PROPERTIES OF THE EXACT PION PROPAGATOR*

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We investigate the properties of the exact pion propagator under the assumption that it is a Herglatz function.

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The exact propagator for a neutral pseudoscalar particle of mass m, e.g., the neutral pion, is defined in terms of the vacuum expectation value of the time-ordered product of two hermitian pseudoscalar fields $\phi(x)$, i.e.,[1]

$$\Delta'_F(x-y) \equiv i\langle 0|T(\phi(x)\phi(y))\,|0\rangle. \tag{1}$$

It can be easily shown that the propagator has the following spectral representation [1],

$$\Delta'_F(x-y) = \int_0^\infty da^2 \varrho(a^2)\Delta_F(x-y;\, a^2) \tag{2}$$

where the spectral function $\varrho(a^2)$ is real and non-negative, and $\Delta_F(x-y;\, a^2)$ is the free propagator for a pseudoscalar meson of mass $a^{\frac{1}{2}}$. If we define the Fourier transform $\Delta'_F(k^2)$ by

$$\Delta'_F(x-y) = \frac{1}{(2\pi)^4}\int dk e^{ik\cdot(x-y)}\Delta'_F(k^2), \tag{3}$$

and use the fact that the propagator for a free field has the Fourier representation

$$\Delta_F(x-y;\, a^2) = \frac{1}{(2\pi)^4}\int dk\, \frac{e^{ik\cdot(x-y)}}{a^2-k^2-i\varepsilon}, \tag{4}$$

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[1] Throughout this paper, we follow the notation of Ref. [1].

then we easily obtain

$$\Delta_F'(k^2) = \int_0^\infty da^2 \frac{\varrho(a^2)}{a^2 - k^2 - i\varepsilon}, \tag{5}$$

which gives the spectral representation of the propagator in momentum space. Separating out the one-particle contribution from the spectral function gives

$$\varrho(a^2) = \delta(a^2 - m^2) + \sigma(a^2), \tag{6}$$

where $\sigma(a^2)$ represents the contribution of the two- and more-particles intermediate states to the spectral function. If equation (6) is substituted into equation (5), we obtain

$$\Delta_F'(k^2) = \frac{1}{m^2 - k^2 - i\varepsilon} + \int_{9m^2}^\infty da^2 \frac{\sigma(a^2)}{a^2 - k^2 - i\varepsilon}. \tag{7}$$

Note for the neutral pion that the simplest many-particle state which can contribute is the three-pion state; thus the continuum starts at $9m^2$. (There are also two-particle contributions which come from nucleon-antinucleon pairs, however, $4M^2 > 9m^2$.)

Let us now define the function

$$\Delta_F'(z) = \frac{1}{m^2 - z} + \int_{9m^2}^\infty \frac{\sigma(a^2) da^2}{a^2 - z}. \tag{8}$$

This function is easily seen to be analytic in the complex z-plane with the exception of a pole at $z = m^2$ and a cut along the positive real axis from $z = 9m^2$ to $+\infty$. A comparison with equation (7) shows that for any real k^2 the exact propagator $\Delta_F'(k^2)$ is the limit of the analytic function $\Delta_F'(z)$ when z approaches the real axis from above, i.e., when $z \to k^2 + i\varepsilon$ with $\varepsilon > 0$. Another consequence of equation (8) is that $\Delta_F'(z)$ is a real analytic function, i.e.,

$$\Delta_F'^*(z) = \Delta_F'(z^*), \tag{9a}$$

where the star (*) denotes complex conjugation. In addition, we have

$$\sigma(a^2) = \frac{1}{\pi} \operatorname{Im} \Delta_F'(a^2) \geqslant 0 \quad (a^2 \geqslant 9m^2). \tag{9b}$$

It should be pointed out that the unsubtracted dispersion relation for $\Delta_F'(z)$, given by equation (8), implies that $\Delta_F'(z)$ vanishes for $|z| \to \infty$ which, in turn, implies that $\sigma(a^2) \to 0$ when $a^2 \to \infty$.

An important property of $\Delta_F'(z)$, as given by equation (8), which is not well-known, is that $\Delta_F'(z)$ is a Herglotz function. A Herglotz function $F(z)$ is a function which is analytic in the half-plane $\operatorname{Im} z > 0$ and satisfies there the condition [2, 3, 4]

$$(\operatorname{Im} z)(\operatorname{Im} F(z)) > 0. \tag{10}$$

(To show that $\Delta'_F(z)$ is a Herglotz function, we substitute $z = x+iy$ into equation (8) and verify directly that the condition of equation (10) is satisfied. This procedure also shows that the condition is satisfied in the lower plane, $\operatorname{Im} z < 0$.)

It is of interest to generalize this result and assume that the exact propagator is, in general, a Herglotz function. Now the fact that a function belongs to the class of Herglotz functions places great restrictions on both its asymptotic behavior in the complex z-plane and its functional form [2, 3, 4]. The main purpose of this paper is to investigate the general properties of the exact pion propagator under the assumption that it is a Herglotz function.

Before we present our results, the following comments are in order. The exact form of the spectral representation for the propagator depends directly on the asymptotic behavior of the two-point function in momentum space. If this quantity goes to zero as $|k^2| \to \infty$, then the unsubtracted form of the spectral representation, as given by equation (7), is correct. However, a priori there is nothing to preclude that perhaps $\Delta'_F(k^2)$ is bounded by a polynomial function of finite degree at infinity. In this case $\Delta'_F(z)$ satisfies a subtracted dispersion relation and a number of uncalculable subtraction constants appear [1, 5]. In the following, we shall show, among other things, that if $\Delta'_F(z)$ is a Herglotz function, then the maximum number of subtractions required for writing down a dispersion relation for $\Delta'_F(z)$ is a priori determined to be two.

