LETTERS TO THE EDITOR

AN EMPIRICAL SPECTRAL FORMULA FOR FERMION GENERATIONS

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We propose a (hopefully) empirical spectral formula for lepton and quark generations as they appear in the standard model. Then the top quark is predicted at about 20 GeV, while the charged lepton, up quark and down quark of the hypothetic fourth generation at $28.5^{+0.2}_{-0.5}$ GeV, at about 250 GeV and at about 62 GeV, respectively.

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As is well known, the discovery of Bohr quantum rules was preceded and stimulated by Balmer empirical spectral formula for hydrogen-like atoms. One can argue that some aspects of the present situation in particle physics are similar to those of the pre-Bohrian situation in atomic spectroscopy. Especially, the intense proliferation of discrete physical states is common to both cases, though the relevant experimental data available to Ritz, Rydberg and Balmer were technically easier to obtain and so more extended and precise. At any rate, a good-working empirical spectral formula for lepton and quark generations would be very helpful in our search for the proper dynamics of fermion generations. However, a serious trouble with establishing and proving such a formula is that it necessarily requires a vast extrapolation beyond the comparatively scarce experimental information about fermion generations which is now at our disposal.

In this note we will propose a (hopefully) empirical spectral formula for lepton and quark generations as they appear in the standard model. So we shall consider four fermion families f = ν, e, u, d, each consisting of several generations $N = 0, 1, 2, \ldots$, viz. $\mathrm{f} = \{\mathrm{f}_N\}$,

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where [1]

$$f_N = \begin{cases} \nu_N = \nu_e, \nu_\mu, \nu_\tau, \ldots \\ e_N = e^-, \mu^-, \tau^-, \ldots \\ u_N = u, c, (t), \ldots \\ d_N = d, s, b, \ldots \end{cases} . \tag{1}$$

Masses and electric charges of f_N particles will be denoted by m_{f_N} and $Q_{f_N} = Q_f$, respectively, where $Q_f = 0, -1, 2/3, -1/3$ for $f = \nu, e, u, d$.

We propose the spectral formula for fermion generations in the form of the following simple recurrence equation for the mass excitation $m_{f_N} - m_{f_0}$ [2]:

$$m_{f_{N+1}} - m_{f_0} = \lambda_f^2 (m_{f_N} - m_{f_0}) + \varepsilon_f Q_f^2, \tag{2}$$

where λ_f and ε_f are constants, universal within f family, which may be determined from the relations

$$\lambda_f^2 = \frac{m_{f_2} - m_{f_1}}{m_{f_1} - m_{f_0}}, \quad \varepsilon_f Q_f^2 = m_{f_1} - m_{f_0}. \tag{3}$$

It can be easily seen that Eq. (2) implies (as its unique solution) the following spectral formula

$$m_{f_N} - m_{f_0} = \frac{\varepsilon_f Q_f^2}{\lambda_f^2 - 1} (\lambda_f^{2N} - 1) \quad (N = 0, 1, 2, \ldots). \tag{4}$$

Note also that Eq. (2) or (4) leads to the mass relations

$$\frac{m_{f_{N+2}} - m_{f_{N+1}}}{m_{f_{N+1}} - m_{f_N}} = \text{const} \equiv \lambda_f^2 \tag{5}$$

and

$$m_{f_{N+1}} - \lambda_f^2 m_{f_N} = \text{const} \equiv \varepsilon_f Q_f^2 - (\lambda_f^2 - 1) m_{f_0}. \tag{6}$$

In the case of leptons, making use of the masses m_e, m_μ and $m_\tau = 1782^{+2}_{-7}$ MeV [3] of e^-, μ^- and τ^- leptons, one gets from Eq. (3)

$$\lambda_e = 3.993^{+0.004}_{-0.009}, \quad \varepsilon_e = m_\mu - m_e. \tag{7}$$

Then Eq. (5) leads to the predictions for charged leptons of the hypothetic fourth and fifth generations:

$$m_{e_3} = 28.5^{+0.2}_{-0.5} \text{ GeV}, \quad m_{e_4} = 455^{+3}_{-10} \text{ GeV}. \tag{8}$$

For neutrinos of all generations Eq. (4) gives

$$m_{\nu_N} = m_{\nu_e}, \tag{9}$$

thus $m_{\nu_N} = 0$ if $m_{\nu_e} = 0$. The mass degeneracy (9) (if valid strictly enough) implies the absence of neutrino oscillations even if $m_{\nu_e} \neq 0$.

