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INTRODUCTION TO THE QUANTIZATION OF GENERAL GAUGE THEORIES*

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The key ingredients of Dirac's constraint formalism are reviewed. Some important properties of general gauge theories are indicated. The canonical quantization of gauge theories, both within the operator approach and in terms of Feynman path integrals, is described. The most general setting for the covariant quantization procedure, given by Fradkin and Vilkovisky, is presented.

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Contents

1. *Introduction*

In these lectures I shall not talk about perturbation expansions, Feynman diagrams, renormalization etc. for gauge theories. There are several review articles and books which cover these aspects. Rather I shall talk about the formal quantization of general gauge theories. After all there are other gauge theories than Yang-Mills which are of physical interest like e.g. gravity, relativistic particles and strings. Now the most general mathematical characterization of a gauge theory is in terms of Dirac's constraint formalism [1–4] within the Hamiltonian formulation. There a gauge theory is a Hamiltonian system

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with first class constraints. Of course, such a formulation has geometrical consequences inherent in it which must be consistent with the physical interpretation. (In Ref. [5] an example of such an inconsistency is described.) However, these geometrical and physical interpretational aspects will not be discussed in these lectures.

In many gauge theories the fermionic degrees of freedom play an important role. The supersymmetric theories are e.g. represented by supergravity, supersymmetric Yang--Mills, spinning relativistic particles and strings. In these theories the fermionic degrees of freedom are classically represented by odd Grassmann variables. Such fermionic degrees are straight-forward to include within the general formalism provided they are of even number such that one may define a phase space as in the bosonic case [6], and provided that the leading terms in the constraints are at most linear in the odd Grassmann variables. If the latter properties do not hold the fermionic degrees introduce completely new properties in the quantization procedure.

The content of the lectures may shortly be described as follows: First the key ingredients of Dirac's constraint formalism are reviewed. Then some important properties of general gauge theories are mentioned. Since we want to relate Hamiltonian properties to Lagrangian ones, we find it convenient to define standard Lagrangians for general gauge theories. Next we turn to the canonical quantization of gauge theories both within the operator approach and in terms of Feynman path integrals. The transition to standard Lagrangian forms of the path integrals are also given. Then we consider the covariant quantization of gauge theories which played such a decisive role in the development of a consistent quantum Yang-Mills theory. The most general setting for the covariant quantization procedure has been given by Fradkin and Vilkovisky. We review their construction and show that the end result is equivalent to that of canonical quantization. Since the global BRS-invariance is the crucial ingredient of this approach, BRS-quantization is a more appropriate name than covariant quantization.

Throughout these lectures the general treatment will be in terms of systems with finite degrees of freedom. The generalization to infinite degrees of freedom (field theories) is straight-forward.

2. *Dirac's constraint formalism* [1–4]

Consider a general mechanical system with n degrees of freedom described by the generalized coordinates $q^i(t), i = 1, \ldots, n$. If the Lagrangian satisfies the condition

$$\text{Det}\,\frac{\partial^2 L}{\partial\dot{q}^i\partial\dot{q}^j} \neq 0 \tag{2.1}$$

we have the standard case in which the Euler-Lagrange equations may be given the form $\ddot{q}^i = f^i(q, \dot{q}),\ i = 1, \ldots, n$. The transition to the Hamiltonian formulation is obtained by replacing the velocities $\dot{q}^i$ by the conjugate momenta p_i defined by

$$p_i \equiv \frac{\partial L}{\partial\dot{q}^i}\,. \tag{2.2}$$

Due to (2.1) we may invert (2.2) to get $\dot{q}^s = F^s(q, p)$. The Hamiltonian is defined by the Legendre transformation

$$H \equiv p_s \dot{q}^s - L \tag{2.3}$$

which due to (2.2) is independent of $\dot{q}^s$ and whose explicit form in terms of q and p is obtained by inserting $\dot{q}^s = F^s(q, p)$ into (2.3).

Now a Lagrangian for a general gauge theory is a singular one, characterized by

$$\text{Det}\,\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} = 0 \tag{2.4}$$

and in this case some of the Euler-Lagrange equations will not involve $\ddot{q}^i$ but reduce to constraint equations among q's and $\dot{q}$'s. In fact if R is the rank of the matrix $\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}$ ($R < n$ due to (2.4)), then we have $n-R$ constraint equations. If we try to make the transition to the Hamiltonian formulation, we find from (2.2) that only R velocities may be expressed in terms of q's, p's and the remaining $n-R$ velocities. Since only Rp's are independent functions of the $\dot{q}$'s, we also find $n-R$ independent relations between the p's and the q's

$$\phi^a(q, p) = 0, \qquad a = 1, \ldots r \equiv (n-R). \tag{2.5}$$

These constraints are called primary constraints and follow only from the form of the Lagrangian. The Hamiltonian as defined by (2.3) is now not uniquely defined on the phase space spanned by the q's and the p's. The appropriate Hamiltonian on the whole phase space is called the total Hamiltonian and is given by

$$H_{\text{tot}} = H + v_a \phi^a \tag{2.6}$$

where v_a are arbitrary functions. The equations of motion may now be cast in a standard form

$$\begin{aligned} \dot{q}^i &= \{q^i, H_{\text{tot}}\}|_{\text{p.c.}} \\ \dot{p}_i &= \{p_i, H_{\text{tot}}\}|_{\text{p.c.}} \end{aligned} \tag{2.7}$$

where the right-hand side is expressed in terms of the standard Poisson bracket (PB)

$$\{q^i, q^j\} = \{p_i, p_j\} = 0, \qquad \{q^i, p_j\} = \delta^i_j. \tag{2.8}$$

Notice that the constraints (2.5) are not satisfied inside this PB. Therefore, it makes a difference whether one uses H or H_{tot} in (2.7). Thus, there is the important calculational rule in (2.7) that the constraints (2.5) are only to be imposed after calculating the PB (which we have indicated by $|_{\text{p.c.}}$). The equations of motion (2.7) are then identical to the original Euler-Lagrange equations and the arbitrary functions v_a are related to the velocities which were not possible to solve out from (2.2).

Now in order for the theory to be consistent all constraints must hold for all times. One has to require the consistency conditions

$$\dot{\phi}^a = \{\phi^a, H_{\text{tot}}\}|_{\text{p.c.}} = 0. \tag{2.9}$$

These equations may lead to that some functions v_a will be determined and/or that new constraints are generated. In bad cases one may obtain completely inconsistent equations.

