ON THE EQUATIONS OF STATE FOR IRROTATIONAL PERFECT FLUID IN GENERAL RELATIVITY

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Equations of state for an irrotational perfect fluid are investigated. It is shown that the equations of state in the form a) $\varrho = \text{const}$, $P \neq \text{const}$, b) $P = \text{const}$, $\varrho_{,k} = \varrho_{,n}u^n u_k \neq 0$, c) $P_{,k} = P_{,n}u^n u_k$, $\varrho_{,k} = \varrho_{,n}u^n u_k$, d) $\varrho_{,k} = \alpha P_{,k}$, $\varrho_{,n}u^n = P_{,n}u^n = 0$ do not contradict the Einstein field equations. Finally we present the physically realistic equations of state.

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1. *Introduction*

The gravitational fields of an irrotational perfect fluid were discussed in [1–11]. In most cases these papers present the new solutions of Einstein's field equations. Some solutions in [1–11] have been obtained for equations of state of the form $\varrho = \varrho(P)$, where ϱ is the density and P is the pressure (see, for example, [1, 2, 6]). In [4, 6, 8] the solutions of Einstein's field equations present the gravitational fields of a perfect fluid with the equation of state in the form $\varrho \neq \varrho(P)$. On the other hand, as follows from the thermodynamics of a perfect fluid its flow is isentropic (the proof of this statement is given in [12]). Therefore the equations of state for a real perfect fluid may be expressed as $\varrho = \varrho(P)$. From this it follows that the equations of state of the form $\varrho \neq \varrho(P)$ describe physically unrealistic models of the perfect fluid. However, the equations of state $\varrho \neq \varrho(P)$ in [4, 6, 8] satisfy the Einstein field equations. Consequently Einstein's field equations admit solutions for perfect fluid with unrealistic equations of state $\varrho \neq \varrho(P)$. In this case two questions arise: what set of equations of state follows from Einstein's field equations and which equations of state are physically admissible?

This paper presents all possible equations of state for an irrotational perfect fluid. These equations of state have been obtained with the help of Einstein's field equations and the Bianchi identities and the identities $C^{nm}{}_{kl;n;m} = 0$ (the proof of these identities is presented in Appendix A). Finally we present the choice of physically realistic equations of state for an irrotational perfect fluid.

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2. *Consequences resulting from identities* $C^{nm}{}_{kl;n;m} = 0$

The field equations for a perfect fluid are

$$R_{ik} - \tfrac{1}{2} R g_{ik} = -(\varrho + P) u_i u_k + P g_{ik}, \tag{2.1}$$

where the units are chosen so that $8\pi G = c = 1$ and the symbols have the usual meaning. If the vorticity (or rotation) is zero, the tensor $u_{i;k}$ may be split up as follows

$$u_{i;k} = t_i u_k + \frac{\theta}{3} h_{ik} + \sigma_{ik}, \tag{2.2}$$

where u_i is a velocity field of a perfect fluid, $u_n u^n = 1$, $t_i = u_{i;n} u^n$ is an acceleration, σ_{ik} is a shear tensor, $h_{ik} = g_{ik} - u_i u_k$ is a projection tensor, θ is a volume expansion and the semicolon denotes covariant differentiation.

The kinematic quantities which characterize the stream lines of the perfect fluid satisfy the relations

$$t_n u^n = 0, \qquad u^n \sigma_{nk} = 0, \qquad g^{nk} \sigma_{nk} = 0, \qquad \sigma_{ik} = \sigma_{ki}.$$

The Bianchi identities, $R_{i[k;l]} = \tfrac{1}{2} R^n{}_{ikl;n}$, are equivalent to the equations

$$C^n{}_{ikl;n} = R_{i[k;l]} - \tfrac{1}{6} g_{i[k} R_{,l]}, \tag{2.3}$$

where $C^n{}_{ikl}$ is the Weyl tensor, the square brackets denote antisymmetrization and $R_{,l} = \partial R / \partial x^l$.

