SOME ANISOTROPIC HOMOGENEOUS MODELS IN A MODIFIED BRANS-DICKE COSMOLOGY

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Adding the cosmological term, which is assumed to be variable, in Brans-Dicke theory we have discussed different cylindrically symmetric models which are of Petrov type I or Petrov type D. The physical and geometrical properties of these models have been discussed. Finally these models have been transformed to the original form (1961) of Brans-Dicke theory (including a variable cosmological term).

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1. Introduction

After the cosmological constant was first introduced into general relativity by Einstein, its significance was studied by various cosmologists (for example, [1]), but no satisfactory results of its meaning have been reported so far. Żeldovich [2] has tried to visualize the meaning of this term from the theory of elementary particles. Further Linde [3] has argued that the cosmological term arises from spontaneous symmetry breaking and suggested that the term is not a constant but a function of temperature. Also Drietlein [4] connects the mass of Higg's scalar boson with both the cosmological term and the gravitational constant. In cosmology, the term may be understood by incorporation with Mach's principle, which suggests the acceptance of Brans-Dicke Lagrangian as a realistic case [5]. The investigation of particle physics within the context of the Brans-Dicke Lagrangian [6] has stimulated the study of the cosmological term with a modified Brans-Dicke Lagrangian in cosmology and elementary particle physics. Recently Endo and Fukui [7] have studied the variable cosmological term from the point of view of cosmology in Brans--Dicke theory [5] and elementary particle physics (especially in the context of Dirac's large number hypothesis [8, 9]).

In this paper we have considered the modified Brans-Dicke theory with the variable cosmological term as an explicit function of a scalar field Φ as proposed by Bergmann [10] and Wagoner [11] and discussed in detail by Endo and Fukui [7].

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The Brans-Dicke field equations with cosmological term Q are [7]:

$$G_{ij}+g_{ij}Q = \frac{8\pi}{\Phi}T_{ij}+\frac{\omega}{\Phi^2}(\Phi_{,i}\Phi_{,j}-\tfrac{1}{2}g_{ij}\Phi_{,k}\Phi^{,k})+\frac{1}{\Phi}(\Phi_{i;j}-g_{ij}\Box\Phi), \tag{1.1}$$

$$\Box\Phi = \frac{8\pi}{(2\omega+3)}\mu T, \tag{1.2}$$

$$Q = \frac{(2\omega+3)}{4}\cdot\frac{(1-\mu)}{\mu}\frac{\Box\Phi}{\Phi} = \frac{8\pi(1-\mu)}{4\Phi}T, \tag{1.3}$$

where the constant μ shows how much our theory including $Q(\Phi)$ deviates from that of Brans and Dicke and as usual ω is coupling constant and T_{ij} is energy-momentum tensor. Semicolons denote covariant derivative with respect to the metric g_{ij} and commas mean partial derivatives with respect to the coordinate x^i. The theory can also be represented in a different form under a unit transformation (UT) [12] in which length, time and reciprocal mass are scaled by the function $\lambda^{1/2}(x)$. Then under the conformal transformation

$$g_{ij} \to \bar{g}_{ij} = \Phi g_{ij} \tag{1.4}$$

equations (1.1)–(1.3) have the form

$$\bar{G}_{ij}+\bar{g}_{ij}\bar{Q} = (8\pi)\bar{T}_{ij}+\tfrac{1}{2}(2\omega+3)(\Lambda_{,i}\Lambda_{,j}-\tfrac{1}{2}\bar{g}_{ij}\Lambda_{,k}\Lambda^{,k}), \tag{1.5}$$

$$\bar{\Box}\Lambda = \frac{8\pi}{(2\omega+3)}\mu\bar{T}, \qquad \Lambda = \log\Phi, \tag{1.6}$$

$$\bar{Q} = \frac{(2\omega+3)}{4}\frac{(1-\mu)}{\mu}\bar{\Box}\Lambda = \frac{8\pi(1-\mu)}{4}\bar{T}; \tag{1.7}$$

where the barred quantities are defined in terms of $\bar{g}_{ij}$ as their unbarred counterparts are defined in terms of the unbarred metric g_{ij} and all barred operations are performed with respect to the barred metric and barred Christoffel symbols.

In Section 3 we have obtained a cosmological solution which is of nondegenerate Petrov type I. In Sections 4 and 5 we have obtained two othei models both of which are of Petrov type D. Finally in Section 6 we have transformed these models to the 1961 form of Brans-Dicke theory [5].

