SOLVING NONPERTURBATIVELY THE BREIT EQUATION FOR PARAPOSITRONIUM

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Taking the Breit equation at its face value we solve for parapositronium the corresponding radial equations with the Breit terms treated nonperturbatively. As expected from known perturbative arguments, the resulting fine-structure formula differs in the α^4-order from the familiar correct formula derived by treating the Breit terms in the lowest order only.

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As is well known, the Coulomb potential plus Breit terms describe in the Coulomb gauge the instantaneous one-photon electromagnetic interaction of two spin-1/2 charged particles. It was observed several years ago (cf. e.g. Ref. [1], p. 259) that the Breit terms involving Dirac α matrices of both particles can be used only as a lowest-order perturbation because in higher orders they strongly couple positive- and negative-energy unperturbed states, leading to a fast zitterbewegung which makes retardation effects important in contradiction with the instantaneous character of the Breit terms (where retardation effects have been neglected). Since the above argument does not apply directly to a nonperturbative treatment of the Breit terms where they are diagonalized (simultaneously with the rest of the Breit hamiltonian), it may be interesting to look for differences between the lowest-order results and the results of a nonperturbative approach (if the latter are expanded into powers of $\alpha = e^2/4\pi$). One may see here a suggestive analogy with the familiar zitterbewegung of a free Dirac particle which becomes "invisible" when the operator $\vec{\alpha} \cdot \vec{p}$ is diagonalized (simultaneously with the rest of the Dirac hamiltonian $\vec{\alpha} \cdot \vec{p} + \beta m$).

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To attain our purpose let us consider the Breit equation (in the centre-of-mass frame) including the Breit terms:

$$\left\{E-(\vec{\alpha}_1\cdot\vec{p}+\beta_1 m_1)-(-\vec{\alpha}_2\cdot\vec{p}+\beta_2 m_2)-V+\tfrac{1}{2}\left[\vec{\alpha}_1\cdot\vec{\alpha}_2+\frac{(\vec{\alpha}_1\cdot\vec{r})(\vec{\alpha}_2\cdot\vec{r})}{r^2}\right]V'\right\}\psi(\vec{r})=0, \tag{1}$$

where in the case of positronium

$$V=V'=-\frac{\alpha}{r} \tag{2}$$

and $m_1=m_2\equiv m$. Applying the method of elimination of angular coordinates described in Appendix in Ref. [2] we can obtain the system of 2×8 radial equations given in the Table in Ref. [3]. For parapositronium having the states ${}^1j_j$ (i.e., $j=l$ and $s=0$ for large-large components), we get from this Table the system of 4 radial equations:

$$\begin{gathered}
\frac{d}{dr}f_2+\tfrac{1}{2}(E-V)f_3=0,\\
-\left(\frac{d}{dr}+\frac{2}{r}\right)f_3+mf_1+\tfrac{1}{2}(E-V-2V')f_2+\frac{i\sqrt{j(j+1)}}{r}g_4=0,\\
mf_2+\tfrac{1}{2}(E-V+2V')f_1=0,\\
\tfrac{1}{2}(E-V+V')g_4-\frac{i\sqrt{j(j+1)}}{r}f_2=0.
\end{gathered} \tag{3}$$

After eliminating f_1, f_3 and g_4 components of the radial wave function we can write the following second-order equation for f_2:

$$\left\{\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}+\frac{\frac{dV}{dr}}{E-V}\frac{d}{dr}-\left[\frac{m^2(E-V)}{E-V+2V'}-\frac{(E-V-2V')(E-V)}{4}+\frac{j(j+1)(E-V)}{(E-V+V')r^2}\right]\right\}f_2=0. \tag{4}$$

First, let us treat the Breit terms as a lowest-order perturbation only. Expanding Eq. (4) up to the first order in V' we obtain

$$\left\{\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}+\frac{\frac{dV}{dr}}{E-V}\frac{d}{dr}-\left[m^2\left(1-\frac{2V'}{E-V}\right)-\frac{(E-V-2V')(E-V)}{4}+\frac{j(j+1)}{r^2}\left(1-\frac{V'}{E-V}\right)\right]\right\}f_2=0. \tag{5}$$

Expanding Eq. (5) now with respect to V and then using Eq. (2) and writing $E = 2m+\varepsilon$ we get

$$\left[\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}-\frac{j(j+1)}{r^2}+m\left(\varepsilon+\frac{\alpha}{r}\right)\right.$$

$$\left.+\frac{\left(\varepsilon+\dfrac{\alpha}{r}\right)^2}{4}+\left(\varepsilon+\frac{\alpha}{r}\right)\frac{\alpha}{r}-\frac{j(j+1)\alpha}{2mr^3}+O(\alpha^6)\right]f_2=0 \tag{6}$$

because $\langle r^{-1}\rangle \sim O(\alpha)$, $\varepsilon \sim O(\alpha^2)$ and

$$\left\langle \frac{\dfrac{dV}{dr}}{E-V}\frac{d}{dr}\right\rangle = -\tfrac{1}{2}\frac{\alpha|f_2^{(0)}(r)|^2}{E+\dfrac{\alpha}{r}}\Bigg|_{r=0} -\tfrac{1}{2}\left\langle \frac{\alpha^2}{r^4\left(E+\dfrac{\alpha}{r}\right)^2}\right\rangle \sim O(\alpha^6), \tag{7}$$

where $f_2^{(0)}$ is the Schrödinger unperturbed radial wave function and $\langle\ \rangle$ denotes the expectation value calculated with the use of this function. From Eq. (6) we obtain the familiar fine-structure formula for parapositronium (cf. e.g., Ref. [1], p. 203):

$$E = 2m-\frac{\alpha^2 m}{4n^2}+\frac{\alpha^4 m}{4n^4}\left(\tfrac{1}{16}-\frac{n}{j+\frac{1}{2}}\right)+O(\alpha^6). \tag{8}$$

We can see that this formula follows from the radial equation if V and V', though equal, are treated in a *different* way.

