REMARKS ON THE PHYSICAL MEANING OF THE "IMPROVED" ENERGY-MOMENTUM TENSOR

BY U. KASPER

Zentralinstitut für Astrophysik der AdW der DDR, Potsdam*

(Received February 4, 1981)

In the framework of classical field theory, we look for the physical meaning of the "improved" energy-momentum tensor for a scalar field. We study the motion of a point-like particle interacting with a scalar field. Together with the "improved" energy-momentum tensor, we introduce a force, which depends on the velocity of the particle and is, if certain conditions are fulfilled, repulsive.

PACS numbers: 04.20.Me

In [1] the question is considered, whether there is an energy-momentum tensor for a renormalizable field theory which gives finite matrix elements in each order of the perturbation series. In the case of a self-interacting scalar field such an "improved" energy-momentum tensor is the quantity

$$\Theta_{kl} = T_{kl} - \tfrac{1}{6}(\partial_k\partial_l - g_{kl}\Box)\varphi^2, \tag{1}$$

where

$$T_{kl} = \varphi_{,k}\varphi_{,l} - g_{kl}L' \quad \left(\varphi_{,a} := \frac{\partial\varphi}{\partial x^a}\right) \tag{2}$$

with

$$L' \doteq \tfrac{1}{2}\,\partial_k\varphi\partial_l\varphi\eta^{kl} - \tfrac{1}{2}\,\mu_0^2\varphi^2 - \lambda_0\varphi^4 \tag{3}$$

is the conventional energy-momentum tensor. We obtain the "improved" energy-momentum tensor Θ_{kl} in the framework of the canonical formalism, if we start from the Lagrangian [2]

$$L'' = \tfrac{1}{6}\,\partial_k\varphi\partial_l\varphi\eta^{kl} - \tfrac{1}{3}\,\varphi\eta^{kl}\partial_k\partial_l\varphi - \tfrac{1}{2}\,\mu_0^2\varphi^2 - \lambda_0\varphi^4. \tag{4}$$

---

* Address: Zentralinstitut für Astrophysik der AdW der DDR, Rosa-Luxemburg-Str. 17a, 1502 Potsdam, DDR.

This expression differs from the one in (3) only by a total divergence

$$-\tfrac{1}{3}(\eta^{kl}\varphi\partial_k\varphi)_{,l}. \tag{5}$$

It is well-known that a total divergence does not change the field equations, if we add it to a Lagrangian. But it will change our understanding of what the flow of energy and momentum and tensions are.

In (3) and (4), we meet the same mass term and self-interaction ansatz. Therefore, we will approach the physical meaning of the "improved" energy-momentum tensor, if we concentrate on those terms of the Lagrangian containing derivatives of the scalar field only.

To be as general as possible, we start from a Lagrangian

$$L = a\varphi\varphi_{,kl}\eta^{kl} + b\varphi_{,k}\varphi_{,l}\eta^{kl}, \tag{6}$$

where a and b are two parameters.

By well-known methods we obtain from (6) the scalar field equation

$$(2a-2b)\eta^{kl}\varphi_{,kl} = 0 \tag{7}$$

and the canonical energy-momentum tensor

$$t_{kl} = -a(\varphi_{,k}\varphi_{,l} - \varphi\varphi_{,kl} + \eta_{kl}\varphi\Box\varphi) + 2b(\varphi_{,k}\varphi_{,l} - \tfrac{1}{2}\eta_{kl}\varphi_{,p}\varphi_{,q}\eta^{pq}). \tag{8}$$

The second term of (8) is proportional to the conventional energy-momentum tensor of a scalar field with vanishing rest mass and without self-interaction. The first term can be written as follows

$$\begin{aligned} A_{kl} := {} & -a(\varphi_{,k}\varphi_{,l} - \varphi\varphi_{,kl} + \eta_{kl}\varphi\Box\varphi) \\ = {} & -a\eta_{ke}U_l{}^{ec}{}_{,c} - 2a(\varphi_{,k}\varphi_{,l} - \tfrac{1}{2}\eta_{kl}\eta^{pq}\varphi_{,p}\varphi_{,q}) \end{aligned} \tag{9}$$

with

$$U_l{}^{ec} = -U_l{}^{ce} = \tfrac{1}{2}(\delta_l{}^e\delta_d{}^c - \delta_d{}^e\delta_l{}^c)\eta^{pd}\partial_p\varphi^2. \tag{10}$$

We recognize that the first term in (8) is, up to a total divergence of an antisymmetric quantity, proportional to the conventional energy-momentum tensor, too.

The tensor A_{kl} has got an interesting property. In the case of a static scalar field, the energy density A_{00} is equal to zero wherever the field equation $\Delta\varphi = 0$ is satisfied. This makes us expect, that the tensor A_{kl} does not contribute to the self-energy of a point-like particle creating a scalar field only.

To obtain information on the physical meaning of the "improved" energy-momentum tensor let us have a look at the interaction of a point-like particle creating a scalar field φ with a scalar field Φ. This problem was considered from different points of view several times in the past. Some authors [3] used the conventional energy-momentum tensor as a starting point. Another approach [4] starts from the equation of motion without a force term depending on the velocity of the particle.

