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PHASE-VARIABLE EQUATION FOR THE BOUND-STATE PROBLEM OF CONFINEMENT POTENTIALS

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The method of phase-variable equations has been used for the bound-state problem of confinement potentials. An equation yielding the bound-state spectrum for the confinement potential has been obtained.

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1. Introduction

Several long-range models of quark confinement have been considered with particular emphasis on gluon exchange analogous to Coulomb and linear potenials. It is often instructive in particle physics to keep nonrelativistic analogues describing the bound-state system by the Schrödinger equation in mind. It has been suggested that the study of the dynamics of a nonrelativisic two-particle system under the influence of confinement potentials may have some interesting bearing on the phenomenological aspects of a hadronic system such as charmonium [1].

Various authors [2], [3] have independently given first-order, non-linear differential equations for the phase shift in the case of short-range potentials. They are superior to the Schrödinger equation formulation for numerical computation because the quantity desired is obtained directly rather than inferred from the coefficients of oscillating terms in the wave function. The phase equations are useful for analytical purposes such as deriving approximations, variational principles, and studying complex angular momentum.

The aim of our investigation is to offer a version of the phase method for a bound-state system of two-body nonrelativistic spinless particles interacting by the confinement potential.

The applicability of the phase method to the bound-state problem of the confinement potential has been discussed in Section 2. In Section 3 the formal series method is applied to approximative solving of the phase-variable equation for confinement potential.

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2. *Phase-variable equation*

In order to discuss the bound-state problem for the confinement potential

$$V(r) = -\frac{\alpha}{r} + \beta r, \tag{1}$$

we shall consider the radial Schrödinger equation

$$\frac{d^2\psi}{dr^2} + [E - V(r)]\psi = 0, \tag{2}$$

where as usual $\hbar = c = 1$ and the reduced mass $M = \frac{1}{2}$. If the potential contains a centrifugal term $\frac{l(l+1)}{r^2}$ the same method may be used after substituting $l(l+1) \to (l+\frac{1}{2})^2$.

Consider the boundary conditions [4]

$$\psi(0) = 0, \qquad \psi(\infty) = 0. \tag{3}$$

It is convenient to introduce the two new parameters defined by the relations

$$\psi(r) = A(r)\sin\varphi(r), \qquad \frac{d\psi}{dr} = kA(r)\cos\varphi(r), \tag{4}$$

where $k = E^{1/2}$. Then from Schrödinger's equation (2) it is easy to obtain the phase equation

$$\frac{d\varphi}{dr} = k - \frac{V(r)}{k}\sin^2\varphi(r). \tag{5}$$

Before discussing it further it is worth writing down some expressions for the wave function at infinity and in the neighbourhood of the classical turning point. At distances well above the point $\sqrt{\frac{|\alpha|}{|\beta|}}$ the Coulomb term $-\frac{\alpha}{r}$ is negligible in comparison to the linear potential. Therefore, at $r \gg \sqrt{\frac{|\alpha|}{|\beta|}}$ the solution of (2) can be approximated by

$$\psi_a(r) = \mathrm{Ai}\left(\beta^{1/3}\left(r - \frac{E}{\beta}\right)\right), \tag{6}$$

where $\mathrm{Ai}(z)$ is the Airy function, defined by

$$\mathrm{Ai}(z) = \frac{1}{\pi}\int_0^\infty \cos\left(zx + \frac{x^3}{3}\right)dx. \tag{7}$$

In the neighbourhood of the classical turning point the equation (2) can be written as

$$\frac{d^2\psi_t}{dr^2} - \gamma(r-a)\psi_t = 0, \tag{8}$$

where

$$\gamma = \left(\frac{dV}{dr}\right)_{r=a} = -\frac{\alpha}{a^2}+\beta. \tag{9}$$

The solution of (8) is of the type (6):

$$\psi_t = \mathrm{Ai}\,(\gamma^3(r-a)). \tag{10}$$

The values of ψ_t and ψ_t' at $r = a$ are

$$\psi_t(a) = \frac{\pi}{3^{2/3}\Gamma(\frac{2}{3})}, \qquad \psi_t'(a) = -\frac{\gamma 3^{1/6}\Gamma(\frac{2}{3})}{2}. \tag{11}$$

Therefore the solutions (6) and (10) are connected by the relation

$$\psi_t = \psi_a(r\gamma^{1/3}\beta^{-1/3}+E\beta^{-1}-\gamma^{1/3}\beta^{-1/3}a) \tag{12}$$

and both tend to 0 at infinity.

To obtain an equation for the eigenvalues E_n we consider the solutions of (5) satisfying the boundary conditions:

$$\varphi_0(0) = 0, \qquad \varphi_\infty(\infty) = 0. \tag{13}$$

Having in mind that the wave function ψ is continuous at every intermediate point r^* within the interval $0 < r < \infty$ we have

$$\varphi_0(E_n, r^*)-\varphi_\infty(E_n, r^*) = \pi(n+1). \tag{14}$$

When the intermediate point r^* coincides with the classical turning point the equation (14) for the eigenvalues should be written down as:

$$\varphi_0(a)+\mathrm{arc\ tg}\,(k\gamma^{-1/3}1.370) = \pi(n+1). \tag{15}$$

The phase function $\varphi_0(r)$, determined by a boundary condition at the origin can be approximated by iterations or by the variational procedure. In the next Section we shall consider the formal series method for approximating the phase function.

