VECTOR POTENTIAL IN A UNIFIED FIELD THEORY

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The paper gives a modification of a unified theory of gravitation and electromagnetism described previously by the author. The modification leads to a geometric interpretation of the electromagnetic potential A_μ : A_μ is proportional to h^5_μ, a component of the five-dimensional equivalent of a tetrad.

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1. Introduction

The unified field theory described in reference [1] introduces directly the electromagnetic field tensor as the fifth component of the Lorentz gauge potential. There is no natural geometric interpretation of the electromagnetic vector potential. This could be considered as an unpleasant feature of the theory when, for example, the Dirac equation in the presence of the electromagnetic field is to be used. It is well known, that under certain circumstances the potential is directly measurable in experiments involving quantum mechanics (see [2] or [3]). A modification of the theory presented here gives an interesting geometric meaning to A_μ, namely as proportional to h^5_μ, a component of the five-dimensional equivalent of a tetrad.

2. Geometrical structure

The basic geometrical structure leading naturally to the appearance of the electromagnetic field tensor is a de Sitter structured connection based on a five-dimensional manifold, as described in reference [1]. The interpretation of the components of the connection is as follows: A^{ik}_μ are connected with the Christoffel symbols by the usual relation

$$\Gamma^\sigma_{\mu\nu} = h^\sigma_k(\partial_\mu h^k_\nu)+h^\sigma_k A^{kl}_\mu h^j_\nu g_{lj}, \tag{1}$$

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A_5^{ik} is the electromagnetic field tensor in the local unholonomic Minkowski coordinates, $A_5^{i5} = 0$ guarantees the four-dimensionality of the observable space-time, and $A_\mu^{i5} = l^{-1}h_\mu^i$, where h_μ^i, $i = 1, \ldots, 4$, $\mu = 1, \ldots, 4$, is the tetrad, and l is a small fundamental length. The four-dimensional tetrad was complemented in [1] by $h_5^i = 0$, $h_\mu^5 = 0$, and $h_5^5 = 1$. We shall now relax this requirement. It seems reasonable to require only that the inverse h_i^μ of the 4×4 matrix h_μ^i forms the 4×4 part of the inverse h_a^α, $\alpha = 1, \ldots, 5$, $a = 1, \ldots, 5$, of the whole pentad h_α^a. Physically, this condition could be interpreted as characterising impossibility of detecting the fifth coordinate by space-time measurements. Mathematically it states

$$h_\mu^a h_a^\sigma = h_\mu^i h_i^\sigma + h_\mu^5 h_5^\sigma = \delta_\mu^\sigma$$

and

$$h_\mu^i h_i^\sigma = \delta_\mu^\sigma,$$

hence

$$h_\mu^5 h_5^\sigma = 0.$$

The geometry used in reference [1] certainly satisfies this condition, but in fact only one of h_μ^5 and h_5^σ must be equal to zero. We put

$$h_5^\sigma = 0 \tag{2}$$

thus reserving $h_\mu^5 \neq 0$ for the possible role of the electromagnetic field potential.

We have to require further that

$$h_\mu^a h_a^5 = h_\sigma^i h_i^5 + h_\mu^5 h_5^5 = 0.$$

Preserving for simplicity $h_5^5 = 1$ we obtain

$$h_\mu^5 = -h_\mu^i h_i^5$$

or

$$h_i^5 = -h_i^\mu h_\mu^5. \tag{3}$$

Finally

$$h_\alpha^i h_k^\alpha = h_\mu^i h_k^\mu + h_5^i h_k^5 = \delta_k^i$$

requires

$$h_5^i = 0. \tag{4}$$

The remaining relations are then satisfied identically.

