frac{\lambda}{8\pi}(\phi_0^2 e^{-\nu_0})^{\cdot}$$

or

$$\dot{\varrho} + \frac{\lambda}{4\pi}(\phi_0^2 e^{-\nu_0})^{\cdot} = -\tfrac{3}{2}\dot{\mu}_0\left(\varrho + \frac{\lambda\phi_0^2}{4\pi}e^{-\nu_0}\right) + \frac{\lambda}{8\pi}(\phi_0^2 e^{-\nu_0})^{\cdot}.$$

Putting $\xi = \varrho + \dfrac{\lambda}{4\pi}(\phi_0^2 e^{-\nu_0})$, this equation becomes

$$\dot{\xi} = -\tfrac{3}{2}\dot{\mu}_0\xi + \frac{\lambda}{8\pi}(\phi_0^2 e^{-\nu_0})^{\cdot}. \tag{3.11}$$

Eq. (3.11) is a first-order linear differential equation whose solution is, after putting $\xi = \varrho + (\lambda/4\pi)\phi_0^2 e^{-\nu_0}$,

$$8\pi\varrho = 3K^2(Bf_0^2 + C)^3 - \lambda\phi_0^2 e^{-\nu_0} - \frac{3\lambda}{2}\eta(t)e^{-3\mu_0/2}, \tag{3.12}$$

where we have defined $f_0 = f(r_0)$, $3K^2 = 8K_1$, K_1 a constant and $\eta(t)$ by

$$\eta(t) = \int \dot{\mu}_0\phi_0^2 e^{-\nu_0}e^{3\mu_0/2}dt. \tag{3.13}$$

Comparing Eqs. (3.10) and (3.12) we find the equation relating B and C for given Q_0 and ϕ_0 in the form

$$3\{(\dot{B}/B)^2 + 4BC\} - 3K^2(Bf_0^2 + C)^3 = Q_0^2 e^{-2\mu_0} - \frac{3\lambda}{2}\eta(t)e^{-3\mu_0/2}. \tag{3.14}$$

The function $\eta(t)$ defined by Eq. (3.13) is quite general. However, under certain assumptions the integral (3.13) may be evaluated explicitly to determine $\eta(t)$. One interesting case is when $\phi e^{-\nu/2} = \Gamma_0$, or $\phi = \Gamma_0 e^{\nu/2}$, Γ_0 a constant. This is the case we have already discussed at length; and we have shown that for this value of ϕ the solution becomes static. In a more general situation ϕ must be determined from Eq. (2.13). It is easy to show, using Eqs. (2.13), (2.20) and (3.3), that

$$\phi = -(B/\dot{B})\int^{r} Q(r)\,(\dot{B}f + \dot{C}/f)e^{-a/2}dr. \tag{3.15}$$

Thus given the charge distribution $Q(r)$ and the function $f(r)$ Eq. (3.15) enables us to find ϕ. Eq. (3.14) may therefore be considered as the equation relating B and C. Eqs. (2.20), (3.3), (3.4) and (3.15) may now be used to determine J_0 and the matter creation rate q from Eqs. (2.32) and (2.33).

4. *Conditions for a physically realistic model*

Our results have been written in terms of the functions $B(t)$ and $C(t)$. For a physically realistic model these functions cannot be completely arbitrary. We now seek other conditions that these functions must satisfy to make our solutions physically meaningful. On physical grounds we expect p and ϱ to be positive and finite; and as already pointed out the pressure must decrease from the centre outwards to the surface of the sphere where it must vanish. Further, the solution of interest to us is that for which the model sphere continually expands. We find that the following conditions must be imposed on our solution:

(a) $0 < \varrho < \infty$,

(b) $-\infty < p' < 0$,

(c) $e^{\mu/2} = R(r, t) > 0, \quad r > 0$,

(d) $(\dot{\mu}/2)e^{\mu/2} = \dot{R}(r, r) > 0, \quad r > 0; \quad$ i.e. $\dot{\mu} > 0, \quad r > 0$.

For the model to continually expand we must require that $R(r, t) > 0$ and $\dot{R}(r, t) > 0$. Now $\dot{R}(r, t) = (\dot{\mu}/2)e^{\mu/2}$; this requires that $\dot{\mu} > 0$. To ensure that $\dot{\mu} > 0$, we shall choose B and C so that $B > 0$, $C > 0$, $\dot{B} < 0$ and $\dot{C} < 0$; so that B and C are chosen to be positive decreasing functions. It is seen from Eq. (3.8) that this choice makes $\dot{\mu} > 0$. In fact this choice ensures, according to Eq. (3.3), that $R(r, t) > 0$.