Our starting point is the following integral representation which holds for any Herglotz function $H(z)$ [2, 3, 4]

$$H(z) = A+Bz+\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\operatorname{Im} H(z)\,(1+zx)dx}{(1+x^2)\,(x-z)}, \tag{11}$$

with A and B real constants, $B \geqslant 0$ and $\operatorname{Im} H(x) \geqslant 0$. Combining the results of equation (9b) and the fact that $\Delta'_F(z)$ has a simple pole at $z = m^2$ with unit residue (see equation (6)), we obtain the following general representation for the exact pion propagator

$$\Delta'_F(z) = A+Bz+\frac{1+m^2 z}{(1+m^2)\,(m^2-z)}+\int_{9m^2}^{\infty}\frac{\varrho(x)\,(1+zx)dx}{(1+x^2)(x-z)}. \tag{12}$$

The constants A and B may be replaced by two quantities which are related to $\Delta'_F(0)$ and $d\Delta'_F(0)/dz$:

$$\Delta'_F(0) = A+\frac{1}{m^2(1+m^4)}+\int_{9m^2}^{\infty}\frac{\varrho(x)dx}{x(1+x^2)}, \tag{13a}$$

$$\frac{d\Delta'_F(0)}{dz} = B+\frac{1}{m^4}+\int_{9m^2}^{\infty}\frac{\varrho(x)dx}{x^2}. \tag{13b}$$

Substitution of equations (13) into equation (12) and simplifying gives

$$\Delta_F'(z) = \Delta_F'(0) + \frac{d\Delta_F'(0)}{dz} z + \frac{z^2}{m^2(m^2-z)} + z^2 \int_{9m^2}^{\infty}{}' \frac{\varrho(x)dx}{x^2(x-z)}, \tag{14}$$

which is the general representation of the exact pion propagator function under the assumption that it is a Herglotz function.

We now list the most important consequences which follow from equations (12) and (14):

(*i*) In general, $\Delta_F'(z)$ satisfies a dispersion relation with two subtractions. This means that, in general, neither $\Delta_F'(0)$ nor its first derivative at $z = 0$ can be calculated; they must be given as input data. Furthermore [4],

$$\Delta_F'(z)/z \to B \quad \text{as} \quad |z| \to \infty,$$
$$(0 < \varepsilon \leqslant \operatorname{Arg} z \leqslant 2\pi - \varepsilon). \tag{15}$$

(*ii*) In the fortunate case where $\Delta_F'(z) \to 0$ as $|z| \to \infty$, then $\Delta_F'(z)$ can be represented by an unsubtracted dispersion relation. In addition, there exists a constant C_1 such that [4]

$$|\Delta_F'(z)| \geqslant C_1/|z| \quad \text{as} \quad |z| \to \infty,$$
$$(0 < \varepsilon \leqslant \operatorname{Arg} z \leqslant 2\pi - \varepsilon). \tag{16}$$

(*iii*) The following integral converges

$$\int_{9m^2}^{\infty} \frac{\sigma(x)dx}{x^2}, \tag{17}$$

which implies that the spectral function satisfies the following upper bound

$$\sigma(x) < \left(\frac{C_2 x}{\log x}\right) \quad \text{as} \quad x \to \infty. \tag{17a}$$

(*iv*) The exact pion propagator is a real analytic function, i.e.,

$$\Delta_F'^*(z) = \Delta_F'(z^*), \tag{18}$$

and is real only for real z. In particular, this means that $\Delta_F'(z)$ has no complex zeros; if $\Delta_F'(z)$ has zeros, then they must lie on the real axis.

(*v*) For the case where $\Delta_F'(z)$ requires a twice subtracted dispersion relation, it follows from equations (13) that $\Delta_F'(0)$ can have any real value, while its derivative $d\Delta_F'(0)/dz$ has to be real and positive. The higher derivatives of $\Delta_F'(z)$ at $z = 0$ can be determined from a knowledge of the spectral function $\varrho(x)$; however, they, like the first derivative, must be positive, i.e.,

$$\frac{d^N \Delta_F'(0)}{dz^N} > 0. \tag{19}$$

These five results clearly show that under the assumption of this paper a large degree of the arbitrariness a priori inherent in the definition of Δ'_F through its spectral representation has been eliminated. A possible area for further investigation is to determine which types of field theories lead to the exact pion propagator being a Herglotz function.

In summary, we have obtained a number of restrictions on the behavior of the exact pion propagator under the assumption that it is a Herglotz function. In particular, we found, that in the worst case, $\Delta'_F(z)$ satisfies a twice subtracted dispersion relation.

The results of this paper may be of use in investigations on the pion field renormalization constant [1, 5].

REFERENCES

[1] P. Roman, *Introduction to Quantum Field Theory*, Wiley, New York 1969; see Chapter 11.

[2] J. A. Shohat, J. D. Tarmarkin, *The Problem of Moments*, American Mathematical Society, New York 1943; see pp. 23–25.

[3] H. M. Nussenzveig, *Causality and Dispersion Relations*, Academic, New York 1972; see Appendix C.

[4] Y. S. Jin, A. Martin, *Phys. Rev.* **135B**, 1369 (1964).

[5] G. Barton, *Introduction to Dispersion Techniques in Field Theory*, Benjamin, New York 1965; see Chapter 6.