2. *The field equations*

The cylindrically symmetric metric is considered in the form given by Marder [13]:

$$ds^2 = A^2(dt^2-dx^2)-B^2dy^2-C^2dz^2, \tag{2.1}$$

where A, B, C are functions of $x^4 \equiv t$ only. This ensures that the model is spatially homogeneous. The transformation $t \to \int A(t)dt$ brings the metric (2.1) into Bianchi type I form. However, for mathematical convenience we retain the metric in the form (2.1). The energy-

-momentum tensor $\bar{T}_{ij}$ for perfect fluid distribution is given by

$$\bar{T}_{ij} = (\bar{\varrho}+\bar{p})\bar{v}_i\bar{v}_j - \bar{p}\bar{g}_{ij} \tag{2.2}$$

together with

$$\bar{g}_{ik}\bar{v}^i\bar{v}^j = 1 \tag{2.3}$$

where $\bar{p}$ and $\bar{\varrho}$ are the proper pressure and energy density respectively and $\bar{v}^i$ are the components of the fluid four-velocity. We assume the coordinates to the comoving so that $\bar{v}^1 = \bar{v}^2 = \bar{v}^3 = 0$ and $\bar{v}^4 = \dfrac{1}{A}$. Scalar field Λ is also taken to be a function of t only. The field equations (1.5) and (1.6) turn into

$$\frac{1}{A^2}\left[\frac{B_{44}}{B}+\frac{C_{44}}{C}+\frac{B_4C_4}{BC}-\frac{A_4}{G}\left(\frac{B_4}{B}+\frac{C_4}{C}\right)\right]+\bar{Q} = -8\pi\bar{p}-\frac{(2\omega+3)}{4A^2}\Lambda_{,4}^2, \tag{2.4}$$

$$\frac{1}{A^2}\left[\left(\frac{A_4}{A}\right)_4+\frac{C_{44}}{C}\right]+\bar{Q} = -8\pi\bar{p}-\frac{(2\omega+3)}{4A^2}\Lambda_{,4}^2, \tag{2.5}$$

$$\frac{1}{A^2}\left[\left(\frac{A_4}{A}\right)_4+\frac{B_{44}}{B}\right]+\bar{Q} = -8\pi\bar{p}-\frac{(2\omega+3)}{4A^2}\Lambda_{,4}^2, \tag{2.6}$$

$$\frac{1}{A^2}\left[\frac{B_4C_4}{BC}+\frac{A_4}{A}\left(\frac{B_4}{B}+\frac{C_4}{C}\right)\right]+\bar{Q} = 8\pi\bar{\varrho}+\frac{(2\omega+3)}{4A^2}\Lambda_{,4}^2, \tag{2.7}$$

$$\left[\Lambda_{,44}+\Lambda_{,4}\left(\frac{B_4}{B}+\frac{C_4}{C}\right)\right] = \frac{8\pi\mu A^2}{(2\omega+3)}(\bar{\varrho}-3\bar{p}). \tag{2.8}$$

The suffix 4 after A, B, C denotes ordinary differentiation with respect to t. Equations (2.4)-(2.8) are five equations in six unknowns A, B, C, $\bar{\varrho}$, $\bar{p}$ and Λ. For complete determinancy of the system one extra condition is needed. One way is to impose an equation of state. The other alternative is a mathematical assumption on the space-time and then the discussion of the physical nature of the universe. We shall confine ourselves to the latter method in this paper and attempt three cases

(*i*) $C_{14}{}^{14} = C_{23}{}^{23} = 0$,

(*ii*) $C_{12}{}^{12} = C_{13}{}^{13}$,

(*iii*) $C_{12}{}^{12} = C_{23}{}^{23}$,

where C_{nijk} is Weyl's conformal curvature tensor. In cases (*ii*) and (*iii*) the space-time is of Petrov type D while in case (*i*) the space-time is of non-degenerate Petrov type I. The conditions (*ii*) and (*iii*) are identically satisfied if $B = C$ and $A = C$ respectively. However, we shall assume A, B, C to be unequal on account of the supposed anisotropy.

3. The first model

This model is obtained under the condition $C_{14}{}^{14} = C_{23}{}^{23} = 0$ which leads to

$$\left(\frac{A_4}{A}\right)_4 = \frac{B_{44}}{2B} + \frac{C_{44}}{2C} - \frac{B_4 C_4}{BC}. \tag{3.1}$$

When we consider equation (3.1) along with (2.4)-(2.7) and proceed on lines similar to Roy and Singh [14] we have the metric

$$ds^2 = L^2 t^{1-a^2}(dt^2 - dx^2) - t^{1+a} dy^2 - t^{1-a} dz^2, \tag{3.2}$$

a and L being constants. The pressure $\bar{p}$ and density $\bar{\varrho}$ in the model (3.2) are given by

$$8\pi\bar{p} = \frac{3}{4L^2}(1-a^2)t^{a^2-3}\sec^2\left\{\sqrt{\frac{3(a^2-1)}{4(2\omega+3)}}\log(k_1 t)\right\} - \bar{Q}, \tag{3.3}$$

$$8\pi\bar{\varrho} = \frac{3}{4L^2}(1-a^2)t^{a^2-3}\sec^2\left\{\sqrt{\frac{3(a^2-1)}{4(2\omega+3)}}\log(k_1 t)\right\} + \bar{Q}. \tag{3.4}$$