The conservation equations, $T^n{}_{k;n} = 0$, read

$$\varrho_{,n} u^n + (\varrho + P)\theta = 0, \qquad t_k = \frac{P_{,k} - P_{,n} u^n u_k}{\varrho + P}. \tag{2.4}$$

Substituting R_{ik}, R from (2.1), $u_{i;k}$ from (2.2) and t_k from (2.4) into (2.3), we obtain

$$\tfrac{1}{3} g_{i[k} \varrho_{,l]} - u_i u_{[k} \varrho_{,l]} - \frac{(\varrho + P)\theta}{3} g_{i[l} u_{k]} - (\varrho + P) \sigma_{i[l} u_{k]} = C^n{}_{ikl;n}. \tag{2.5}$$

Differentiating (2.5) covariantly with respect to x^s, contracting the result and using (2.2), (2.4) and $C^{nm}{}_{k;n;m} = 0$, we obtain

$$\varrho_{,k} P, - \varrho, P_{,k} = a_k u - a u_k, \tag{2.6}$$

where

$$a_k = (\varrho + P) \Big\{ \frac{P_{,n} u^n}{\varrho + P} \varrho_{,k} - \theta \varrho_{,k} - \varrho_{,k;n} u^n + \tfrac{1}{3} (\varrho_{,n} u^n)_{,k}$$

$$- \tfrac{1}{3} \frac{\varrho_{,n} u^n}{\varrho + P} P_{,k} - (\varrho + P)_{,n} \sigma^n{}_k - (\varrho + P) \sigma^n{}_{k;n} \Big\}.$$

We can rewrite equation (2.6) as follows

$$a_k = a_n u^n u_k + \varrho_{,k} P_{,n} u^n - P_{,k} \varrho_{,n} u^n.$$

The substitution of this relation into (2.6) yields

$$(\varrho_{,k}-\varrho_{,n}u^n u_k)(P_{,l}-P_{,n}u^n u_l) = (\varrho_{,l}-\varrho_{,n}u^n u_l)(P_{,k}-P_{,n}u^n u_k). \tag{2.7}$$

Generally speaking from (2.7) we have the following possible expressions for the equations of state

$$\varrho_{,k} = \varrho_{,n}u^n u_k, \quad P_{,k} \neq P_{,n}u^n u_k, \quad \varrho, P \neq \text{const}, \tag{2.8}$$

$$P_k = P_{,n}u^n u_k, \quad \varrho_{,k} \neq \varrho_{,n}u^n u_k, \quad \varrho, P \neq \text{const}, \tag{2.9}$$

$$\varrho_{,k} = 0, \quad P_{,k} \neq 0, \tag{2.10}$$

$$P_k = 0, \quad \varrho_{,k} \neq 0, \tag{2.11}$$

$$P_{,k} = P_{,n}u^n u_k, \quad \varrho_{,k} = \varrho_{,n}u^n u_k, \quad \varrho, P \neq \text{const} \tag{2.12}$$

$$\varrho_{,k}-\varrho_{,n}u^n u_k = \alpha(P_{,k}-P_{,n}u^n u_k), \quad \alpha \neq 0. \tag{2.13}$$

In comoving coordinates equation (2.8) corresponds to the equation of state in the form $\varrho = \varrho(t)$, $P = P(t, x^\alpha)$, $\alpha = 1, 2, 3$. Consequently, equation (2.8) describes a perfect fluid with uniform density.

In comoving coordinates equation (2.9) is equivalent to the equation of state $P = P(t)$, $\varrho = \varrho(t, x^\alpha)$, $\alpha = 1, 2, 3$.

The equation (2.10) corresponds to an incompressible perfect fluid.

All the equations (2.8)–(2.11) do not satisfy the condition $\varrho = \varrho(P)$. Therefore the equations (2.8)–(2.11) are physically unrealistic models of state for the irrotational perfect fluid. In the following section the equations (2.8)–(2.11) will be considered in detail. We will find out which of the equations (2.8)–(2.11) are contradictory.