3. *Approximative solving of the phase-variable equation by the formal series method*

To solve equation (15) in the interval $[0, a]$ with the boundary condition $\varphi(0) = 0$, we obtain after integration

$$\varphi(r)+G[r, \varphi(r)] = \theta(r), \tag{16}$$

where

$$\theta(r) = kr, \qquad G[r, \varphi(r)] = \int_0^r \frac{V(r)}{k}\sin^2\varphi(r)dr. \tag{17}$$

The functional $G[r, \varphi(r)]$ can be represented as a Volterra series:

$$G[r, \varphi(r)] = \sum_{n\geq 2} \frac{1}{n!} \int \dots \int G_n(r; x_1, x_2, \dots, x_n)\varphi(x_1) \dots \varphi(x_n)dx_1 \dots dx_n. \tag{18}$$

The solution of equation (16) can be represented by the ratio of two power series of the coupling constant [5]–[7]:

$$\varphi(r) = \frac{P(k, r)}{Q(k)} = \frac{\sum\limits_{n=0}^{\infty} P_n}{\sum\limits_{n=0}^{\infty} Q_n}, \tag{19}$$

where

$$P(k, r) = \Gamma \exp\left\{-\int dy \frac{\delta G[y, \theta(y)]}{\delta\theta(y)}\right\}\theta(r) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^a \dots \int_0^a dx_1 \dots dx_n$$

$$\times \frac{\delta^n}{\delta\theta(x_1) \dots \delta\theta(x_n)} \int_0^{x_1} \dots \int_0^{x_n} dx'_1 \dots dx'_n L(x'_1, \theta(x'_1)) \dots L(x'_n, \theta(x'_n)), \tag{20}$$

$$Q(k) = \Gamma \exp\left\{-\int dy \frac{\delta G[y, \theta(y)]}{\delta\theta(y)}\right\}\cdot 1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^a \dots \int_0^a dx_1 dx_2 \dots dx_n$$

$$\times \frac{\delta^n}{\delta\theta(x_1) \dots \delta\theta(x_n)} \int_0^{x_1} \dots \int_0^{x_n} dx'_1 \dots dx'_n L(x'_1, \theta(x'_1)) \dots L(x'_n, \theta(x'_n)), \tag{21}$$

$$L(x, \theta(x)) = \frac{V(x)}{k} \sin^2 \theta(x) \tag{22}$$

and the symbol Γ indicates that all functional derivatives must be on the left, acting thus on all functionals which are put on their right.

The approximation obtained is valid for the strong coupling constants α and β. When the numerator and denominator of (19) are replaced by N-th degree polynomials instead of the corresponding infinite series of the coupling constant, the N-th diagonal Padé approximant type of approximation is obtained. We shall use this approximation by truncating the coupling constant at the first degree in both the numerator and denominator. It is an easy exercise to verify that

$$P_0(k, d) = ka, \tag{23}$$

$$P_1(k, a) = \frac{\alpha}{2k}(\ln ka - \mathrm{Ci}\,(2ka) + C) - \frac{\beta}{k^3}\left(\frac{k^2a^2}{4} - \frac{ka \sin 2ka}{4} - \cos 2ka + \tfrac{1}{8}\right)$$

$$+ \frac{\alpha a}{2}\left(\mathrm{si}\,(2ka) + \frac{\pi}{2}\right) - \frac{\beta a}{8k^3}(\sin 2ka - 2ka \cos 2ka), \tag{24}$$

$$Q_0 = 1, \tag{25}$$

$$Q_1 = \frac{1}{k}\left[\frac{\alpha}{2}\left(\mathrm{si}\,(2ka)+\frac{\pi}{2}\right)\right] - \frac{\beta}{8k^3}(\sin 2ka - 2ka\cos 2ka), \tag{26}$$

where

$$\mathrm{si}\,(x) = -\int_x^\infty \frac{\sin t}{t}\,dt, \qquad \mathrm{Ci}\,(x) = -\int_x^\infty \frac{\cos t}{t}\,dt, \tag{27}$$

and C is the Euler constant.

In the first approximation the equation for the eigenvalues E_n is:

$$\varphi_0^{[1,1]}(a) + \mathrm{arc\,tg}\,(k\gamma^{-1/3}\,1.370) = \pi(n+1), \tag{28}$$

where

$$\varphi_0^{[1,1]} = \frac{P_0+P_1}{Q_0+Q_1}. \tag{29}$$

As an example we shall discuss the case when $\alpha = 0$. Upon introducing dimensionless energy and distance parameters

$$\varepsilon = \beta^{-2/3}E, \qquad \varrho = \beta^{1/3}r \tag{30}$$

the numerator and denominator in this case can be represented as

$$P_0+P_1 = \varepsilon^{3/2} - \frac{1}{\varepsilon^{3/2}}\left(\frac{\varepsilon^3}{4} - \frac{\varepsilon^{3/2}\sin 2\varepsilon^{3/2}}{4} - \frac{\cos 2\varepsilon^{3/2}}{8} + \frac{1}{8}\right)$$

$$-\tfrac{1}{8}(\sin 2\varepsilon^{3/2} - 2\varepsilon^{3/2}\cos 2\varepsilon^{3/2}), \tag{31}$$

$$Q_0+Q_1 = 1 - \frac{1}{8\varepsilon^{3/2}}(\sin 2\varepsilon^{3/2} - 2\varepsilon^{3/2}\cos 2\varepsilon^{3/2}). \tag{32}$$

By solving the equation (28) numerically, we get $\varepsilon = 1{,}89$ while the first zero of the Airy function $\mathrm{Ai}(-\varepsilon)$ is $\varepsilon = 2{,}34$.

4. *Conclusion*

The energy eigenvalue problem associated with the confinement potential is related to the study of a first-order nonlinear differential equation for the phase shift. The formal series method has been used to solve the phase-variable equation obtained. This method seems to be well adaptable for computational purposes. In the near future we are planning to program it for the next order approximations and describe the charmed particle spectrum.

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