3. *Dirac equation with the electromagnetic interaction*

Consider the Dirac equation in the space of general relativity:

$$\gamma^i h_i^\mu \left(\frac{\partial \psi}{\partial x^\mu} - \tfrac{1}{2} A_\mu^{ik} L_{ik} \psi\right) - \frac{imc}{\hbar}\psi = 0. \tag{5}$$

Here L_{ik} are the generators of the Lorentz transformations in the appropriate representation. This is just the Lorentz covariant equation. In the de Sitter gauge theory it should be a consequence of a more general de Sitter covariant equation when expressed in a specific gauge along the lines of reference [4]. This, however, needs reconsideration in the light of the approach presented in this paper, and at present we shall take equation (5) as the starting point. Nevertheless, the base manifold must be the five-dimensional manifold described in Section 1 and the fifth coordinate yields an additional term

$$\gamma^i h_i^5 \left(\frac{\partial \psi}{\partial x^5} - \tfrac{1}{2} A_5^{ik} L_{ik} \psi \right).$$

Using equation (3) this term can be combined with the corresponding space-time term in equation (5) to yield

$$\gamma^i h_i^\mu \left(\frac{\partial \psi}{\partial x^\mu} - h_\mu^5 \frac{\partial \psi}{\partial x^5} - \tfrac{1}{2} (A_\mu^{ik} - h_\mu^5 A_5^{ik}) L_{ik} \psi \right).$$

We assume now that

$$h_\mu^5 = kA_\mu, \tag{6}$$

where A_μ is the vector potential, and k is a constant of dimension charge/work. It is related to the gravitational interaction constant. The relationship is discussed in the next section.

One should point out that equation (6) leads to

$$g_{\mu 5} = h_\mu^a h_5^b g_{ab} = h_\mu^5 = kA_\mu,$$

which is precisely the interpretation given to $g_{\mu 5}$ in the Kaluza-Klein theory [5]. Some parts of this section are, in fact, contained in reference [5], but without tetrads and the Dirac field.

The usual minimal electromagnetic interaction term

$$\frac{\partial \psi}{\partial x^\mu} - i \frac{e}{\hbar c} A_\mu \psi \tag{7}$$

is obtained by requiring that the dependence of ψ on x^5 is given by

$$\psi(x^\mu, x^5) = e^{ix^5/d} \psi(x^\mu, 0), \tag{8}$$

where

$$d = \frac{\hbar c k}{e},$$

e being the charge of the particle. Constant d has a dimension of length and its size will also be discussed in the next section. There is an extra term

$$h_\mu^5 A_5^{ik}$$

also describing the electromagnetic interaction, but since A_5^{ik} is related to the electromagnetic field tensor via a constant which is also of the order of the gravitational interaction constant, the contribution of this term is negligible.

Consider now a coordinate transformation

$$x^{\mu'} = x^{\mu}, \quad \mu = 1, \ldots, 4,$$
$$x^{5'} = x^5 + \eta(x^{\mu}), \tag{9}$$

where η is an arbitrary smooth function of the space-time coordinates. We have

$$\psi'(x^{\mu'}, x^{5'}) = \psi(x^{\mu}, x^5) = e^{-i\eta/d}\psi(x^{\mu'}, x^{5'})$$

and

$$h_{\mu}^{5'} = \frac{\partial x^{\nu}}{\partial x^{\mu'}} h_{\nu}^5 + \frac{\partial x^5}{\partial x^{\mu'}} h_5^5 = h_{\mu}^5 - \frac{\partial \eta}{\partial x^{\mu}}$$

or

$$A'_{\mu} = A_{\mu} - \frac{1}{k}\frac{\partial \eta}{\partial x^{\mu}}.$$

This is precisely the "gauge" transformation of the vector potential and the wave function that leaves the term (7) invariant. It should be noted that the coordinate transformation (9) also leaves A_5^{ik} invariant:

$$\Gamma_5^{ik'} = \frac{\partial x^5}{\partial x^{5'}} \Gamma_5^{ik} + \frac{\partial x^{\mu}}{\partial x^{5'}} \Gamma_{\mu}^{ik} = \Gamma_5^{ik}.$$

Thus the interpretation of A_5^{ik} as the electromagnetic field tensor is not contradicted.