The equations (2.12)–(2.13) are considered in Section 4.

3. The equations of state with $\varrho \neq \varrho(P)$

First of all we derive some necessary relations. For this purpose we use (2.1), (2.2), (2.4) in the identities

$$R_{i[k;l]} = \tfrac{1}{2} R^n{}_{ikl;n}.$$

The result is

$$\tfrac{1}{2} g_{i[k}(\varrho-P)_{,l]}-u_i u_{[k}\varrho_{,l]}-\frac{(\varrho+P)\theta}{3} g_{i[l}u_{k]}-(\varrho+P)\sigma_{i[l}u_{k]} = \tfrac{1}{2} R^n{}_{ikl;n}. \tag{3.1}$$

From (3.1) it follows that

$$u^m D_{mkl} = 0, \tag{3.2}$$

where

$$D_{mkl} = R^n{}_{mkl;n}+R^n{}_{mls;n}u^s u_k-R^n{}_{mks;n}u^s u_l. \tag{3.3}$$

It is easily seen that from (2.6) and (3.1) follows

$$P^m D_{mkl} = 0, \qquad \varrho^m D_{mkl} = 0, \tag{3.4}$$

where

$$P^m = P_{,n} g^{nm}, \qquad g^m = \varrho_{,n} g^{nm}.$$

With the aid of conditions (3.2)–(3.4) we will consider the validity of equations (2.8)–(2.13) in General Relativity.

a) the state of the fluid with $\varrho_{,k} = \varrho_{,n} u^n u_k$, $P_{,k} \neq P_n u^n u_k$, $\varrho, P \neq$ const.

Generally speaking the equation (2.8) does not contradict the relations (3.2)–(3.4) in the following cases

$$P^m R^n{}_{mkl;n} = 0, \qquad u^m R^n{}_{mkl;n} \neq 0, \tag{3.5}$$

$$u^m R^n{}_{mkl,n} = 0, \qquad P^m R^n{}_{mkl,n} \neq 0, \tag{3.6}$$

$$R^n{}_{mkl;n} = 0, \tag{3.7}$$

$$D_{mkl} = 0. \tag{3.8}$$

At first we examine (2.8) and (3.5). Contracting (3.1) with P^i and using (2.4), (2.8) and (3.5) we get $P^n b_{nkl} = 0$, where

$$b_{nkl} = \tfrac{1}{3}\, \varrho_{,m} u^m g_{n[k} u_{l]} - 2(\varrho + P)\sigma_{n[l} u_{k]}.$$

Simultaneously we have $u^n b_{nkl} = 0$. By virtue of $P^n \neq 0$, $P^n \neq P_{,m} u^m u^n$ (see (2.8)) we obtain $b_{nkl} = 0$ from $P^n b_{nk} = 0$ and $u^n b_{nkl} = 0$. Contraction of $b_{nkl} = 0$ gives $\varrho_{,n} u^n = 0$. If $\varrho_{,n} u^n = 0$, then it follows from (2.8) that $\varrho_{,k} = 0$. But in the case of (2.8) we consider the equations of state with $\varrho, P \neq$ const. Therefore, equation (2.8) does not satisfy the relation (3.5).

Now we consider the condition (3.6). In this case contracting (3.1) with u^i and using (2.8), (3.6) we obtain $P_{,k} = P_{,n} u^n u_k$. The relation $P_{,k} = P_{,n} u^n u_k$ does not satisfy (2.8) and (3.6). Therefore, equation (2.8) does not satisfy the relation (3.6).

Similarly, it can be shown that (2.8) and (3.7) are incompatible.