When new constraints called secondary constraints are generated, one has to repeat the consistency condition (2.9) also for them. Finally one ends up with, let say, m linearly independent constraints $\Phi^a = 0$, $a = 1, \dots m$, which satisfy

$$\dot{\Phi}^a = \{\Phi^a, H_{\text{tot}}\} = c^{ab}\Phi^b. \tag{2.10}$$

The classification into primary and secondary constraints are only with respect to the particular Lagrangian chosen to describe the system. Within the Hamiltonian formalism the important classification is in terms of first and second class constraints [1]. A first class constraint $\psi = 0$ satisfies

$$\{\psi, \Phi^a\} = \lambda^{ab}\Phi^b, \quad a = 1, \dots m \tag{2.11}$$

while a second class one does not. The complete set of constraints $\Phi^a = 0$ may thus be uniquely decomposed into first and second class ones.

$$\Phi^a = 0, \quad a = 1, \dots m \Leftrightarrow \begin{cases} \phi^a = 0, & a = 1, \dots k \\ \psi^r = 0, & r = 1, \dots m-k. \end{cases}$$

Dirac has shown [2] that second class constraints $\phi^a = 0$ satisfy

$$\text{Det}\,\{\phi^a, \phi^b\} \neq 0 \tag{2.12}$$

while first class ones satisfy

$$\{\psi_r, \psi_s\} = c_{rs}{}^t\psi_t. \tag{2.13}$$

The second class constraints may be eliminated by means of the Dirac bracket (DB) technique. The expression [1]

$$\{A, B\}^* = \{A, B\} - \{A, \phi^a\}\Delta^{ab}\{\phi^b, B\} \tag{2.14}$$

where $\Delta^{ab}\{\phi^b, \phi^c\} = \delta^{ac}$, may be shown to be a Poisson bracket inside which $\phi^a = 0$ holds strongly. We may now replace the original PB (2.8) by the DB (2.14) and impose $\phi^a = 0$. The resulting theory will then be defined on a new phase space, the phase space of the DB (2.14), and it will only contain first class constraints. Notice that

$$\begin{aligned} \{\psi_r, \psi_s\}^*|_{\phi=0} &= c'_{rs}{}^t\psi_t \\ \{\psi_r, H_{\text{tot}}\}^*|_{\phi=0} &= c'_r{}^s\psi_s. \end{aligned} \tag{2.15}$$

The treatment of such a theory will be our main concern in what follows.

3. *Properties of general gauge theories* [4]

Consider a Hamiltonian system defined on a phase space Γ of dimension $2n$ with only first class constraints. It is characterized by

$$\begin{aligned} \{\psi_r, \psi_s\} &= c_{rs}{}^t\psi_t \\ \{\psi_r, H_{\text{tot}}\} &= d_r{}^s\psi_s \\ \psi_r = 0, &\quad r = 1, \dots m < n \end{aligned} \tag{3.1}$$

for some functions d_{rs} and c_{rst}.

Now these properties define a general gauge theory. The constraints ψ_r generate the gauge group and their closed PB algebra in (3.1) is the gauge algebra which is a Lie algebra when c_{rst} are constants. $F(\alpha) = \alpha_r\psi_r$ where α_r are infinitesimal parameters generate infinitesimal gauge transformations which also are infinitesimal canonical transformations.

The first class constraints $\psi_r = 0$ trace out a hypersurface M in Γ of dimension $2n-m$. We seek now a PB defined on M inside which $\psi_r = 0$ holds strongly. Such a PB is obtained by means of gauge invariant functions. A function f on Γ is gauge invariant when it satisfies

$$\{f, \psi_r\} = h_r^{\ s}\psi_s, \ r = 1, \dots m \tag{3.2}$$

and gauge invariant functions satisfy a closed PB algebra since

$$\begin{aligned}\{\{f, g\}, \psi_r\} &= -\{\{g, \psi_r\}, f\} \\ +\{\{f, \psi_r\}, g\} &= d_r^{\ s}\psi_s, \ r = 1, \dots m.\end{aligned} \tag{3.3}$$

Hence, for gauge invariant functions f and g we may define a consistent PB on M as follows

$$\{f|_M, g|_M\}' \equiv \{f, g\}|_M. \tag{3.4}$$

From this definition it follows

$$\{f|_M, \psi_r\}' = \{f, \psi_r\}|_M = 0. \tag{3.5}$$

Now we may always find m variables on Γ, χ^r, $r = 1, \dots m$, such that

$$\mathrm{Det}\,\{\chi^s, \psi_r\} \neq 0. \tag{3.6}$$

However, from (3.5) we have

$$\left.\frac{\partial f}{\partial \chi^s}\right|_M = 0, \tag{3.7}$$

i.e. gauge invariant functions are independent of such variables. We may therefore impose the additional constraints

$$\chi^r = 0, \qquad r = 1, \dots m \tag{3.8}$$

called gauge conditions without affecting anything. Now the first class constraints together with the gauge conditions (3.8) form a set of second class constraints $\phi_a = (\psi_r, \chi^s)$ since

$$\begin{aligned}\mathrm{Det}\,\{\phi_a, \phi_b\} &= \mathrm{Det}\begin{pmatrix}\{\psi_r, \psi_s\} & \{\psi_r, \chi^s\} \\ \{\chi^r, \psi_s\} & \{\chi^r, \chi^s\}\end{pmatrix} \\ &= (\mathrm{Det}\,\{\psi_r, \chi^s\})^2 + c_r\psi_r \neq 0.\end{aligned} \tag{3.9}$$

We may therefore construct the Dirac bracket (2.14), and we find on M

$$\{A, B\}^*|_M = \{A, B\}|_M - \{A, \psi_r\}M^r{}_s\{\chi^s, B\}|_M + \{A, \chi^r\}M_r{}^s\{\psi_s, B\}|_M, \tag{3.10}$$

where $\{\chi^t, \psi_r\}M^r_s = \delta_{ts}$. For gauge invariant functions f and g we obviously have

$$\{f, g\}^*|_M = \{f, g\}|_M. \tag{3.11}$$

The definitions (3.4) and (3.10) are therefore equivalent.

We conclude that the PB (3.4) is defined on a phase space Γ^* of dimension $2n-2m$. From Darboux' theorem Γ^* is, at least locally, described in terms of conventional canonical coordinates q^{*s} and p^*_s [7]. From the equivalence of the two definitions (3.4) and (3.10) it follows that there always exist gauge invariant functions $q^{*s}(g, p)$ and $p^*_s(g, p)$ such that

$$\begin{aligned} q^{*s}(q, p)|_M &= q^{*s} \\ p^*_s(q, p)|_M &= p^*_s. \end{aligned} \tag{3.12}$$

The left-hand sides are called gauge invariant extensions of the right-hand sides [8]. The physical variables are certain specific projections from M. In fact gauge transformations trace out so called fibres in M and gauge invariant functions are constant along these fibres. When the gauge group is a Lie group we have the following general formulas for gauge invariant extensions [9].

$$\begin{aligned} q^i_{(\chi)} &= \int d\Omega |\mathrm{Det}\,\{\chi^r(\Omega), \psi_s(\Omega)\}|\delta^m(\chi(\Omega))q^i(\Omega), \\ p_{i(\chi)} &= \int d\Omega |\mathrm{Det}\,\{\chi^r(\Omega), \psi_s(\Omega)\}|\delta^m(\chi(\Omega))p_i(\Omega), \end{aligned} \tag{3.13}$$

where $q^i(\Omega)$ and $p_i(\Omega)$ represent finite gauge transformations parameterized by the group space coordinates. Formally we have the relation

$$\Gamma^* = M/\Omega, \tag{3.14}$$

where Ω is the group space.