4. *Maxwell equations*

Since now A_5^{ik} is associated with the electromagnetic field tensor while h_{μ}^5 is proportional to the vector potential, a relation connecting the two quantities is needed. Such a relation can be expected to involve the torsion tensor as can be seen from the following expressions:

$$T_{\mu\nu}^{\sigma} = t_{\mu\nu}^{\sigma} + l^{-1}(h_{\nu}^5 \delta_{\mu}^{\sigma} - h_{\mu}^5 \delta_{\nu}^{\sigma}), \tag{10a}$$

$$T_{\mu\nu}^5 = -h_{\sigma}^5 t_{\mu\nu}^{\sigma} + \partial_{\mu} h_{\nu}^5 - \partial_{\nu} h_{\mu}^5, \tag{10b}$$

$$T_{\mu 5}^{\sigma} = l^{-1}\delta_{\mu}^{\sigma} - h_i^{\sigma} h_{\mu}^l A_5^{ik} g_{lk}. \tag{10c}$$

These expressions were obtained from the general formula for the torsion

$$T_{\alpha\beta}^{\gamma} = h_a^{\gamma}(\partial_{\alpha} h_{\beta}^a - \partial_{\beta} h_{\alpha}^a + h_{\beta}^b A_{\alpha}^{ac} g_{bc} - h_{\alpha}^b A_{\beta}^{ac} g_{bc}), \tag{11}$$

using equations (2), (3), (4), as well as $A_{\mu}^{i5} = l^{-1} h_{\mu}^i$ and an assumption that all the components of the pentad are independent of x^5. $t_{\mu\nu}^{\sigma}$ is the space-time torsion given by equation (11) in which all the indices are restricted to values from 1 to 4 only. If it is assumed that $t_{\mu\nu}^{\sigma} = 0$, then in the first approximation a relation

$$T_{\alpha\beta\gamma} + T_{\beta\gamma\alpha} + T_{\gamma\alpha\beta} = 0, \tag{12}$$

where

$$T_{\alpha\beta\gamma} = g_{\beta\delta}T^{\delta}_{\alpha\gamma} \tag{13}$$

yields the relation between A^{ik}_5 and h^5_μ we are looking for. In particular, when the index is lowered in equation (13), we use the approximate metric with $g_{55} = 1$ and the off--diagonal elements $g_{\mu 5}$ equal to zero, since the inclusion of $g_{\mu 5} = h^5_\mu = kA_\mu$ leads to terms of higher order in k, and k is assumed to be small. In this approximation

$$T_{\mu\sigma\nu} = kl^{-1}(g_{\sigma\mu}A_\nu - g_{\sigma\nu}A_\mu), \tag{14a}$$

$$T_{\mu 5\nu} = k(\partial_\mu A_\nu - \partial_\nu A_\mu), \tag{14b}$$

$$T_{\mu\nu 5} = kl^{-1}g_{\mu\nu} - g_{\nu\varrho}h^\varrho_i h^l_\mu A^{ik}_5 g_{lk}. \tag{14c}$$

Expression (14a) satisfies (12) identically, while substitution of (14b) and (14c) into (12) yields

$$h^m_\mu h^n_\nu A^{ik}_5 g_{mi} g_{nk} = \frac{k}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu), \tag{15}$$

which expresses the electromagnetic field tensor in terms of the vector potential if

$$h^m_\mu h^n_\nu A^{ik}_5 g_{mi} g_{nk} = \frac{k}{2} F_{\mu\nu}. \tag{16}$$

The meaning of the constant k can now be seen from comparison with the Lagrangian of reference [1] which is of the form

$$-4R^{ik}_{\mu\nu}h^\mu_i h^\nu_k + \frac{k^2}{4} F_{\mu\nu}F^{\mu\nu}. \tag{17}$$

We have

$$k^2 = 16\frac{\kappa}{c^4},$$

where κ is the Newton's gravitational constant. Using the numerical value of κ (6.67×10^{-8} cm^3 g^{-1} sec^{-2}) in evaluation of the length d from Section 2, together with the appropriate values for $\hbar$, c and e (using the electron charge) one obtains

$$d = 7.6 \times 10^{-32} \text{ cm}.$$

So far the fundamental length l of the de Sitter gauge group is independent of d, but a generalisation of the Dirac equation to a de Sitter gauge covariant form should bring a relationship between the two constants into the theory. This will be investigated further and reported in a subsequent publication.

As a final comment it should be mentioned that equation (16) also expresses certain geometric features of the connection when it is considered as a deformed five-dimensional analog of the Poincaré group in a sense described in reference [6]. However, it is not closer to any physical interpretation than equation (12), which is quite appealing because of its simplicity.

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