Now we consider the condition (3.8). If $D_{mkl} = 0$, the contraction, $g^m D_{mkl} = 0$, gives

$$R^n{}_{mks;n} u^m u^s = \tfrac{1}{2}\,(R_{,s} u^s u_k - R_{,k}).$$

Using (2.1) and (2.8) we get

$$R^n{}_{mks;n} u^m u^s = \tfrac{3}{2}\,(P_{,k} - P_{,n} u^n u_k).$$

Contracting (3.1) with u^i and using (2.8) and $R^n{}_{mks;n} u^m u^s = \tfrac{3}{2}\,(P_{,k} - P_{,n} u^n u_k)$ we obtain $P_{,k} = P_{,n} u^n u_k$. This relation is incompatible with (2.8). This incompatibility implies that the equation (2.8) does not satisfy (3.8).

We considered all the cases (3.5)–(3.8). Generally speaking in these cases the equation of state (2.8) should not contradict the conditions (3.2)–(3.4). Yet the detailed analysis shows that (2.8) and (3.5)–(3.8) are incompatible. Therefore, (2.8) and (3.2)–(3.4) are also incompatible. Consequently the equation of state (2.8) does not satisfy the conditions (3.2)–(3.4) or Einstein's field equations.

b) The state of fluid with $P_{,k} = P_{,n}u^n u_k$, $\varrho_{,k} \neq \varrho_{,n}u^n u_k$, $\varrho, P \neq$ const.

Generally speaking (2.9) does not contradict the conditions (3.2)–(3.4) in the following cases

$$\varrho^m R^n{}_{mkl;n} = 0, \qquad u^m R^n{}_{mkl;n} \neq 0, \tag{3.9}$$

$$u^m R^n{}_{mkl;n} = 0, \qquad \varrho^m R^n{}_{mkl;n} \neq 0 \tag{3.10}$$

and in the cases (3.7), (3.8).

At first we consider the condition (3.9). Contracting (3.1) with ϱ^i and using (2.4), (2.9), (3.9) gives $\varrho^n K_{nkl} = 0$, where

$$K_{nkl} = (\tfrac{4}{3}\,\varrho_{,m}u^m - P_{,m}u^m)g_{n[k}u_{l]} - 2(\varrho + P)\sigma_{n[l}u_{k]}.$$

Simultaneously we have $u^n K_{nkl} = 0$. By virtue of $\varrho^n \neq 0$, $\varrho^n \neq \varrho_{,m}u^m u^n$ it follows from $u^n K_{nkl} = 0$ and $\varrho^m K_{nkl} = 0$ that $K_{nkl} = 0$. Contracting $K_{nkl} = 0$ with g^n gives

$$\tfrac{4}{3}\,\varrho_{,m}u^m = P_{,m}u^m, \qquad \sigma_{nk} = 0. \tag{3.11}$$

If $\sigma_{ik} = 0$ and $P_{,k} = P_{,n}u^n u_k$, the velocity field (2.2) can be written as

$$u_{i;k} = \frac{\theta}{3}\,h_{ik}. \tag{3.12}$$

The integrability conditions of (3.12) are equivalent to the equations

$$\tfrac{1}{3}\,g_{i[k}\theta_{,l]} - \frac{u_i}{3}\,u_{[k}\theta_{,l]} - \frac{\theta^2}{9}\,g_{i[l}u_{k]} = \tfrac{1}{2}\,u^n R_{nikl}. \tag{3.13}$$

The contraction of (3.13) with g^{lk} and the use of equations (2.1) gives

$$\theta_{,k}u_i = \theta_{,i}u_k. \tag{3.14}$$

From equations $P_{[,k;l]} = 0$, with the aid of the substitutions $P_{,k} = P_{,n}u^n u_k$, (3.11) and (3.12), it is possible to obtain

$$(\varrho_{,n}u^n)_{,k}u_l = (\varrho_{,n}u^n)_{,l}u_k. \tag{3.15}$$

With the help of (2.9), (3.14) and (3.15) after some simplifications we have from $\varrho_{,n}u^n + (\varrho+P)\theta = 0$ that $u_{[l}\varrho_{,k]}\theta = 0$. If $\theta = 0$, we obtain from $\varrho_{,n}u^n + (\varrho+P)\theta = 0$ that $\varrho_{,n}u^n = 0$. In this case we have $P_{,k} = 0$ from (2.9). The relation $P_{,k} = 0$ contradicts the equation (2.9). Therefore, $\theta \neq 0$. If $\theta \neq 0$, it follows from $u_{[l}\varrho_{,k]}\theta = 0$ that $u_{[l}\varrho_{,k]} = 0$. However, this relation contradicts the equation (2.9). Consequently (2.9) and (3.9) are incompatible.