Eq. (3.10) shows that ϱ is positive and bounded provided that B and C are chosen as prescribed above and also satisfy the condition

$$3\{(\dot{B}/B)^2+4CB\} > Q_0^2 e^{-2\mu_0}+\lambda\phi_0^2 e^{-\nu_0}. \tag{4.1}$$

Using Eq. (2.20) and the condition $A(t) = 1/\dot{\mu}^2(0, t)$, we find from Eq. (3.5) that $p(r, t)$ may be written in the form

$$p(r, t) = \varrho\left\{\frac{\dot{\mu}(r_0, t)}{\dot{\mu}(r, t)}-1\right\}+e^{-\nu/2}\{G(r, t)-G(r_0, t)\}, \tag{4.2}$$

so that

$$p' = -\frac{\dot{\mu}'}{\dot{\mu}}(\varrho+p)+e^{-\nu/2}\{G(r, t)-G(r_0, t)\}\left(\frac{\dot{\mu}'}{\dot{\mu}}-\frac{\nu'}{2}\right)+G'(r, t)e^{-\nu/2}. \tag{4.3}$$

From Eq. (3.8) we find that

$$-\dot{\mu} = \frac{2\dot{B}(x+f^2+B\dot{x}/\dot{B}-f_0^2}{B(f^2+x-f_0^2)}$$

$$\dot{\mu}' = \frac{4\dot{x}ff'}{(f^2+x-f_0^2)^2}, \tag{4.4}$$

where $x(t) = f_0^2 + C/B$. We then find that

$$p' = \frac{2Bff'(\varrho+p)}{(f^2+x-f_0^2)(x+f^2+B\dot{x}/\dot{B}-f_0^2)}\left(\frac{\dot{x}}{\dot{B}}\right)+g(r,t)e^{-\nu/2}. \tag{4.5}$$

By Eq. (3.4), $f > 0$, and since $f = S/r$, with S a power series involving only positive and zero powers of r, $f \to \infty$ as $r \to 0$. Also $f' < 0$, and physical considerations require that $\varrho+p > 0$. Thus to make the first term on the right-hand side of Eq. (4.5) negative, we shall require that $(\dot{x}/\dot{B}) > 0$ and finite. The second term on the right-hand side of Eq. (4.5) is negative if we require that

$$(\dot{Q}Q'e^{-2\mu}-\lambda\phi\phi'e^{-\nu}) < 0 \tag{4.6}$$

so that the necessary and sufficient condition for $p' < 0$ is

$$0 < \dot{x}/\dot{B} < \infty, \qquad QQ'e^{-2\mu} < \lambda\phi\phi'e^{-\nu}. \tag{4.7}$$

We shall now use the fact that $p' < 0$ to establish that we always have

$$\frac{d}{dt}\left(\frac{C}{B}\right) < 0 \tag{4.8}$$

in the case of expansion. Eq. (2.28) may be written in the form

$$p' = -\tfrac{1}{2}\nu'(\varrho+p)+g(r,t), \qquad g(r,t) < 0.$$

Here $\varrho+p > 0$, so that $p' < 0$ if $\nu' > 0$. We have already stated that $\dot{\mu} > 0$ in the case of expansion. From Eq. (2.17) we have putting $\dot{\mu} = \dot{y}$

$$\nu' = 2\dot{\mu}'/\dot{\mu}.$$

So that if $\nu' > 0$, $\dot{\mu} > 0$, then $\dot{\mu}' > 0$. But by Eq. (3.8)

$$\dot{\mu}' = \frac{4ff'(B\dot{C}-\dot{B}C)}{(Bf^2+C)^2}.$$

But because $f > 0$, $f' < 0$, we have $ff' < 0$. Thus $\dot{\mu}' > 0$ provided that $B\dot{C}-\dot{B}C < 0$. This condition implies that

$$\frac{d}{dt}\left(\frac{C}{B}\right) < 0 \tag{4.9}$$

in the case of expansion.

5. *Conclusion*

The general belief prior to this work was that the Lyttleton–Bondi charge imbalance theory can only explain the observed expansion of the Universe and nothing else. However, we have now shown that starting from arbitrary initial conditions, but admitting the possibility of matter creation, our model sphere either grows to a static state or will expand forever. Further, we emphasize that charge distribution alone may be able to halt this expansion. Thus the Universe may not after all expand forever.

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