Also the scalar field Λ is given by

$$\Lambda = \log\sec^2\left\{\sqrt{\frac{3(a^2-1)}{4(2\omega+3)}}\log(k_1 t)\right\} \tag{3.5}$$

and

$$\bar{Q} = \frac{(\mu-1)}{4\mu}\left[\frac{3}{2L^2}(1-a^2)t^{a^2-3}\sec^2\left\{\sqrt{\frac{3(a^2-1)}{4(2\omega+3)}}\log(k_1 t)\right\}\right], \tag{3.6}$$

where k_1 is a constant.

For reality of $\bar{p}$, $\bar{\varrho}$ and Λ and the conditions $\bar{\varrho} > 0$, $\bar{p} > 0$ and $\bar{\varrho} \geqslant 3\bar{p}$ to hold,

$$a^2 < 1, \quad \omega < -\tfrac{3}{2}, \quad \bar{Q} > 0 \quad (\text{i.e. } \mu > 1) \tag{3.7}$$

and

$$\bar{Q} < \frac{3}{4L^2}(1-a^2)t^{a^2-3}\sec^2\left\{\sqrt{\frac{3(a^2-1)}{4(2\omega+3)}}\log(k_1 t)\right\} \leqslant 2\bar{Q}. \tag{3.8}$$

The flow vector $\bar{v}^i$ of the distribution for the model (3.2) is given by

$$\bar{v}^1 = \bar{v}^2 = \bar{v}^3 = 0, \quad \bar{v}^4 = \frac{1}{L}t^{(a^2-1)/2}. \tag{3.9}$$

Clearly $\bar{v}^i_{;j}\bar{v}^j = 0$, so that the flow is geodetic. The rotation tensor $w_{ij} = \bar{v}_{i;j} - \bar{v}_{j;i}$ is zero. The expansion scalar $\theta = \frac{1}{3}\bar{v}^i{}_{;i}$ and shear tensor

$$\sigma_{ij} = \tfrac{1}{2}(\bar{v}_{i;j} + \bar{v}_{j;i}) - \theta(\bar{g}_{ij} - \bar{v}_i \bar{v}_j)$$

are given by

$$\theta = \frac{(3-a^2)}{6L}\, \bar{t}^{(a^2-3)/2} \tag{3.10}$$

and

$$\sigma_{11} = \frac{a^2 L}{3}\, \bar{t}^{(1+a^2)/2},$$

$$\sigma_{22} = -\frac{(a^2+3a)}{6L}\, t^{(a^2+2a-1)/2},$$

$$\sigma_{33} = -\frac{(a^2-3a)}{6L}\, t^{(a^2-2a-1)/2},$$

$$\sigma_{44} = 0. \tag{3.11}$$

Also the shear σ is

$$\sigma^2 = \tfrac{1}{2}\,\sigma_{ij}\sigma^{ij} = \frac{a^2(a^2+3)}{12L^2}\, t^{a^2-3} \tag{3.12}$$

The nonvanishing components of conformal curvature tensor $C_{ni}{}^{jk}$ are

$$C_{12}{}^{12} = C_{34}{}^{34} = \frac{a(a^2-1)}{4L^2}\, t^{a^2-3},$$

$$C_{13}{}^{13} = C_{24}{}^{24} = \frac{a(1-a^2)}{4L^2}\, t^{a^2-3}. \tag{3.13}$$

Thus the model represents an irrotational, expanding universe with shear. The free gravitational field characterized by C_{nij} is also nonvanishing.

The pressure, density, scalar field and cosmological constant are singular at

$$t = \left(\frac{1}{k_1}\right) \exp\left\{\pi \sqrt{\frac{(2\omega+3)}{3(a^2-1)}}\right\}.$$

The model exists for a finite time

$$\left(\frac{1}{k_1}\right) \leqslant t < \left(\frac{1}{k_1}\right) \exp\left\{\pi \sqrt{\frac{(2\omega+3)}{3(a^2-1)}}\right\}. \tag{3.14}$$

When $\mu = 1$, the solution reduces to a simple Brans-Dicke analogue of the solution due to Roy and Singh [14] in general relativity.

4. *The second model*

This model is obtained under the condition $C_{12}{}^{12} = C_{13}{}^{13}$ which leads to

$$\frac{B_{44}}{B} - \frac{C_{44}}{C} + \frac{2A_4}{A}\left(\frac{C_4}{C} - \frac{B_4}{B}\right) = 0. \tag{4.1}$$

So we consider equations (2.4)-(2.8) along with (4.1).