In order to find canonical coordinates q^{*s}, p^{*s} in Γ^* the following tricks are useful: *i*) When the gauge algebra is abelian i.e. when $\{\psi_r, \psi_s\} = 0$ then we may perform the canonical transformation

$$(q^i, p_i) \to (q^{*s}, p^*_s, P_r, Q^r), \tag{3.15}$$

where $P_r \equiv \psi_r$, and eliminate P_r and Q^r trivially since the gauge conditions may be written as $Q^r = f^r(g^*, p^*)$.

ii) In the general case we should choose the gauge conditions $\chi^r = 0$ such that $\{\chi^r, \chi^s\} = 0$, because then it is still possible to perform the transformation (3.15) now with $\chi^r = Q^r$. $\psi_r = 0$ may then be written as $P_r = g_r(g^*, p^*)$. (χ^r should be chosen such that ψ_r is as linear as possible in P_r in order to avoid multiple solutions.)

4. *Standard Lagrangians for general gauge theories*

We have defined a consistent Hamiltonian system with first class constraints (3.1) to be a general gauge theory. Now there are usually several different Lagrangians which yield one and the same Hamiltonian theory. We shall therefore define a standard Lagrangian for a gauge theory given by a particular Hamiltonian system. (In fact, as will be shown later, it is exactly this Lagrangian which is to be used in Feynman's path integral quantization.) First we shall assume that the Hamiltonian system is minimal in the sense that it does not contain any trivial constraints. Trivial constraints we define to be constraints of the form $P_i = 0$. They generate translations in unobservable coordinates and are trivially eliminated. The system is then characterized by

$$\{\psi_r, \psi_s\} = c_{rs}{}^t\psi_t$$

$$\{\psi_r, H_0\} = c_r{}^s\psi_s$$

$$\psi_r = 0, \quad r = 1, \dots m < n \tag{4.1}$$

where the PB is defined on a phase space Γ of dimension $2n$. The Hamiltonian is given by

$$H = H_0 + v^r\psi_r \tag{4.2}$$

where v^r are arbitrary functions. The equations of motion are

$$\dot{q}^i = \{q^i, H\} = \{q^i, H_0\} + v^r\{q^i, \psi_r\} \tag{4.3}$$

and $\dot{p}_i = \{p_i, H\}$. Now these equations together with the constraints (4.1) are derivable from the phase space Lagrangian

$$L = p_i\dot{q}^i - H_0 - v^r\psi_r, \tag{4.4}$$

where v^r are Lagrange multipliers to be treated as dynamical variables. However, making a transition to the Hamiltonian formalism we now also find the primary constraints

$$P_r = 0, \tag{4.5}$$

where P_r are the canonical conjugates to v_r. We have thus the total Hamiltonian

$$H_{\text{tot}} = H + \bar{c}_r P_r \tag{4.6}$$

and the dynamics expressed by

$$\dot{A} = \{A, H_{\text{tot}}\}|_{P_r=0}. \tag{4.7}$$

In particular we have

$$\dot{P}_r = -\psi_r. \tag{4.8}$$

Hence, consistency requires $\psi_r = 0$, i.e. the original constraints appear as secondary constraints. The original Hamiltonian system is obtained by eliminating the trivial primary constraints.

So far we have only a Lagrangian in phase space. However, if we require that (4.3) should be possible to rewrite as

$$p_i = f_i(q, \dot{q}, v) \tag{4.9}$$

then we may cast the Lagrangian (4.4) into the form $L = L(q, \dot{q}, v)$. This prescription, which may impose restrictions on v_r, should work for all gauge theories which allow for a Lagrangian description in configuration space.

What are the gauge symmetries of the Lagrangian $L(q, \dot{q}, v)$? The answer may be found in the following fashion: First we consider a generator of a general infinitesimal gauge transformation

$$G = \alpha^r P_r + \beta^r \psi_r, \tag{4.10}$$

where α_r and β_r are infinitesimal parameter functions of t. Then we require this generator to be conserved

$$\dot{G} \equiv \dot{G}|_{P_r} = 0 \tag{4.11}$$

which yields the solution

$$G(\beta) = (\dot{\beta}^r + \beta^s c_s{}^r + \beta^t v^s c_{ts}{}^r) P_r + \beta^r \psi_r \tag{4.12}$$

since

$$\dot{\psi}_r = c_r{}^s \psi_s + v^s c_{rs}{}^t \psi_t . \tag{4.13}$$

By means of the Jacobi identities involving ψ_r, ψ_s, ψ_t and ψ_r, ψ_s, H_0 one may show that the generator (4.12) satisfies the algebra

$$\{G(\alpha), G(\beta)\} = G(\gamma) \tag{4.14}$$

where $\gamma_t = \alpha^r \beta^s c_{rst}$. Now using the inverse Noether theorem it follows that transformations generated by (4.12) leave the action $\int dt L(q, \dot{q}, v)$ invariant. Furthermore, since the generator of infinitesimal gauge transformations in the original Hamiltonian formulation, i.e. $F(\alpha) = \alpha^r \psi_r$, also satisfies the algebra (4.14), we conclude that the gauge invariance of $\int dtL$ is the same as that of the original Hamiltonian system. However, now this invariance holds "off-shell", i.e. not just on the level of the equations of motion where the Hamiltonian formulation is given.

Ex. 1. A free relativistic particle

may be described by the Hamiltonian system

$$H_0 \equiv 0, \qquad \psi = p_\mu p^\mu - m^2. \tag{4.15}$$

The phase space Lagrangian (4.4) becomes here

$$L = p_\mu \dot{x}^\mu - v(p_\mu p^\mu - m^2). \tag{4.16}$$

From the equations

$$\dot{x}^\mu = \{x^\mu, v(p_\mu p^\mu - m^2)\} = 2vp^\mu \tag{4.17}$$

we find

$$p_\mu = \frac{1}{2v} \dot{x}_\mu \tag{4.18}$$

which when inserted into (4.16) yields

$$L = \frac{1}{2V}\dot{x}^2 + \tfrac{1}{2}Vm^2 \tag{4.19}$$

where $V = 2v$. This is the so called einbein Lagrangian for the free relativistic particle [10]. V is the einbein variable. When the solution of the equation for V is inserted back into (4.19) one obtains the usual Lagrangian $L = m\sqrt{\dot{x}^2}$.