Now consider the case (3.10). The contraction of (3.1) and the use of (2.9), (3.10) gives $u_{[l}\varrho_{,k]} = 0$. The relation $u_{[l}\varrho_{,k]} = 0$ contradicts equation (2.9). Therefore (2.9) and (3.10) are incompatible.

Similarly it can be shown that (2.9) and (3.7) are also incompatible.

Further we consider the case (3.8). The contraction of $D_{mkl} = 0$ with g^{ml} and the use of (2.9), (2.1) gives $R^{n}{}_{mks;n}u^m u^s = \frac{1}{2}(\varrho_{,n}u^n u_k - \varrho_{,k})$. Contracting (3.1) with $u^i u^l$ and using (2.9) and $R^{n}{}_{mks;n}u^m u^s = \frac{1}{2}(\varrho_{,n}u^n u_k - \varrho_{,k})$ we obtain $\varrho_{,k} = \varrho_{,n}u^n u_k$. This relation contradicts equation (2.9). Consequently equations (2.9) and (3.8) are incompatible.

We considered all cases (3.9), (3.10), (3.7) and (3.8). In all the cases we have obtained contradictory relations. Therefore the equation (2.9) does not satisfy either (3.2)–(3.4) or Einstein's field equations.

c) The state of fluid with $\varrho = \text{const}$, $p \neq \text{const}$.

If $P_{,k} \neq P_{,n}u^n u_k$, the equation (2.10) does not contradict the conditions (3.2)–(3.4) when the relations (3.5)–(3.8) are valid.

If $P_{,k} = P_{,n}u^n u_k$ equation (2.10) satisfies (3.2)–(3.4) when the relations (3.7)–(3.8) are valid.

However, for a definite conclusion it is necessary to consider the details.

Let $P_{,k} = P_{,n}u^n u_k$ and $\varrho_{,k} = 0$. If $\varrho_{,k} = 0$, it follows from $\varrho_{,n}u^n + (\varrho + p)\theta = 0$ that $\theta = 0$. If $\theta = 0$ and $P_{,k} = P_{,n}u^n u_k$, it follows from (2.2) that

$$u_{i;k} = \sigma_{ik}. \tag{3.16}$$

The integrability conditions of (3.16) are

$$\sigma_{i[k;l]} = \frac{1}{2} u^n R_{nikl}. \tag{3.17}$$

Contracting (3.17) with g^{ik} and using (2.1) gives

$$\sigma^{n}{}_{l;n} = \frac{\varrho + 3P}{2} u_l. \tag{3.18}$$

The contraction of (3.18) with u and the use of $u^l \sigma_{kl} = 0$ gives $\sigma = -\dfrac{\varrho + 3P}{2}$. It is impossible because $\sigma > 0$, $\varrho + 3P > 0$. Consequently $P_{,k} = P_{,n}u^n u_k$ and (2.10) are incompatible.

Now let us assume that $P_{,k} \neq P_{,n}u^n u_k$ and the condition (3.5) is valid. Contracting (3.1) with P^i and using (3.5), (2.10) gives $P^n \sigma_{n[l}u_{k]} = 0$ or $P^n \sigma_{nk} = 0$. Simultaneously we have $u^n \sigma_{nk} = 0$ and $P^n \sigma_{nk} = 0$; that could be possible if $\sigma_{nk} = 0$. In this case it can be shown that the conditions (3.5) and (2.10) are compatible. Consequently the equation of state (2.10) does not contradict Einstein's field equations.