From (2.4) and (2.5), we have

$$\left(\frac{A_4}{A}\right)_4 + \frac{A_4}{A}\left(\frac{B_4}{B} + \frac{C_4}{C}\right) - \frac{B_{44}}{B} - \frac{B_4 C_4}{BC} = 0. \tag{4.2}$$

Also from (2.5) and (2.6), we get

$$\frac{B_{44}}{B} = \frac{C_{44}}{C}. \tag{4.3}$$

Equations (4.1) and (4.3) lead to

$$\frac{A_4}{A}\left(\frac{B_4}{B} - \frac{C_4}{C}\right) = 0. \tag{4.4}$$

Since $B \neq C$, equation (4.4) gives

$$A = N \text{ (const)}. \tag{4.5}$$

From (4.2) and (4.5), we have

$$\frac{B_{44}}{B} + \frac{B_4 C_4}{BC} = 0. \tag{4.6}$$

Equation (4.3) after integration gives

$$B_4 C - B C_4 = k_2, \tag{4.7}$$

k_2 being the constant of integration. On substituting $B/C = \alpha$, $BC = \beta$ so that $B^2 = \alpha\beta$, $C^2 = \beta/\alpha$ equation (4.7) reduces to

$$\left(\frac{\alpha_4}{\alpha}\right)\beta = k_2. \tag{4.8}$$

From (4.6) we have

$$\left[\left(\frac{\alpha_4}{\alpha} + \frac{\beta_4}{\beta}\right)\beta\right]_4 = 0. \tag{4.9}$$

From (4.8) and (4.9), we get

$$\beta_{44} = 0,$$

which gives

$$\beta = (k_3 t + b), \tag{4.10}$$

where k_3 and b are constants of integration. From (4.8) and (4.10) one has

$$\alpha = k_4 \beta^{k_2/k_3}.$$

Therefore

$$B^2 = k_4(k_3 t + b)^{1 + k_2/k_3}$$

and

$$C^2 = \frac{1}{k_4}(k_3 t+b)^{1-k_2/k_3} \tag{4.12}$$

Consequently the line-element takes the form

$$ds^2 = N^2(dt^2-dx^2)-k_4(k_3t+b)^{1+k_2/k_3}dy^2-\frac{1}{k_4}(k_3t+b)^{1-k_2/k_3}dz^2. \tag{4.13}$$

By the following transformation of coordinates

$$Nx \to x, \quad Nt \to t, \quad k_4^{1/2}y \to y, \quad k_4^{-1/2}z \to z$$

this line-element reduces to the form

$$ds^2 = dt^2-dx^2-\left(\frac{k_3}{N}t+b\right)^{1+k_2/k_3}dy^2-\left(\frac{k_3}{N}t+b\right)^{1-k_2/k_3}dz^2. \tag{4.14}$$

Now for the metric (4.14) the pressure and density are given by

$$8\pi\bar{p} = \frac{(k_3^2-k_2^2)}{4\left(\frac{k_3}{N}t+b\right)^2}\sec^2\left[\sqrt{\frac{(k_2^2-k_3^2)}{4k_3^2(2\omega+3)}}\log\left\{k_5\left(\frac{k_3}{N}t+b\right)\right\}\right]-\bar{Q}, \tag{4.15}$$

$$8\pi\bar{\rho} = \frac{(k_3^2-k_2^2)}{4\left(\frac{k_3}{N}t+b\right)^2}\sec^2\left[\sqrt{\frac{(k_2^2-k_3^2)}{4k_3^2(2\omega+3)}}\log\left\{k_5\left(\frac{k_3}{N}t+b\right)\right\}\right]+\bar{Q}. \tag{4.16}$$

Also the scalar field Λ is given by

$$\Lambda = \log\sec^2\left[\sqrt{\frac{(k_2^2-k_3^2)}{4k_3^2(2\omega+3)}}\log\left\{k_5\left(\frac{k_3}{N}t+b\right)\right\}\right], \tag{4.17}$$

and

$$\bar{Q} = \frac{(\mu-1)}{4\mu}\,\frac{(k_3^2-k_2^2)}{2\left(\frac{k_3}{N}t+b\right)^2}\sec^2\left[\sqrt{\frac{(k_2^2-k_3^2)}{4k_3^2(2\omega+3)}}\log\left\{k_5\left(\frac{k_3}{N}t+b\right)\right\}\right]. \tag{4.18}$$

The model is real and the conditions $\bar{\rho} > 0$, $\bar{p} > 0$, $\bar{\rho} \geqslant 3\bar{p}$ hold when

$$k_3^2 > k_2^2, \quad \omega < -3/2, \quad \bar{Q} > 0 \quad (\text{i.e. } \mu > 1), \tag{4.19}$$

and

$$\bar{Q} < \frac{(k_3^2-k_2^2)}{4\left(\frac{k_3}{N}t+b\right)^2}\sec^2\left[\sqrt{\frac{(k_2^2-k_3^2)}{4k_3^2(2\omega+3)}}\log\left\{k_5\left(\frac{k_3}{N}t+b\right)\right\}\right] \leqslant 2\bar{Q}. \tag{4.20}$$