In the case of quarks, if assuming

$$\lambda_u = \lambda_d, \quad \varepsilon_u = \varepsilon_d, \tag{10}$$

one concludes from Eq. (4) that

$$m_{u_N} - m_u = 4(m_{d_N} - m_d) = \frac{4}{9}\frac{\varepsilon_u}{\lambda_u^2 - 1}(\lambda_u^{2N} - 1). \tag{11}$$

So, taking the small current masses $m_u \simeq 0 \simeq m_d$ (of a few MeV), one obtains for $N > 0$

$$m_{u_N} \simeq 4m_{d_N} \simeq \frac{4}{9}\frac{\varepsilon_u}{\lambda_u^2 - 1}(\lambda_u^{2N} - 1). \tag{12}$$

Hence, for $N = 1$ and 2 [4]

$$m_c \simeq 4m_s \simeq \frac{4}{9}\varepsilon_u \tag{13}$$

and

$$m_t \simeq 4m_b \simeq \frac{4}{9}\varepsilon_u(\lambda_u^2 + 1). \tag{14}$$

With $m_c \simeq 1.5$ GeV and $m_b \simeq 5$ GeV, Eqs. (13) and (14) give

$$\lambda_u \simeq 3.5, \quad \varepsilon_u \simeq 3.4 \text{ GeV} \tag{15}$$

and

$$m_s \simeq 0.38 \text{ GeV}, \quad m_t \simeq 20 \text{ GeV}, \tag{16}$$

predicting toponium $t\bar{t}$ at about 40 GeV (perhaps at ca. 38 GeV since $m_{t\bar{t}} \simeq 4m_{b\bar{b}}$). Then Eq. (5) leads to the predictions for up and down quarks of the hypothetic fourth and fifth generations:

$$m_{u_3} \simeq 250 \text{ GeV}, \quad m_{d_3} \simeq 62 \text{ GeV}, \quad m_{u_4} \simeq 3100 \text{ GeV}, \quad m_{d_4} \simeq 770 \text{ GeV}. \tag{17}$$

We can see that the values of λ_f determined precisely for leptons and estimated for quarks are roughly equal, while those of ε_f are very different. Phenomenologically one may write

$$\varepsilon_f = \varepsilon + \varepsilon' C_f, \tag{18}$$

where C_f is the quadratic Casimir operator for colour SU(3) which is equal to 0 or 4/3 for leptons or quarks, respectively. Then

$$\varepsilon = \varepsilon_e = m_\mu - m_e, \quad \varepsilon' = \frac{3}{4}(\varepsilon_u - \varepsilon_e) \simeq 2.5 \text{ GeV}. \tag{19}$$

Note that $\varepsilon'/\varepsilon \simeq 24 \simeq \alpha'/\alpha$, where $\alpha' \simeq 0.17$ and $\alpha = 1/137$. The pertinent question, why in the proposed recurrence equation (2) there appears the term $\varepsilon_f Q_f^2 = \varepsilon Q_f^2 + \varepsilon' Q_f^2 C_f$ and not the phenomenologically more complete term $\varepsilon Q_f^2 + \varepsilon' Q_f^2 C_f + \varepsilon'' C_f$ with a significant $\varepsilon'' > 0$, may be related to the fact that in the case of quarks Eq. (2) describes the comparatively small *effective* mass of quarks inside hadrons and not a very large "true mass" of abstract free quarks, the latter mass certainly including $\varepsilon'' C_f$ with $\varepsilon'' \gg 0$. Our fitting of Eq. (2) to quark effective masses shows indeed that then ε'' is rather negligible because the relation $m_c \simeq 4m_s$ is reasonably good (cf. Eqs. (11) and (12)).

When operating with the recurrence equation (2) or the spectral formula (4), we do not know whether the number of fermion generations is finite or infinite. Obviously a natural possibility would be the existence of gravitational cut-off N_f+1 for the number of generations in each massive f family, since with growing N the Compton wave length $\hbar/m_{f_N}c$ becomes smaller than the Schwarzschild gravitational radius $2Gm_{f_N}/c^2$ if only N exceeds some N_f (then f_N particles become black holes with spin 1/2). Thus, in this case, we have the upper bound

$$m_{f_N} < \sqrt{\frac{\hbar c}{2G}} \equiv \frac{m_{PL}}{\sqrt{2}}, \quad N \leqslant N_f, \tag{20}$$

where $m_{PL} = 1.2211027 \times 10^{19}$ GeV/c^2 is the Planck mass. Making use of the spectral formula (4) and the values (7) and (15) of λ_f and ε_f obtained by their fitting to experimental data, we can show that

$$m_{e_{17}} < \frac{m_{PL}}{\sqrt{2}} < m_{e_{18}}, \quad m_{u_{18}} < \frac{m_{PL}}{\sqrt{2}} < m_{u_{19}}, \quad m_{d_{18}} < \frac{m_{PL}}{\sqrt{2}} < m_{d_{19}}. \tag{21}$$