Ex. 2. A free relativistic string

The relativisitic string is usually given in terms of the Nambu-Goto action

$$A = \int_{-\infty}^{\infty} d\tau \int_{0}^{\pi} d\sigma \mathscr{L}(\tau, \sigma)$$

$$\mathscr{L}(\tau, \sigma) = -\frac{1}{2\pi\alpha'}\sqrt{(\dot{x}\cdot x')^2 - (x')^2(\dot{x})^2}. \tag{4.20}$$

In the Hamiltonian formulation one finds the following two primary constraints (and $\mathscr{H}_0 = 0$)

$$\psi_1 \equiv \mathscr{P}^2 + \frac{(x')^2}{(2\pi\alpha')^2} = 0, \qquad \psi_2 \equiv \mathscr{P}_\mu x'^\mu = 0, \tag{4.21}$$

which are first class. The standard Lagrangian density in phase space is

$$\mathscr{L}(\tau, \sigma) = \mathscr{P}_\mu \dot{x}^\mu - \lambda_1\psi_1 - \lambda_2\psi_2. \tag{4.22}$$

Varying $\mathscr{P}_\mu$ one finds (requiring $\lambda_1 \neq 0$)

$$\mathscr{P}_\mu = \frac{1}{2\lambda_1}(\dot{x}_\mu - \lambda_2 x'_\mu) \tag{4.23}$$

which when inserted into (4.22) yields

$$\mathscr{L}(\tau, \sigma) = \frac{1}{4\lambda_1}\dot{x}^2 - \frac{\lambda_2}{2\lambda_1}\dot{x}\cdot x' + \left(\frac{\lambda_2^2}{4\lambda_1} - \frac{\lambda_1}{(2\pi\alpha')^2}\right)(x')^2. \tag{4.24}$$

This Lagrangian is equal to [11]

$$\mathscr{L}(\tau, \sigma) = -\frac{1}{4\pi\alpha'}\sqrt{-\mathrm{Det}\, g_{\alpha\beta}}\; g^{\gamma\delta}\partial_\gamma x^\mu \partial_\delta x_\mu, \tag{4.25}$$

where $\alpha, \beta, \gamma, \delta = 1, 2$ and $\partial_1 = \partial_\tau$, $\partial_2 = \partial_\sigma$. Superficially the last expression contains three Lagrange multipliers $g^{\alpha\beta}(= g^{\beta\alpha})$. However, due to the conformal invariance $g^{\gamma\delta} \to \varrho g^{\gamma\delta}$ it depends only on two, as is made manifest in the form (4.24). Varying the Lagrange multipliers and solving the resulting equations and inserting back the solutions into the Lagrangian (4.24) or (4.25) the Nambu-Goto expression (4.20) is obtained.

Ex. 3. A pure Yang-Mills theory

with a compact semi-simple gauge group is given by the Lagrangian density

$$\mathscr{L}(x) = -\tfrac{1}{4} F_{a\mu\nu}(x) F_a^{\mu\nu}(x) \tag{4.26}$$

where

$$F_{a\mu\nu}(x) \equiv \partial_\mu A_{a\nu} - \partial_\nu A_{a\mu} + g c_{abc} A_{b\mu} A_{c\nu}. \tag{4.27}$$

c_{abc} are the totally antisymmetric structure constants of the gauge group. The conjugate momenta to $A_{a\mu}$ are

$$E_a^\mu(x) \equiv \frac{\partial \mathscr{L}}{\partial \dot{A}_{a\mu}} = F_a^{\mu 0}(x) \tag{4.28}$$

from which we find the primary constraints

$$E_a^0(x) = 0. \tag{4.29}$$

Since the canonical Hamiltonian density is

$$\mathscr{H}_c(x) = \mathscr{H}_0(x) - g A_a^0(x) G_a(x) \tag{4.30}$$

where

$$\mathscr{H}_0(x) = -\tfrac{1}{2} E_a^i(x) E_{ai}(x) + \tfrac{1}{4} F_a^{ij}(x) F_{aij}(x)$$

$$G_a(x) - \frac{1}{g} D_i E_a^i \equiv \frac{1}{g} \partial_i E_a^i(x) - c_{abc} E_b^i(x) A_{ci}(x) \tag{4.31}$$

the consistency conditions $\dot{E}_a^0 = 0$ requires the secondary constraints

$$G_a(x) = 0. \tag{4.32}$$

These constraints satisfy the local Lie algebra of the gauge group

$$\{G_a(x), G_b(y)\} = \delta^3(x-y) c_{abc} G_c(x). \tag{4.33}$$

Since the primary constraints are trivial, E_a^0 and A_a^0 are trivially eliminated leaving the Hamiltonian theory described by the Hamiltonian density (4.31) and the constraints (4.32). The Lagrangian (4.26) is obviously in the standard form.

5. *Canonical quantization of gauge theories*

In the operator version of canonical quantization one turns q^i and p_i into operators $\hat{q}^i$ and $\hat{p}_i$ (carets will denote the corresponding quantum operators in what follows), satisfying

$$[\hat{q}^i, \hat{p}_j]_- = -i\hbar \delta^i_j \tag{5.1}$$

and the corresponding eigenstates

$$\hat{q}^i|q\rangle = q^i|q\rangle,\ \hat{p}_i|p\rangle = p_i|p\rangle \tag{5.2}$$

which we assume normalized according to

$$\langle q|q'\rangle = \delta^n(q-q'),\ \langle p|p'\rangle = \delta^n(p-p'),$$
$$\langle q|p\rangle = (2\pi\hbar)^{-n/2}e^{\frac{ip\cdot q}{\hbar}},$$
$$\int d^nq|q\rangle\langle q| = 1,\ \int d^np\ |p\rangle\langle p| = 1. \tag{5.3}$$

This transition may be performed at any time instant. States at arbitrary times are obtained by means of unitary transformations generated by the Hamiltonian:

$$|q, t\rangle = e^{\frac{it\hat{H}}{\hbar}}\ |q\rangle, \tag{5.4}$$

where $|q\rangle$ is assumed to be an eigenstate at $t = 0$. The Green function for the wave function $\psi(q, t) \equiv \langle q, t\,|\psi\rangle$ is thus

$$\langle q', t'\,|q, t\rangle = \langle q'|\,e^{i\frac{(t-t')}{\hbar}\hat{H}}\ |q\rangle. \tag{5.5}$$