The nonvanishing components of the Weyl conformal curvature tensor $C_{ni}{}^{jk}$ are

$$C_{14}{}^{14} = C_{23}{}^{23} = \frac{(k_2^2-k_3^2)}{6\left(\frac{k_3}{N}t+b\right)^2}, \tag{4.21}$$

$$C_{12}{}^{12} = C_{34}{}^{34} = \frac{(k_3^2-k_2^2)}{12\left(\frac{k_3}{N}t+b\right)^2},$$

$$C_{13}{}^{13} = C_{24}{}^{24} = \frac{(k_3^2-k_2^2)}{12\left(\frac{k_3}{N}t+b\right)^2}.$$

The flow vector $\bar{v}^i$ is given by

$$\bar{v}^1 = \bar{v}^2 = \bar{v}^3 = 0, \quad \bar{v}^4 = 1. \tag{4.22}$$

It satisfies $\bar{v}^i_{;j}\bar{v}^j = 0$, so that the flow is geodetic. Also $w_{ij} = 0$.

The scalar of expansion is

$$\theta = \frac{k_3}{3N}\left(\frac{k_3}{N}t+b\right)^{-1}. \tag{4.23}$$

The non-zero components of shear tensor σ_{ij} are

$$\sigma_{11} = \frac{k_3}{3N}\left(\frac{k_3}{N}t+b\right)^{-1},$$

$$\sigma_{22} = -\frac{(3k_2+k_3)}{6N}\left(\frac{k_3}{N}t+b\right)^{k_2/k_3},$$

$$\sigma_{33} = \frac{(3k_2-k_3)}{6N}\left(\frac{k_3}{N}t+b\right)^{-k_2/k_3} \tag{4.24}$$

and the shear σ is

$$\sigma^2 = \frac{(k_3^2+3k_2^2)}{12N^2}\left(\frac{k_3}{N}t+b\right)^{-2}. \tag{4.25}$$

Thus the model represents an irrotational, expanding universe with shear.

The pressure, density, scalar field and cosmological constant are singular at

$$t = \left(\frac{N}{k_3}\right)\left[-b+\left(\frac{1}{k_5}\right)\exp\left\{\pi k_3\sqrt{\frac{(2\omega+3)}{(k_2^2-k_3^2)}}\right\}\right].$$

The model exists for a finite time

$$\frac{N}{k_3}\left[-b+\frac{1}{k_5}\right] \leqslant t < \frac{N}{k_3}\left[-b+\left(\frac{1}{k_5}\right)\exp\left\{\pi k_3\sqrt{\frac{(2\omega+3)}{(k_2^2-k_3^2)}}\right\}\right]. \tag{4.26}$$

When $\mu = 1$, the cosmological term $\bar{Q}$ vanishes and the model (4.14) reduces to a Brans-Dicke analogue of one of the models due to Roy and Prakash [15] in general relativity.

5. *The third model*

This model is obtained under the condition $C_{12}{}^{12} = C_{23}{}^{23}$ which leads to

$$\left(\frac{A_4}{A}\right)_4 = \frac{C_{44}}{C} - \frac{B_4C_4}{BC} + \frac{A_4}{A}\left(\frac{B_4}{B} - \frac{C_4}{C}\right). \tag{5.1}$$

From (2.4)-(2.6) we obtain

$$\left(\frac{A_4}{A}\right)_4 + \frac{A_4}{A}\left(\frac{B_4}{B} + \frac{C_4}{C}\right) = \frac{B_{44}}{B} + \frac{B_4C_4}{BC} \tag{5.2}$$

and

$$\frac{B_{44}}{B} = \frac{C_{44}}{C}. \tag{5.3}$$

From (5.1), (5.2) and (5.3) we get

$$\frac{B_4}{B}\left(\frac{A_4}{A} - \frac{C_4}{C}\right) = 0. \tag{5.4}$$

Since $A \neq C$ equation (5.4) gives

$$B = N' \text{ (const)}. \tag{5.5}$$

Equations (5.1) and (5.5) give

$$\left(\frac{A_4}{A}\right)_4 + \frac{A_4C_4}{AC} = 0. \tag{5.6}$$

From (5.3) and (5.5) we get

$$C = k_6t+k_7 \tag{5.7}$$

k_6 and k_7 being constants of integration.