Similarly it can be shown that equation (2.10) does not satisfy the conditions (3.6)–(3.8).

d) The state of fluid with $P_{,k} = 0$, $\sigma_{,k} \neq 0$.

Generally speaking in this case we have either $\varrho_{,k} = \varrho_{,n}u^n u_k$ or $\varrho_{,k} \neq \varrho_{,n}u^n u_k$. If $\varrho_{,k} \neq \varrho_{,n}u^n u_k$, equation (2.11) does not contradict the conditions (3.2)–(3.4) when the relations (3.9), (3.10), (3.7) and (3.8) are valid.

We first consider the case $\varrho_{,k} \neq \varrho_{,k}u^n u_k$ and (3.9). Contracting (3.1) with ϱ^i and using (2.4), (2.11) gives $\varrho^n r_{nkl} = 0$, where

$$r_{nkl} = \frac{4}{3} \varrho_{,m}u^m g_{n[k}u_{l]} - 2(\varrho + P)\sigma_{n[l}u_{k]}.$$

From $\varrho^n r_{nkl} = 0$, $u^n r_{nkl} = 0$ we have $r_{nkl} = 0$. It follows from $r_{nkl} = 0$ that $\varrho_{,n}u^n = 0$ and $\sigma_{nk} = 0$. Simultaneously it follows from $\varrho_{,n}u^n = 0$ and $\varrho_{,n}u^n + (\varrho + P)\theta = 0$ that $\theta = 0$.

If $\theta = 0$, $P_{,k} = 0$, $\sigma_{nk} = 0$, the equation (2.2) takes the form $u_{i;k} = 0$. From the integrability conditions of the relations $u_{i;k} = 0$, $u_{i[;k;l]} = \frac{1}{2} u^n R_{nikl} = 0$, it follows that $\varrho + 3P = 0$. This is impossible. Consequently $\varrho_{,k} \neq \varrho_{,n} u^n u_k$, (2.11) and (3.9) are incompatible.

Similarly it can be shown that the conditions (3.10), (3.7), (3.8) and $\varrho_{,k} \neq \varrho_{,n} u^n u_k$, (2.11) are also incompatible.

If $\varrho_{,k} = \varrho_{,n} u^n u_k$, the equation of state (2.11) does not contradict the conditions (3.2)–(3.4).

Thus we have shown that the following equations of state satisfy the conditions (3.2)–(3.4) or Einstein's field equations: 1. $\varrho_{,k} = 0$, $P_{,k} \neq P_{,n} u^n u_k \neq 0$, 2. $P_{,k} = 0$, $\varrho_{,k} = \varrho_{,n} u^n u_k$.

4. *The equations of state with* $\varrho_{,k} = \varrho_{,n} u^n u_k$, $P_{,k} = P_{,n} u^n u_k$ *and* $\varrho_{,k} - \varrho_{,n} u^n u_k = \alpha (P_{,k} - P_{,n} u^n u_k)$

The equation of state (2.12) and (3.2)–(3.4) are compatible in all cases. Moreover the equation of state (2.12) satisfies the thermodynamically correct condition $\varrho = \varrho(P)$. For this class of equations of state it can be shown that

$$u^m R^n{}_{mkl;n} = 0, \qquad P^m R^n{}_{mkl;n} = 0, \qquad \varrho^m R^n{}_{mkl;n} = 0. \tag{4.1}$$

The conditions (4.1) imply that the gravitational field of an irrotational perfect fluid satisfying (2.12) are conformally flat.

Now we consider the equation of state (2.13). Here we use the formalism of the differential forms. Besides, we use some results of calculations which were obtained in [13].

Let

$$\omega = u_i dx^i \tag{4.2}$$

be the differential form of the 1-st order, where u_i is the velocity field satisfying (2.2) and dx^i is a basic form of the 1-st order.