By means of the completeness relations (5.3) we may subdivide the time interval $t'-t$ into infinitely many pieces

$$\langle q', t'|q, t\rangle = \lim_{N\to\infty}\prod_{m=1}^{N-1}\int d^nq_m\prod_{k=0}^{N-1}\langle q_{k+1}, t_{k+1}|q_k,t_k\rangle$$

$$= \lim_{N\to\infty}\prod_{m=1}^{N-1}\int d^nq_m\prod_{k=0}^{N-1}\int\frac{d^np_k}{(2\pi\hbar)^n}\exp\left(\frac{i}{\hbar}p_k\cdot\Delta q_k-\frac{i}{\hbar}\Delta tH(p_k, \bar{q}_k)\right)$$

$$= \int_{\text{Path}}\prod_t dq\,\frac{dp}{2\pi\hbar}\exp\left\{\frac{i}{\hbar}\int_t^{t'}dt(p\dot{q}-H(p, q))\right\}, \tag{5.6}$$

where $q_0 = q$, $t_0 = t$, $q_\infty = q'$, $t_\infty = t'$, $\Delta q_k = q_{k+1}-q_k$ and $\bar{q}_k = \frac{1}{2}(q_{k+1}+q_k)$. The right-hand side is the phase space version of the Feynman path integral [12]. In what follows we shall assume that $H(p, q)$ is the classical Hamiltonian and that we may perform Gaussian integrations inside the path integral although there are examples where the above definition does not allow this [13]. (This difference is only formal. Renormalization by means of different regularization procedures restores the equivalence [14].) Thus, if $H(p, q)$ is at most quadratic in p we may integrate over p to obtain the standard configuration space version of the path integral

$$\langle q', t'|q, t\rangle = \int_{\text{Path}}\prod_t dq\,\frac{1}{N(q)}\exp\left\{\frac{i}{\hbar}\int_t^{t'}dtL(q, \dot{q})\right\} \tag{5.7}$$

where $N(q)$ is a normalization coefficient. For instance, if $H(q, p) = \frac{1}{2} p \cdot A \cdot p$ where A is a matrix, possibly q-dependent, then

$$\prod_t dq \int \prod_t \frac{dp}{2\pi\hbar} \exp\left\{\frac{i}{\hbar} \int_t^{t'} dt (p\dot{q} - \tfrac{1}{2} p \cdot A \cdot p)\right\}$$

$$= \prod_t \frac{dq}{\sqrt{2\pi i\hbar}} \frac{1}{\sqrt{\operatorname{Det} A}} \exp\left\{\frac{i}{\hbar} \int_t^{t'} dt \, \tfrac{1}{2} \dot{q} \cdot A^{-1} \cdot \dot{q}\right\}$$

$$= \prod_t \frac{dq}{\sqrt{2\pi i\hbar}} \exp\left\{\frac{i}{\hbar} \int_t^{t'} dt (\tfrac{1}{2} \dot{q} \cdot A^{-1} \cdot \dot{q} - \tfrac{1}{2} i\hbar \delta(0) \ln \operatorname{Det} A)\right\}. \tag{5.8}$$

This is the standard quantization procedure of a regular classical theory. However, exactly the same procedure is applicable even in the case of a general gauge theory, namely if one first reduce the original phase space to the physical one by means of the Dirac bracket technique. We have

$$\langle q^{*\prime}, t' | q^*, t \rangle = \int_{\text{Path}} \prod_t \frac{dq^* dp^*}{2\pi\hbar} \exp\left\{\frac{i}{\hbar} \int_t^{t'} dt (p^* \cdot \dot{q}^* - H^*(p^*, q^*))\right\}. \tag{5.9}$$

The right-hand side may be rewritten as

$$\langle q^{*\prime}, t' | q^*, t \rangle = \int_{\text{Path}} \prod_t \frac{d^k q^* d^k p^*}{(2\pi\hbar)^k} d^m Q d^m P \delta^m(Q)$$

$$\times \delta^m(P - f(q^*, p^*)) \exp\left\{\frac{i}{\hbar} \int_t^{t'} dt (p^* \dot{q}^* - H^*)\right\}$$

$$= \int_{\text{Path}} \prod_t \frac{d^k q^* d^k p^*}{(2\pi\hbar)^k} d^m Q d^m P \delta^m(Q) \delta^m(\psi)$$

$$\times |\operatorname{Det} \{Q^r, \psi_s\}| \exp\left\{\frac{i}{\hbar} \int_t^{t'} dt (p^* \dot{q}^* + P\dot{Q} - H_0)\right\}, \tag{5.10}$$

where $Q^r \equiv \chi^r = 0$ are the gauge conditions which in this construction is required to satisfy $\{\chi^r, \chi^s\} = 0$. Formally one may now perform a canonical transformation on the right-hand side obtaining Faddeev's expression

$$\int_{\text{Path}} \prod_t \frac{d^n q d^n p}{(2\pi\hbar)^{n-m}} \delta^m(\chi) \delta^m(\psi) |\operatorname{Det} \{\psi_r, \chi^s\}| \exp\left\{\frac{i}{\hbar} \int_t^{t'} d\tau (p\dot{q} - H_0)\right\}. \tag{5.11}$$

A gauge transformation changes only the form of the gauge conditions. However, since $\chi \to \chi'$ is a canonical transformation, $\{\chi^r, \chi^s\} = 0$ implies $\{\chi'^r, \chi'^s\} = 0$. Even when $\{\chi^r, \chi^s\} = 0$ (5.11) does not always reduce to (5.10), since this also requires χ to be chosen linear in those components of q (or p) which are to be eliminated. Anyhow, if this is assumed one may derive (5.11) from the operator quantization as in (5.6) in the following fashion: First one projects out physical states by the conditions [1]

$$\hat{\psi}_r(p, q)\,|\text{phys}\rangle = 0, r = 1, \ldots, m, \tag{5.12}$$

where $\hat{\psi}_r$ are the operator form of the first class constraints. They are required to satisfy the commutator algebra

$$[\hat{\psi}_r, \hat{\psi}_s]_- = C_{rs}{}^t\hat{\psi}_t \tag{5.13}$$

which is a non-trivial requirement if C_{rst} are q-numbers. Then one construct the gauge invariant extension of q^*, i.e. $q^*(q, p)$ and requires

$$\hat{q}^*(q, p)|q^*, \text{phys}\rangle = q^*|q^*, \text{phys}\rangle. \tag{5.14}$$

This together with (5.12) yield the solution

$$|q^*, \text{phys}\rangle = \int d^nqf(q^*, q)\,|q\rangle. \tag{5.15}$$

The function $f(q^*, q)$ is then normalized by the condition that the resulting states should satisfy the standard normalization (5.3). Finally the Green function is

$$\langle q^{*\prime}, t'|q^*, t\rangle = \langle q^{*\prime}, \text{phys}\,|e^{\frac{i}{}\frac{(t-t')}{\hbar}\hat{H}_0}|q^*, \text{phys}\rangle. \tag{5.16}$$

This formula only holds when the constraints have no explicit time dependence (cf. the relativistic particle). The expression (5.11) is now obtained by cutting the interval $t-t'$ in infinitely many pieces as in (5.6).