Equations (5.6) and (5.7) after integration give

$$A = M(k_6t+k_7)^{K/k_3}, \tag{5.8}$$

where M and K are arbitrary constants of integration. Consequently the line-element takes the form

$$ds^2 = M^2(k_6t+k_7)^{2K/k_6}(dt^2-dx^2)-N'^2dy^2-(k_6t+k_7)^2dz^2. \tag{5.9}$$

By the following transformation of coordinates

$$k_6t+k_7 \to t, \quad k_6x \to x, \quad N'y \to y, \quad z \to z, \quad k_6^{-1}M \to M$$

the line-element (5.9) reduces to the form

$$ds^2 = M^2t^{2K/k_6}(dt^2-dx^2)-dy^2-t^2dz^2. \tag{5.10}$$

Now for the metric (5.10) $\bar{p}$, $\bar{\varrho}$ and Λ are given by

$$8\pi\bar{p} = \frac{K}{M^2}\,\bar{t}^{2(1+K/k_6)}\sec^2\left\{\sqrt{\frac{(-K)}{(2\omega+3)}}\log(k_8t)\right\}-\bar{Q}, \tag{5.11}$$

$$8\pi\bar{\varrho} = \frac{K}{M^2}\,\bar{t}^{2(1+K/k_6)}\sec^2\left\{\sqrt{\frac{(-K)}{(2\omega+3)}}\log(k_8t)\right\}+\bar{Q}. \tag{5.12}$$

Also the scalar field Λ is given by

$$\Lambda = \log\sec^2\left\{\sqrt{\frac{(-K)}{(2\omega+3)}}\log(k_8t)\right\}, \tag{5.13}$$

and

$$\bar{Q} = \frac{(\mu-1)}{4\mu}\left[\frac{2K}{M^2}\,\bar{t}^{2(1+K/k_6)}\sec^2\left\{\sqrt{\frac{(-K)}{(2\omega+3)}}\log(k_8t)\right\}\right], \tag{5.14}$$

k_8 being a constant.

The model is real and the conditions $\bar{\varrho} > 0$, $\bar{p} > 0$, $\bar{\varrho} \geqslant 3\bar{p}$ hold when

$$K > 0, \quad \omega < -\tfrac{3}{2}, \quad \bar{Q} > 0 \quad (\text{i.e. } \mu > 1) \tag{5.15}$$

and

$$\bar{Q} < \frac{K}{M^2}\,\bar{t}^{2(1+K/k_6)}\sec^2\left\{\sqrt{\frac{(-K)}{(2\omega+3)}}\log(k_8t)\right\} \leqslant 2\bar{Q}. \tag{5.16}$$

The nonvanishing components of Weyl's conformal curvature tensor are

$$C_{14}{}^{14} = C_{23}{}^{23} = \frac{K}{3M^2}\,\bar{t}^{2(1+K/k_6)},$$

$$C_{12}{}^{12} = C_{34}{}^{34} = \frac{K}{3M^2}\,\bar{t}^{2(1+K/k_6)},$$

$$C_{13}{}^{13} = C_{24}{}^{24} = -\frac{2K}{3M^2}\,\bar{t}^{2(1+K/k_6)}. \tag{5.17}$$

The velocity vector $\bar{v}^i$ is given by

$$\bar{v}^1 = \bar{v}^2 = \bar{v}^3 = 0, \quad \bar{v}^4 = \frac{1}{M}\,\bar{t}^{K/k_6}. \tag{5.18}$$

Clearly $\bar{v}^i_{;j}\bar{v}^j = 0$, so that the flow is geodetic. The scalar of expansion θ is given by

$$\theta = \frac{(1+K/k_6)}{3M}\,\bar{t}^{(1+K/k_6)}. \tag{5.19}$$

The tensor of rotation w_{ij} is zero. The non-zero components of shear tensor σ_{ij} are

$$\sigma_{11} = \frac{\left(1-\dfrac{2K}{k_6}\right)}{3}\,Mt^{\left(\frac{K}{k_6}-1\right)},$$

$$\sigma_{22} = \frac{\left(1+\dfrac{K}{k_6}\right)}{3M}\,\bar{t}^{(1+K/k_6)},$$

$$\sigma_{33} = \frac{\left(\dfrac{K}{k_6'}-1\right)}{3M}\,t^{(1-K/k_6)} \tag{5.20}$$

and the shear σ is

$$\sigma^2 = \frac{1}{3M^2}\left\{\left(1-\frac{K}{k_6}\right)^2+\frac{K}{k_6}\right\}\bar{t}^{2(1+K/k_6)}. \tag{5.21}$$

Thus the model represents an irrotational, expanding universe with shear.

The pressure, density, scalar field and cosmological constant are singular at

$$t = \left(\frac{1}{k_8}\right)\exp\left\{\frac{\pi}{2}\sqrt{\frac{(2\omega+3)}{(-K)}}\right\}.$$

The model exists for a finite time

$$\left(\frac{1}{k_8}\right) \leqslant t < \left(\frac{1}{k_8}\right)\exp\left\{\frac{\pi}{2}\sqrt{\frac{(2\omega+3)}{(-K)}}\right\}. \tag{5.22}$$

When $\mu = 1$, we get a Brans-Dicke analogue of one of the models due to Roy and Prakash [15].