According to the definition we have $d\omega = \frac{\partial u_i}{\partial x^k} dx^k \wedge dx^i = u_{i,k} dx^k \wedge dx^i$, where $d\omega$ is the differential form of the 2-nd order. Using (4.2), we see that

$$-2d\omega = a_{ik} dx^i \wedge dx^k, \tag{4.3}$$

where $a_{ik} = u_{i,k} - u_{k,i}$. Moreover, from (2.2) it follows that $a_{ik} = t_i u_k - t_k u_i$. Hence we obtain det $(a_{ik}) = 0$. In this case, by virtue of theorem 2 (see [13], p. 415), we have Pf$(a_{ik}) = 0$, where Pf(a_{ik}) is the pfaffian of the tensor a_{ik}.

The relation Pf$(a_{ik}) = 0$ means that $a_{[ik} a_{lm]} = 0$. This, together with (4.3), gives $d\omega \wedge d\omega = 0$. Consequently, the differential form ω is the form of class 3 (see [13], p. 414). In this case the Darboux Theorem implies that there exist three independent functions τ, φ, η such that $\omega = \tau_{,i} + \eta \varphi_{,i}$ (see [13]).

It follows from $u_i = \tau_{,i} + \eta \varphi_{,i}$ that $a_{ik} = \eta_{,k} \varphi_{,i} - \eta_{,i} \varphi_{,k}$. Moreover, we have $a_{ik} = t_i u_k$

$-t_k u_i$ from (2.2). The relations $a_{ik} = \eta_{,k}\varphi_{,i} - \eta_{,i}\varphi_{,k}$ and $a_{ik} = t_i u_k - t_k u_i$ are different presentations of the same tensor a_{ik}. Comparing $a_{ik} = \eta_{,k}\varphi_{,i} - \eta_{,i}\varphi_{,k}$ and $a_{ik} = t_i u_k - t_k u_i$ we get $u_{i,k} = u_{k,i}$ and $t_{i,k} = t_{k,i}$. If $u_{i,k} = u_{k,i}$, we have by virtue of $u_n u^n = 1$ that $t_i = u_{i;n} u^n = 0$. This is valid only in the case when we could construct the antisymmetric tensor $a_{ik} = u_{i;k} - u_{i;k}$ from $u_{i;k}$. It is impossible only in the static gravitational fields of a perfect fluid. Consequently the following theorem is valid.

In nonstationary gravitational fields of an irrotational perfect fluid the acceleration of fluid particles is zero.

According to this theorem for nonstationary gravitational fields the equations of state (2.13) are equivalent to (2.12).

For static gravitational fields we have $\varrho_{,k} \neq \varrho_{,n} u^n u_k$, $P_{,k} \neq P_{,n} u^n u_k$, $\varrho_{,n} u^n = P_{,n} u^n = 0$. In this case (2.13) becomes $\varrho_{,k} = \alpha P_{,k}$.

Generally speaking, the equations of state $\varrho_{,k} = \alpha P_{,k}$ do not contradict relations (3.2)–(3.4), if either $D_{mkl} = 0$ or $u^m R^n{}_{mkl;n} = 0$, $P^m R^n{}_{mkl;n} \neq 0$, $\varrho^m R^n{}_{mkl;n} \neq 0$ or $P^m R^n{}_{mkl;n} = 0$, $\varrho^m R^n{}_{mkl;n} = 0$, $u^m R^n{}_{mkl;n} \neq 0$.

After more careful calculations one can show that the condition $u^m R^n{}_{mkl;n} = 0$ gives the contradictory relation $u_{[i}(\varrho + P)_{,k]} = 0$. This relation does not satisfy the conditions

$$P^m R^n{}_{mkl;n} \neq 0, \; \varrho^m R^n{}_{mkl;n} \neq 0.$$

If $D_{nkl} = 0$, we have $\sigma_{ik} = 0$, $\theta = 0$ and $\alpha = \dfrac{d}{dp} = 1$. Consequently, in this case the equations of state are $\varrho = p + a$, $a = \text{const}$ and the velocity field equation (2.2) may be written as

$$u_{i;k} = \frac{P_{,i} u_k}{\varrho + P}. \tag{4.4}$$

If $P^m R^n{}_{mkl;n} = \varrho^m R^n{}_{mkl;n} = 0$, $u^m R^n{}_{mkl;n} \neq 0$ we have from (3.1) that $\sigma_{ik} = 0$, $\theta = 0$ and $\alpha = \dfrac{d\varrho}{dP} \neq 1$. In this case equation (4.4) is also valid.