This operator quantization is possible to perform even if $\{\chi_r, \chi_s\} \neq 0$ in which case the expression (5.11) is not obtained (cf. the spinning particle in the time-like gauge [15]).

We now turn to some formal equalities satisfied by the path integral formula (5.11). Obviously it may be rewritten as [4]

$$\int_{\text{Path}}\prod_t \frac{d^nqd^npd^mv}{(2\pi\hbar)^n}\,\delta^m(x)|\text{Det}\,\{\chi^r, \psi_s\}|\,\exp\left\{\frac{i}{\hbar}\int_t^{t'} dt(p\dot{q}-H_0-v^r\psi_r)\right\} \tag{5.17}$$

which is a Lagrangian phase space version of the Hamiltonian one (5.11). Again if H_0 and ψ_r are at most quadratic in p we may perform the integration over p obtaining

$$\int_{\text{Path}}\prod_t d^nqd^mv\,\frac{1}{N(q, v)}\,(\delta^m(\chi)|\text{Det}\,\{\chi_r, \psi_s\}|)_{p\equiv\frac{\partial L}{\partial\dot{q}}}\exp\left\{\frac{i}{\hbar}\int_t^{t'} dtL(q, \dot{q}, v)\right\}. \tag{5.18}$$

Notice that

$$\int\prod_t dpf(p)\exp\left\{\frac{i}{\hbar}\int_t^{t'} dt(p\dot{q}-H)\right\} = f\left(-i\hbar\frac{\delta}{\delta\dot{q}}\right)\int\prod_t dp\exp\left\{\frac{i}{\hbar}\int_t^{t'} dt(p\dot{q}-H)\right\}$$

$$= f\left(-i\hbar\frac{\delta}{\delta\dot{q}}\right)\frac{1}{N(q)}\exp\left\{\frac{i}{\hbar}\int_t^{t'} dtL(q,\dot{q})\right\} = f\left(\frac{\partial L(q,\dot{q})}{\partial\dot{q}}\right)\frac{1}{N(q)}\exp\left\{\frac{i}{\hbar}\int_t^{t'} dtL(q,\dot{q})\right\}. \tag{5.19}$$

Thus, the expression (5.18) is obtained from (5.17) by eliminating $\prod_t d^n p$ and replacing p_i by $\partial L/\partial\dot{q}^i$ everywhere and inserting the normalization factor $1/N(q, v)$.

The expression (5.17) may be obtained from the naive expression in the following formal way when the gauge group is a Lie group

$$\int_{\text{Path}}\prod_t \frac{d^n q d^n p d^m v}{(2\pi\hbar)^n}\exp\left\{\frac{i}{\hbar}\int_t^{t'} dt(p\cdot\dot{q}-H_0-v^r\psi_r)\right\}$$

$$= \int_{\text{Path}}\prod_t \frac{d^n q d^n p d^m v}{(2\pi\hbar)^n} d^m\Omega\delta^m(\chi(\Omega))|\text{Det}\{\chi_r(\Omega),\psi_s(\Omega)\}|$$

$$\times\exp\left\{\frac{i}{\hbar}\int_t^{t'} dt(p\cdot\dot{q}-H_0-v^r\psi_r)\right\} = V_\Omega\times(5.17), \tag{5.20}$$

where $d^m\Omega$ is the group measure and $\chi^r(\Omega)$, $\psi_s(\Omega)$ are the gauge transformed variables. $V_\Omega \equiv \int\prod_t d^m\Omega$. The first equality follows from the fact that [9]

$$\int d^m\Omega\delta^m(\chi(\Omega))|\text{Det}\{\chi_r(\Omega),\psi_s(\Omega)\}| = 1 \tag{5.21}$$

and the last equality follows since the naive expression is gauge invariant under gauge transformations generated by (4.12). (By adding $\int\prod_t d^m P\delta^m(P) = 1$ the measure becomes canonically invariant and since P_r is strictly invariant the statement follows.) Eq. (5.20) is a particular form of the Faddeev-Popov trick.

Ex. Quantization of the free relativistic particle [15]

Here we turn x^μ and p_μ into operators $\hat{x}^\mu$, $\hat{p}_\mu$ satisfying

$$[\hat{x}^\mu, \hat{p}_\nu]_- = -i\hbar\delta^\mu_\nu \tag{5.22}$$

and corresponding eigenstates

$$\hat{x}^\mu|x\rangle = x^\mu|x\rangle, \quad \hat{p}_\mu|p\rangle = p_\mu|p\rangle$$

$$\int d^4x|x\rangle\langle x| = 1, \quad \int d^4p|p\rangle\langle p| = 1. \tag{5.23}$$

Physical states are projected out by the condition ($\psi \equiv \frac{1}{2}(p^2-m^2) = 0$)

$$(\hat{p}_\mu \hat{p}^\mu - m^2)\,|\text{phys}\rangle = 0. \tag{5.24}$$

The gauge choice

$$\chi \equiv x^0 - t = 0 \tag{5.25}$$

leave x^i and p_j as the canonical variables that span the physical phase space. The gauge invariant extension of x^i is obtained by means of the formula (3.13)

$$\begin{aligned} &\int d\theta |\text{Det}\,\{\chi_\theta, \psi_\theta\}|\delta(\chi_\theta) x^i_\theta \\ &= \int d\theta |P_0| \delta(x^0 + \theta P^0 - t)\,(x^i + \theta P^i) \\ &= \frac{J^{i0}}{P^0} + \frac{P^i}{P^0}\, t \equiv x^i(x, p, t) \end{aligned} \tag{5.26}$$

which is gauge invariant since the Poincaré generators are gauge invariant. Imposing the further condition

$$\hat{x}^i(x, p, t)|\vec{x}, t, \text{phys}\rangle = x^i|\vec{x}, t, \text{phys}\rangle \tag{5.27}$$

and the normalization $\langle \vec{x}|\vec{x}'\rangle = \delta^3(x-x')$ we find the solution [15] ($\hbar = 1$)

$$\begin{aligned} |\vec{x}, t, \text{phys}\rangle &= (2\pi)^{-3/2}(\delta(0))^{-1/2} \\ &\times \int dx^0 \delta(x^0 - t) \int d^4p\,\delta(\tfrac{1}{2}(P^2 - m^2))\,(|P_0|)^{1/2} \\ &\times \exp(ip_\mu x^\mu)|p\rangle, \end{aligned} \tag{5.28}$$

where $\delta(0)$ is defined by $(\delta(p^2-m^2))^2 = \delta(0)\delta(p^2-m^2)$ and is connected to the infinite group volume. Obviously

$$\begin{aligned} &\langle \vec{x}', t', \text{phys}|\vec{x}, t, \text{phys}\rangle \\ &= (2\pi)^{-3} \int d^4p |P^0| \delta(\tfrac{1}{2}(P^2 - m^2)) \exp(-ip_\mu(x'^\mu - x^\mu)) \end{aligned} \tag{5.29}$$

where $x^0 = t,\ \ x'^0 = t'$.