We choose as the scalar field created by a point-like particle the retarded potential

$$\varphi(x) = -\frac{g}{(\dot{z}(s), x-z(s))}, \tag{11}$$

g is a coupling number and $z^k(s)$ means the world line of the particle the parameter of which is the proper-time s of the particle. We choose the notation

$$(\dot{z}(s), x-z(s)) = \eta_{kl}\frac{dz^k}{ds}(x^l - z^l(s)) \tag{12}$$

and

$$v^k = \frac{dz^k}{ds} = \dot{z}^k. \tag{13}$$

According to Dirac's procedure [5], we obtain from the energy-momentum tensor (8), written with $\varphi+\Phi$ instead of φ, the following expression, which is proportional to the rate of flow of energy and momentum

$$b\left[-g^2\dot{v}_k \lim_{\varepsilon\to 0}\frac{1}{\varepsilon} + \tfrac{2}{3}g^2(\ddot{v}_k+\dot{v}^2v_k)+2g\Phi_{,k}\right]$$
$$-a\left[\tfrac{2}{3}g^2(\ddot{v}_k+\dot{v}^2v_k)-g\Phi\dot{v}_k+\tfrac{4}{3}g\Phi_{,k}-\tfrac{1}{3}g\Phi_{,p}v^pv_k\right]. \tag{14}$$

This expression has to be the derivative of a quantity — let us call it B_k — with respect to the proper-time s. We put

$$B_k = (2b-a)g\Phi v_k - bg^2v_k\lim_{\varepsilon\to 0}\frac{1}{\varepsilon} + m(2b-2a)v_k \tag{15}$$

and obtain the equation of motion of a point-like particle interacting with a scalar field

$$\frac{d}{ds}[(m+g\Phi)v^k] - \tfrac{1}{3}g^2(\ddot{v}^k+\dot{v}^2v^k)$$
$$-\frac{3b-2a}{3(b-a)}g\Phi_{,p}\eta^{pk} + \frac{ag}{3(b-a)}\Phi_{,p}v^pv^k = 0. \tag{16}$$

m is the renormalized mass the origin of which is not the self-field of the particle. As one could expect, the calculations show that the non-renormalized mass contains a divergent term coming only from the part of the energy-momentum (8) proportional to the conventional one.

To get the "improved" energy-momentum tensor we have to put

$$a = -\tfrac{2}{6}, \quad b = \tfrac{1}{6}. \tag{17}$$

In this case the equation of motion reads

$$\frac{d}{ds}[(m+g\Phi)v^k] - \tfrac{1}{3}g^2(\ddot{v}^k+\dot{v}^2v^k) - \tfrac{7}{9}g\Phi_{,p}\eta^{pk} - \tfrac{2}{9}g\Phi_{,p}v^pv^k = 0. \tag{18}$$

The equation of motion (18) does not give us any hint why we should choose the parameter values (17).

Now, we compare (18) with the equation of motion resulting in the case where we start from the conventional energy-momentum tensor. It reads

$$\frac{d}{ds}\left[(m+g\Phi)v^k\right]-g\Phi_{,p}\eta^{pk}-\tfrac{1}{3}\,g^2(\ddot{v}^k+\dot{v}^2v^k) = 0. \tag{19}$$

We recognize that there is no change of the "Abraham vector", but the force acting on the particle is changed in an important way. A force term enters (18) depending on the velocity of the particle. In the case $g > 0$ and $\Phi_{,\alpha} > 0$ ($\alpha = 1, 2, 3$), the conventional force term which is independent of particle's velocity is weakened. Additionally, if $\Phi_{,k}v^k < 0$, there is a force acting in the direction opposite to that given by the motion of the particle.

It would be most satisfactory to consider the case where Φ is the sum of a pure radiation field and retarded fields of other point-like particles. This case shall be postponed and, instead of it, we suppose that Φ is a static field of the form

$$\Phi = -\frac{M}{r} \quad (M > 0) \tag{20}$$

which means formally, the breaking of Newton's third law. Then, the force term of (18) is given by

$$\boldsymbol{K} = -\tfrac{7}{9}\,gM\frac{\boldsymbol{r}}{r^3}+\tfrac{2}{9}\,gM\frac{1}{r^3}(\boldsymbol{r}\cdot\boldsymbol{v})\boldsymbol{v}. \tag{21}$$

In the case $g > 0$, the particle is attracted by a force

$$\boldsymbol{K}_1 = \tfrac{7}{9}\,gM\frac{\boldsymbol{r}}{r^3} \tag{22}$$

and, moving in the direction of the source of the field given by (20), is repelled by the force

$$\boldsymbol{K}_2 = +\tfrac{2}{9}\,gM\frac{1}{r^3}(\boldsymbol{r}\cdot\boldsymbol{v})\boldsymbol{v}. \tag{23}$$

One could expect that this repulsion, *depending on the velocity* of the particle, is the origin of the fact that momentum space integrations give finite matrix elements for the "improved" energy-momentum tensor in quantum field theory.

Remark. Although the "Abraham vector" enters the "improved" and conventional equation of motion in the same way, this need not mean that the radiation rate is the same in both cases as was pointed out in [6]. This subject was not considered here because there is no necessity to obtain the same radiation rate. Contrary to electrodynamics, we have no experimental evidence concerning the motion of a point-like particle creating a scalar field. Emphasis here was on the introduction of a velocity dependent force term if we start from the "improved" energy-momentum tensor.

I am indebted to Dr. R. W. John for drawing my attention to the papers [3] and [4] and checking my calculations leading to (14).

REFERENCES

[1] C. G. Callan Jr., S. Coleman, R. Jackiw, *Ann. Phys.* (USA) **59**, 42 (1970).
[2] U. Kasper, in Abstracts of Contr. Papers to the 9th Intern. Conf. on Gen. Rel. and Gravitation, Jena 1980, Vol. 3, p. 566.
[3] C. Teitelboim, *Phys. Rev.* **D1**, 1572 (1970); D. Leiter, G. Szamosi, *Phys. Rev.* **D13**, 2762 (1976).
[4] A. O. Barut, D. Villarroel, *J. Phys.* **A8**, 156 (1975).
[5] P. A. M. Dirac, *Proc. R. Soc.* **A167**, 148 (1938).
[6] D. Villarroel, *Ann. Phys.* (USA) **89**, 241 (1975).