5. *Conclusions*

It is shown that the equations of state (2.10), (2.12), $P = \text{const}$, $\varrho_{,k} = \varrho_{,n} u^n u_k \neq 0$ and $\varrho_{,k} = \alpha P_{,k}$, $\alpha = \dfrac{d\varrho}{dp}$, $P_{,n} u^n = \varrho_{,n} u^n = 0$ do not contradict Einstein's field equations. However, the perfect fluid satisfying (2.10) and $P_{,k} = 0$, $\varrho_{,k} = \varrho_{,n} u^n u_k \neq 0$ has unrealistic equations of state. Such a fluid does not satisfy the condition $\varrho = \varrho(P)$. Therefore, the thermodynamics of this fluid does not correspond to the thermodynamics of a real perfect fluid.

The gravitational fields of the irrotational perfect fluid with the equations of state (2.12) and $\varrho_{,k} = \alpha P_{,k}$, $\alpha = \dfrac{d\varrho}{dP}$ are physically realistic. These fields satisfy the condition $\varrho = \varrho(P)$.

APPENDIX A

In this appendix we prove the validity of identities $C^{nm}{}_{kl;n;m} = 0$. From (2.3) we have

$$\frac{1}{3}(C^{n}{}_{ikl;n;m}+C^{n}{}_{imk;n;l}+C^{n}{}_{ilm;n;k}) = R_{i[k;l;m]}.$$

Using the Ricci identities in the above mentioned relation we obtain

$$C^{n}{}_{ikl;n;m}+C^{n}{}_{imk;n;l}+C^{n}{}_{ilm;n;k} = R_{nk}R^{n}{}_{ilm}+R_{nl}R^{n}{}_{imk}+R_{nm}R^{n}{}_{ikl}. \quad \text{(A1)}$$

The contraction of (A1) gives

$$C^{nm}{}_{kl;n;m} = 0.$$

Similarly we may obtain

$$R^{n}{}_{k;l;n} = R^{n}{}_{l;k;n}, \qquad R^{nm}{}_{kl;n;m} = 0.$$

REFERENCES

[1] I. H. Thompson, G. I. Whitrow, *Mon. Not. R. Astron. Soc.* **136**, 207 (1967).
[2] G. G. Mc Vittie, *Ann. Inst. Henri Poincaré* **6**, 1 (1967).
[3] W. B. Bonnor, M. C. Faulkes, *Mon. Not. R. Astron. Soc.* **137**, 239 (1967).
[4] M. C. Faulkes, *Prog. Theor. Phys.* **42**, 1139 (1969).
[5] H. Nariai, *Prog. Theor. Phys.* **40**, 1013 (1968).
[6] A. Barnes, *Gen. Relativ. Gravitation* **4**, 105 (1973).
[7] R. M. Misra, D. C. Srivastava, *Phys. Rev.* **8**, 1653 (1973).
[8] D. A. Szafron, I. Wainwright, *J. Math. Phys.* **18**, 1668 (1977).
[9] V. I. Obozov, *Izv. VUZ MV SSO SSSR, Ser. Fiz.* N2, 12 (1974).
[10] V. I. Obozov, *Izv. VUZ MV SSO SSSR, Ser. Fiz.* N8, 71 (1979).
[11] V. I. Obozov, *Izv. VUZ MV SSO SSSR, Ser. Fiz.* N1, 27 (1975).
[12] C. W. Misner, K. S. Thorne, I. A. Wheeler, *Gravitation*, W. H. Freeman and Company, San Francisco 1973, vol. 2.
[13] A. Krasiński, *Acta Phys. Pol.* **B5**, 411 (1974).