The gauge condition (5.25) requires the first class constraint to be solved in such a fashion that P_0 becomes a dependent variable. However, since ψ is quadratic in p there are two solutions ($P_0 > 0$ and $P_0 < 0$) and hence two disconnected physical phase spaces which have to be treated separately in a standard quantization (5.9). The above method on the other hand takes into account both physical solutions since (5.28) and (5.19) are linear combinations of both physical states and Green functions. Cutting the time interval in infinitely many pieces as in (5.6) we obtain the path integral expression

$$\langle \vec{x}', t'|\vec{x}, t\rangle = \int\limits_{\text{Path}} \prod_t \frac{d^4x d^4p}{(2\pi\hbar)^3}\, \delta(x^0 - \tau)|P^0|\delta(\tfrac{1}{2}(P^2 - m^2)) \exp\left\{\frac{i}{\hbar}\int_t^{t'} d\tau p_\mu \dot{x}^\mu\right\} \tag{5.30}$$

which is exactly Faddeev's expression (5.11) for this case. Eq. (5.30) may also be written as

$$\langle \vec{x}', t' | \vec{x}, t\rangle = \int_{\text{Path}} \prod_t \frac{d^4x d^4p dV}{(2\pi\hbar)^4} \delta(x^0 - \tau)|P^0|$$

$$\times \exp\left\{\frac{i}{\hbar}\int_t^{t'} d\tau (p_\mu \dot{x}^\mu - \tfrac{1}{2} V(p^2 - m^2))\right\}. \tag{5.31}$$

Since the Hamiltonian is quadratic in p we may integrate away the momenta to obtain

$$\langle \vec{x}', t' | \vec{x}, t\rangle = \int_{\text{Path}} \prod_t d^4x dV \frac{1}{N(V)} \delta(x^0 - \tau) \left|\frac{\dot{x}^0}{V}\right|$$

$$\times \exp\left\{\frac{i}{\hbar}\int_t^{t'} d\tau \left(\frac{\dot{x}^2}{2V} + \tfrac{1}{2} Vm^2\right)\right\}, \tag{5.32}$$

where $N(V) \propto V^2$ and where we recognize the einbein Lagrangian (4.19).

The non-covariant form of the Green function (5.29) is due to the chosen normalization of the physical states (5.27). A covariant form is obtained when the physical states are constructed according to the rules

$$|q^*, \text{phys}\rangle \propto (V_\Omega)^{-1/2} \int d^m Q \delta^m(Q) \int d^n p \delta(\psi(q,p)) |\text{Det}'\{Q, \psi\}| \exp(ip \cdot q)|p\rangle. \tag{5.33}$$

This rule was used throughout in Ref. [16].

6. Covariant quantization

The Faddeev-Popov trick [17] consists of adding to the naive path integral expression the factor

$$\frac{1}{V_\Omega} \Delta(q, p) \int \prod_t d^m\Omega \delta^m(\chi(\Omega)) \tag{6.1}$$

where

$$V_\Omega \equiv \int \prod_t d^m\Omega$$

$$\Delta^{-1}(q,p) = \int \prod_t d^m\Omega\delta^m(\chi(\Omega)) \tag{6.2}$$

and where $\Delta(q, p)$ is the Faddeev-Popov determinant which is gauge invariant due to the invariant group measure. This is equivalent to the construction (5.20). However, the above construction may formally be generalized to gauge conditions χ which also contain

v and $\dot{v}$. Furthermore, it is possible to lift up all the inconvenient factors in the measure to an effective Lagrangian: The Faddeev-Popov determinant by means of Faddeev-Popov ghosts [17] and the delta function in the gauge fixing conditions by a kind of average expression. The resulting effective Lagrangian has then e.g. the following form for Yang--Mills theories

$$\mathscr{L}(x) = -\tfrac{1}{4} F_a^{\mu\nu} F_{a\mu\nu} - \frac{1}{2\alpha} (\partial_\mu A_a^\mu)^2 + \partial^\mu \bar{\eta}_a (\partial_\mu \eta_a - g C_{abc} A_{b\mu} \eta_c) \tag{6.3}$$

where η_a and $\bar{\eta}_a$ are the anticommuting scalar Faddeev-Popov ghost fields. The possibility of a covariant quantization of Yang-Mills theories was first shown by Feynman at the one-loop level [18], and was then further developed by de Witt [19]. Faddeev-Popov treated two covariant theories: one given by (6.3) and one with an explicit delta function $\delta(\delta_\mu A_a^\mu)$ (formally the $\alpha \to 0$ limit of (6.3)). In the last version the Lorentz conditions hold off-shell while this is not the case in (6.3). (However, they hold on-shell in (6.3) as we shall see.) One crucial property of the gauge fixing term is that it must contain $(\dot{A}_a{}^0)^2$, i.e. a kinetic term for the Lagrange multiplier field $A_a{}^0$. Otherwise (6.3) would still have a local gauge invariance which has to be fixed. As a consequence the term $(\partial_\mu A_a^\mu)^2$ must always be there in a covariant treatment, but other covariant terms may be added. Another crucial property of (6.3) is its invariance under the following global supertransformation (so called BRS-invariance [20])

$$\begin{aligned} \delta A_{a\mu}(x) &= \lambda(\partial_\mu \eta_a + g C_{abc} A_{b\mu} \eta_c) \\ \delta \eta_a(x) &= \lambda \frac{g}{2} C_{abc} \eta_b \eta_c \\ \delta \bar{\eta}_a(x) &= -\lambda \frac{1}{\alpha} \partial_\mu A_a^\mu \end{aligned} \tag{6.4}$$

where λ is an x^μ-independent odd Grassmann parameter anticommuting with η and $\bar{\eta}$. Since repeated transformations yield zero, the corresponding conserved supercharge $\hat{Q}$ satisfies

$$\hat{Q}^2 = 0. \tag{6.5}$$

Now one may show that, in the quantum theory of (6.3), the unitarity in the physical subspace is ensured if one requires [21]

$$\hat{Q}\,|\text{phys}\rangle = 0 \tag{6.6}$$

which also generates the Slavnov-Taylor identities [21]. Fradkin and Vilkovisky [22–23] have given a Hamiltonian formulation of the covariant quantization scheme for an arbitrary, general gauge theory as defined in (3.1). In what follows we shall review their formulation and show that (6.6) is a sufficient condition for the determination of physical states and that the resulting theory is equivalent to the non-covariant canonical quantization.