Now, we will treat V and V' in Eq. (4) on *the same* footing. Then, if we use Eq. (2) and separate the interactions proportional to V and V^2 from the rest, we can rewrite Eq. (4) as follows:

$$\left\{\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}\right.$$

$$\left.-\left[m^2-\frac{E^2}{4}-\frac{\left(1-\dfrac{2m^2}{E^2}\right)\alpha E}{r}+\frac{j(j+1)-\left(3-\dfrac{8m^2}{E^2}\right)\dfrac{\alpha^2}{4}}{r^2}\right]+P\right\}f_2=0, \tag{9}$$

where

$$P=\frac{\alpha}{r(Er+\alpha)}\frac{d}{dr}-\frac{2m^2\alpha^3}{E^2r^2(Er-\alpha)}-\frac{j(j+1)\alpha}{Er^3}$$

$$=-\frac{j(j+1)\alpha}{Er^3}+O(\alpha^6) \tag{10}$$

will be considered as a perturbation, what establishes some ladder approximation. Eq. (9) without the term P gives the following asymptotic behaviour:

$$f_2 \underset{r\to 0}{\sim} r^{p-1}, \quad f_2 \underset{r\to\infty}{\sim} e^{-\frac{r}{a}}, \tag{11}$$

where

$$p = \tfrac{1}{2} + \sqrt{(j+\tfrac{1}{2})^2 - \left(3 - \frac{8m^2}{E_0^2}\right)\frac{\alpha^2}{4}}, \quad \frac{1}{a} = \sqrt{m^2 - \frac{E_0^2}{4}}, \tag{12}$$

the energy E_0 being the unperturbed energy, subject to the correction from the P term. The substitution

$$f_2 = r^{p-1} e^{-\frac{r}{a}} v \tag{13}$$

in Eq. (9) without the term P leads to the equation

$$\left[\frac{d^2}{dr^2} + 2\left(\frac{p}{r} - \frac{1}{a}\right)\frac{d}{dr} - \frac{\frac{2p}{a} - \left(1 - \frac{2m^2}{E_0^2}\right)\alpha E_0}{r}\right] v = 0 \tag{14}$$

which is the confluent hypergeometric equation for

$$F(a, b, z) = F\left[2p,\ p - \left(1 - \frac{2m^2}{E_0^2}\right)\frac{\alpha E_0 a}{2},\ \frac{2r}{a}\right]. \tag{15}$$

Hence we obtain the following bound-state condition:

$$p - \left(1 - \frac{2m^2}{E_0^2}\right)\frac{\alpha E_0 a}{2} = -n_r, \tag{16}$$

where $n_r = 0, 1, 2, \ldots$. This condition implies the fourth-degree algebraic equation

$$\frac{\alpha^2}{4}x^4 + \frac{\alpha}{2}(2n_r + 1 + \alpha)x^3 + \left[(n_r + \tfrac{1}{2})^2 - (j+\tfrac{1}{2})^2 - \frac{\alpha^2}{4}\right]x^2 - \frac{\alpha}{2}(2n_r+1)x + \frac{\alpha^2}{4} = 0 \tag{17}$$

for

$$x = \sqrt{\left(\frac{2m}{E_0}\right)^2 - 1}. \tag{18}$$

Solving Eq. (17) up to the third order in α we get

$$x = \frac{\alpha}{2n} - \frac{\alpha^3}{8n^3}\left(1 - \frac{n}{2j+1}\right) + O(\alpha^5) \tag{19}$$

and hence

$$E_0 = \frac{2m}{\sqrt{1+x^2}} = 2m - \frac{\alpha^2 m}{4n^2} + \frac{\alpha^4 m}{16n^4}\left(\tfrac{11}{4} - \frac{n}{j+\frac{1}{2}}\right) + O(\alpha^6), \tag{20}$$

where $n = n_r + j + 1$. The perturbation term P in Eq. (9) leads to the correction

$$\Delta E = -\frac{1}{m}\langle P\rangle = \frac{\alpha^4 m}{16n^4}\,\frac{n}{j+\frac{1}{2}} + O(\alpha^6). \tag{21}$$

Thus we come to the fine-structure formula for parapositronium

$$E = E_0 + \Delta E = 2m - \frac{\alpha^2 m}{4n^2} + \frac{\alpha^4 m}{4n^4}\,\frac{11}{16} + O(\alpha^6) \tag{22}$$

which differs by the term

$$-\frac{\alpha^4 m}{4n^4}\,\frac{n}{j+\frac{1}{2}} \tag{23}$$

from the familiar formula (8) derived by treating the Breit terms as a lowest-order perturbation only.

In conclusion we can say that the Breit equation, if taken at its face value, leads to the fine-structure formula for parapositronium which differs in the α^4-order from the familiar formula. This discrepancy is connected with the fact that the Breit equation, similarly as the Dirac equation for one spin-1/2 particle, is not fully consistent with the hole theory when, as in the present case, the role played by negative-energy states is important. In such a case the Salpeter equation [4] is more adequate (for a review cf. Ref. [5]). Unfortunately, it leads to a very involved system of radial equations [6] which can hardly be treated nonperturbatively.

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