So, excitingly enough, we get in this case very *close* numbers of generations in e, u and d families:

$$N_e+1 = 18, \quad N_u+1 = N_d+1 = 19. \tag{22}$$

They become even *identical*, $N_f+1 = 18$ for f = e, u, d, if we take $\lambda_u = \lambda_d \simeq 3.8$ instead of $\lambda_u = \lambda_d \simeq 3.5$ (then we obtain $m_b \simeq 5.8$ GeV and $m_t \simeq 23$ GeV instead of $m_b \simeq 5$ GeV and $m_t \simeq 20$ GeV). It is well known, however, that there are various arguments for a much lower number of lepton and quark generations, as e.g. the important perturbative argument [5] based on one-loop corrections to the relation $m_W^2/m_Z^2 \cos^2\theta_W = 1$ in the standard Glashow-Weinberg-Salam model.

In conclusion, we can say that the proposed spectral formula (4) reproduces neatly all actually known features of lepton and quark spectra. This formula is equivalent to the relation

$$m_{f_N} - m_{f_M} = \frac{\varepsilon_f^2 Q_f^2}{\lambda_f^2 - 1} (\lambda_f^{2N} - \lambda_f^{2M}) \quad (N, M = 0, 1, 2, \ldots) \tag{23}$$

which may be considered as our counterpart of the Balmer formula. The constants λ_e, ε_e, $\lambda_u = \lambda_d$ and $\varepsilon_u = \varepsilon_d$ are fitted to m_μ, m_τ, m_c and m_b, while the masses of the first or ground generation, m_{ν_e}, m_e, m_u and m_d, are treated as initial conditions. Then the masses m_{ν_μ}, m_{ν_τ}, m_s and m_t and all masses of the fourth and higher generations are predicted, providing a possible experimental check of our spectral formula. The spectra of e, u and d families have rapidly growing exponential behaviour exp $(\alpha_f N)$, where $\alpha_f = 2 \ln \lambda_f$ (numerically $\alpha_e = 2.769$ and $\alpha_u = \alpha_d \simeq 2.5$). These spectra, if continued to larger and larger N, exceed Planck mass/$\sqrt{2}$ for $N > N_f$, where $N_f = 17-18$ is practically equal for all three families. The spectrum of ν family is constant, possibly with the value zero (if $m_{\nu_e} = 0$).

The growing exponential mass spectra do not remind us of any known energy spectra of bound dynamical systems [6]. It may be a signal that a dynamics of excitations of a new kind ("mass excitations") is responsible for fermion generations. Then $N = 0, 1, 2, \ldots$ might be equal to the number of some "elementary mass excitations".

Finally, in connection with the last remark, we would like to emphasize for our spectral formula (4) a formal statistical analogy valid in the case of a finite number of generations N_f+1. In this case Eq. (4) can be rewritten as

$$m_{f_N} - m_{f_0}^{(0)} = \frac{Z_f \varepsilon_f Q_f^2}{\lambda_f^2 - 1} \varrho_{f_N} \quad (N = 0, 1, 2, \ldots), \tag{24}$$

where $m_{f_0}^{(0)} = m_{f_0} - \varepsilon_f Q_f^2 (\lambda_f^2 - 1)^{-1}$ and

$$\varrho_{f_N} = Z_f^{-1} \exp(\alpha_f N), \quad Z_f = \sum_{N=0}^{N_f} \exp(\alpha_f N) = (\lambda_f^{2N_f+2} - 1)(\lambda_f^2 - 1)^{-1} \tag{25}$$

and $\alpha_f = 2 \ln \lambda_f > 0$. Since ϱ_{f_N} satisfies the normalization condition in N-space, it has a formal analogy with a statistical distribution in this space. If we put $\alpha_f \equiv (\mu_f - E_f)(kT)^{-1} > 0$, ϱ_{f_N} takes the form of a grand canonical ensemble which implies the N-number conservation for a whole system. It is interesting to note that in the real world of fermion generations the N-number conservation (considered on the level of the second quantization) is not incompatible with experimental data. It obviously forbids the unwanted processes $\mu^- \to e^- e^+ e^-$ and $\mu^- \to e^- \gamma$ as well as $\tau^- \to e^- e^+ e^-$ or $\mu^- e^+ e^-$ or $\mu^- \mu^+ \mu^-$ and $\tau^- \to e^- \gamma$ or $\mu^- \gamma$ but it allows for $\tau^- \to \mu^- \bar{\nu}_e \nu_\mu$ and/or $\tau^- \to \mu^- e^+ \mu^-$ if only there are charged and/or neutral intermediate bosons of the generation $N = 1$ (which, however, would get presumably much larger masses than the familiar $W^\pm$ and Z of the generation $N = 0$ and would lead, therefore, to much weaker effective Fermi interactions responsible for these exotic decays). In the case of quarks, the N-number conservation is violated by Cabibbo inter-generation mixing.

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