6. *Transformations of the solutions and discussion*

Under the transformations

$$\bar{g}_{ij} \rightarrow g_{ij} = \frac{1}{\Phi}\,\bar{g}_{ij}, \qquad \bar{T}_{ij} \rightarrow T_{ij} = \Phi\bar{T}_{ij},$$

$$\bar{T} \rightarrow T = \Phi^2\bar{T}, \qquad \bar{p} \rightarrow p = \Phi^2\bar{p},$$

$$\bar{\varrho} \rightarrow \varrho = \Phi^2\bar{\varrho}, \qquad \bar{\Phi} \rightarrow \Phi = e^{A},$$

$$\bar{Q} \rightarrow Q = \Phi\bar{Q}, \qquad \bar{v}^i \rightarrow v^i = \Phi^{1/2}\bar{v}^i \tag{6.1}$$

the field equations (1.5)–(1.7) are changed into (1.1)–(1.3). We now apply these transformations to the solutions obtained in Sections 3,4 and 5.

The first model is transformed into

$$\Phi = \sec^2\left\{\sqrt{\frac{3(a^2-1)}{4(2\omega+3)}}\log(k_1 t)\right\}, \tag{6.2a}$$

$$g_{ij} = \cos^2\left\{\sqrt{\frac{3(a^2-1)}{4(2\omega+3)}}\log(k_1 t)\right\}\bar{g}_{ij}, \tag{6.2b}$$

i.e.

$$g_{11} = -\cos^2\left\{\sqrt{\frac{3(a^2-1)}{4(2\omega+3)}}\log(k_1 t)\right\}L^2 t^{1-a^2},$$

$$g_{22} = -\cos^2\left\{\sqrt{\frac{3(a^2-1)}{4(2\omega+3)}}\log(k_1 t)\right\}t^{1+a},$$

$$g_{33} = -\cos^2\left\{\sqrt{\frac{3(a^2-1)}{4(2\omega+3)}}\log(k_1 t)\right\}t^{1+a},$$

$$g_{44} = \cos^2\left\{\sqrt{\frac{3(a^2-1)}{4(2\omega+3)}}\log(k_1 t)\right\}L^2 t^{1-a^2};$$

$$8\pi p = \frac{3}{4L^2}(1-a^2)t^{a^2-3}\sec^6\left\{\sqrt{\frac{3(a^2-1)}{4(2\omega+3)}}\log(k_1 t)\right\}\left[1-\frac{(\mu-1)}{2\mu}\right],$$

$$8\pi\varrho = \frac{3}{4L^2}(1-a^2)t^{a^2-3}\sec^6\left\{\sqrt{\frac{3(a^2-1)}{4(2\omega+3)}}\log(k_1 t)\right\}\left[1+\frac{(\mu-1)}{2\mu}\right]; \tag{6.2c}$$

$$v^1 = v^2 = v^3 = 0, \quad v^4 = \Phi^{1/2}\bar{v}^4 \tag{6.2d}$$

$$v^4 = \frac{1}{L}t^{\frac{(a^2-1)}{2}}\sec\left\{\sqrt{\frac{3(a^2-1)}{4(2\omega+3)}}\log(k_1 t)\right\},$$

$$Q = \frac{(\mu-1)}{4\mu}\left[\frac{3}{2L^2}(1-a^2)t^{a^2-3}\sec^4\left\{\sqrt{\frac{3(a^2-1)}{4(2\omega+3)}}\log(k_1 t)\right\}\right]. \tag{6.2e}$$

The second model is transformed into

$$\Phi = \sec^2\left[\sqrt{\frac{(k_2^2-k_3^2)}{4k_3^2(2\omega+3)}}\log\left\{k_5\left(\frac{k_3}{N}t+b\right)\right\}\right], \tag{6.3a}$$

$$g_{ij} = \cos^2\left[\sqrt{\frac{(k_2^2-k_3^2)}{4k_3^2(2\omega+3)}}\log\left\{k_5\left(\frac{k_3}{N}t+b\right)\right\}\right]\bar{g}_{ij}, \tag{6.3b}$$

i.e.

$$g_{11} = -\cos^2\left[\sqrt{\frac{(k_2^2-k_3^2)}{4k_3^2(2\omega+3)}}\log\left\{k_5\left(\frac{k_3}{N}t+b\right)\right\}\right],$$

$$g_{22} = -\cos^2\left[\sqrt{\frac{(k_2^2-k_3^2)}{4k_3^2(2\omega+3)}}\log\left\{k_5\left(\frac{k_3}{N}t+b\right)\right\}\right]\left(\frac{k_3}{N}t+b\right)^{1+k_2/k_3},$$

$$g_{33} = -\cos^2\left[\sqrt{\frac{(k_2^2-k_3^2)}{4k_3^2(2\omega+3)}}\log\left\{k_5\left(\frac{k_3}{N}t+b\right)\right\}\right]\left(\frac{k_3}{N}t+b\right)^{1-k_2/k_3},$$

$$g_{44} = \cos^2\left[\sqrt{\frac{(k_2^2-k_3^2)}{4k_3^2(2\omega+3)}}\log\left\{k_5\left(\frac{k_3}{N}t+b\right)\right\}\right],$$

$$8\pi p = \frac{(k_3^2-k_2^2)}{4\left(\frac{k_3}{N}t+b\right)^2}\sec^6\left[\sqrt{\frac{(k_2^2-k_3^2)}{4k_3^2(2\omega+3)}}\log\left\{k_5\left(\frac{k_3}{N}t+b\right)\right\}\right]\left[1-\frac{(\mu-1)}{2\mu}\right],$$

$$8\pi \varrho = \frac{(k_3^2-k_2^2)}{4\left(\frac{k_3}{N}t+b\right)^2}\sec^6\left[\sqrt{\frac{(k_2^2-k_3^2)}{4k_3^2(2\omega+3)}}\log\left\{k_5\left(\frac{k_3}{N}t+b\right)\right\}\right]\left[1+\frac{(\mu-1)}{2\mu}\right]; \tag{6.3c}$$

$$v^1 = v^2 = v^3 = 0, \quad v^4 = \sec\left[\sqrt{\frac{(k_2^2-k_3^2)}{4k_3^2(2\omega+3)}}\log\left\{k_5\left(\frac{k_3}{N}t+b\right)\right\}\right], \tag{6.3d}$$

$$Q = \frac{(\mu-1)}{4\mu}\,\frac{(k_3^2-k_2^2)}{2\left(\frac{k_3}{N}t+b\right)^2}\sec^4\left[\sqrt{\frac{(k_2^2-k_3^2)}{4k_3^2(2\omega+3)}}\log\left\{k_5\left(\frac{k_3}{N}t+b\right)\right\}\right]. \tag{6.3e}$$

In the third model we have

$$\Phi = \sec^2\left\{\sqrt{\frac{(-K)}{(2\omega+3)}}\log(k_8 t)\right\}, \tag{6.4a}$$

$$g_{ij} = \cos^2\left\{\sqrt{\frac{(-K)}{(2\omega+3)}}\log(k_8 t)\right\}\bar{g}_{ij}, \tag{6.4b}$$

i.e.

$$g_{11} = -\cos^2\left\{\sqrt{\frac{(-K)}{(2\omega+3)}}\log(k_8 t)\right\}M^2 t^{2K/k_6},$$

$$g_{22} = -\cos^2\left\{\sqrt{\frac{(-K)}{(2\omega+3)}}\log(k_8 t)\right\},$$

$$g_{33} = -\cos^2\left\{\sqrt{\frac{(-K)}{(2\omega+3)}}\log(k_8 t)\right\}t^2,$$

$$g_{44} = \cos^2\left\{\sqrt{\frac{(-K)}{(2\omega+3)}}\log(k_8 t)\right\} M^2 t^{2K/k_6};$$

$$8\pi p = \frac{K}{M^2}\, \bar{t}^{2(1+K/k_6)} \sec^6\left\{\sqrt{\frac{(-K)}{(2\omega+3)}}\log(k_8 t)\right\}\left[1-\frac{(\mu-1)}{2\mu}\right],$$

$$8\pi \varrho = \frac{K}{M^2}\, \bar{t}^{2(1+K/k_6)} \sec^6\left\{\sqrt{\frac{(-K)}{(2\omega+3)}}\log(k_8 t)\right\}\left[1+\frac{(\mu-1)}{2\mu}\right]; \quad (6.4c)$$

$$v^1 = v^2 = v^3 = 0, \quad v^4 = \frac{1}{M}\, \bar{t}^{K/k_6} \sec\left\{\sqrt{\frac{(-K)}{(2\omega+3)}}\log(k_8 t)\right\}, \quad (6.4d)$$

$$Q = \frac{(\mu-1)}{4\mu}\left[\frac{2K}{M^2}\, \bar{t}^{2(1+K/k_6)} \sec^4\left\{\sqrt{\frac{(-K)}{(2\omega+3)}}\log(k_8 t)\right\}\right]. \quad (6.4e)$$

The reality conditions should also be imposed on the solutions in (6.2), (6.3) and (6.4) similar to those in Sections 3,4 and 5.

7. *Discussion*

When $\mu = 1$, the cosmological term vanishes and the models (6.2), (6.3) and (6.4) reduce to cylindrically symmetric universes with $p = \varrho$ in the 1961 form of Brans-Dicke theory discussed by Singh and Rai [16]. All the models obtained in this paper are new and like other models with $p = \varrho$ they may be used in the relativistic cosmology for the description of very early stages of the universe expansion.

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