7. *Fradkin-Vilkovisky formulation* [22–23]

In the formulation of Fradkin and Vilkovisky the BRS-charge Q plays a fundamental rôle. They give the following general form of Q for an arbitrary gauge theory

$$Q = P_r\bar{\mathscr{P}}^r + \psi_r\eta^r - \tfrac{1}{2}\, c^r{}_{st}\mathscr{P}_r\eta^s\eta^t \tag{7.1}$$

where P_r and ψ_r are the primary and secondary constraints of the standard Lagrangian defined in Sec. 4. $\bar{\mathscr{P}}^r$, $\bar{\eta}^r$, η^r and $\mathscr{P}^r$ are $4m$ additional phase space variables of the odd Grassmann type satisfying the PB relations

$$\{\bar{\eta}^r, \bar{\mathscr{P}}_s\} = \delta^r{}_s, \qquad \{\eta^r, \mathscr{P}_s\} = \delta^r{}_s \tag{7.2}$$

with remaining PB's zero. The algebraic properties of PB's with Grassmann variables follows from the relations [24]

$$\{E, \varepsilon O\} = \varepsilon\{E, O\} = \{\varepsilon E, O\}_+$$

$$\{\varepsilon O, E\} = \varepsilon\{O, E\} = -\{O, \varepsilon E\}_+ \tag{7.3}$$

where E is an even and O an odd Grassmann variable and ε an odd constant. Thus, εO is even and εE odd. In particular it follows that

$$\{\mathscr{P}_s, \eta^r\}_+ = \{\eta^r, \mathscr{P}_s\}_+ = \delta^r{}_s. \tag{7.4}$$

Using the Jacobi identities involving ψ_r, ψ_s, ψ_t one finds

$$\{Q, Q\}_+ = \tfrac{1}{4}\{C_{st}{}^r, C_{uv}{}^x\}\eta^s\eta^t\eta^u\eta^v\mathscr{P}_r\mathscr{P}_x. \tag{7.5}$$

For bosonic theories the right-hand side is claimed to be zero in Refs. [22] and [23]. A BRS--invariant Hamiltonian is defined by

$$H_\varrho \equiv H_1 + \{\varrho, Q\} \tag{7.6}$$

where ϱ is an odd gauge fixing variable and where

$$H_1 = H_0 + \eta^r C_r{}^s\mathscr{P}_s \tag{7.7}$$

satisfying (from Jacobi identities between ψ_r, ψ_s, H_0)

$$\{Q, H_1\} = \tfrac{1}{2}\{C_{st}{}^r, C_q{}^v\}\eta^s\eta^t\eta^q\mathscr{P}_v\mathscr{P}_r \tag{7.8}$$

which is also claimed to be zero in Refs. [22] and [23] for bosonic theories. Thus, assuming the right-hand sides of (7.5) and (7.8) to be zero, we have (a more general treatment is given in Ref. [25])

$$\{H_\varrho, Q\} = 0. \tag{7.9}$$

Notice that $H_\varrho = H_{\varrho'}$ where $\varrho' = \varrho + \lambda\{\varrho, Q\}_+$ is a (finite) generalized BRS-transformation. Hence H_ϱ is independent of the gauge fixing variable ϱ when chosen among the equivalence class generated by Q. The corresponding Lagrangian is

$$L_\varrho = p_i\dot{q}^i + P_r\dot{v}^r + \mathscr{P}_r\dot{\eta}^r + \bar{\mathscr{P}}_r\dot{\bar{\eta}}^r - H_\varrho. \tag{7.10}$$

A suitable form of ϱ is

$$\varrho = \mathscr{P}_r v^r + \bar{\eta}^r(\chi_r + \tfrac{1}{2}\,\alpha P_r) \tag{7.11}$$

which leads to

$$H_\varrho = H_1 + \{\varrho, Q\} = H_0 + \eta^r C_r{}^s \mathscr{P}_s + \mathscr{P}_r \bar{\mathscr{P}}^r + P^r(\chi_r + \tfrac{1}{2}\,\alpha P_r) + v^r \psi_r$$
$$+ \bar{\eta}^r\{\chi_r, \psi_s\}\eta^s + C^r{}_{st}\mathscr{P}_r v^s \eta^t - \tfrac{1}{2}\,\bar{\eta}^r\{\chi_r, C^u{}_{st}\}\mathscr{P}_u \eta^s \eta^t. \tag{7.12}$$

Now varying P_r, $\mathscr{P}_r$ and $\bar{\mathscr{P}}_r$ in the action of (7.10) one finds the equations

$$P_r = \frac{1}{\alpha}(\dot{v}_r - \chi_r),\ \mathscr{P}^r = -\dot{\bar{\eta}}^r$$

$$\bar{\mathscr{P}}^r = \dot{\eta}^r + \eta^s C_s{}^r + C^r{}_{st} v^s \eta^t + \tfrac{1}{2}\,\bar{\eta}^u\{\chi_u, C^r{}_{st}\}\eta^s \eta^t. \tag{7.13}$$

Inserting these expressions into (7.10) one finds

$$L_\varrho = L + L_{\text{g.f.}} + L_{\text{FP}} \tag{7.14}$$

where

$$L = p_i \dot{q}^i - H_0 - v^r \psi_r$$

$$L_{\text{g.f.}} = \frac{1}{2\alpha}(\dot{v}_r - \chi_r)(\dot{v}^r - \chi^r)$$

$$L_{\text{FP}} = \dot{\eta}^r \dot{\bar{\eta}}_r + \eta^r C_r{}^s \dot{\bar{\eta}}_s + C^r{}_{st} \dot{\bar{\eta}} v^s \eta^t - \bar{\eta}^r\{\chi_r, \psi_s\}\eta^s - \tfrac{1}{2}\,\bar{\eta}^r \dot{\bar{\eta}}^u\{\chi_r, C_{ust}\}\eta^s \eta^t \tag{7.15}$$

which is the conventional decomposition of the effective Lagrangian in the covariant quantization: L is the original Lagrangian, $L_{\text{g.f.}}$ is the gauge fixing part and L_{FP} the Faddeev-Popov ghost part. In the path integral quantization one may now make use of L_ϱ which has no gauge invariance. However, L_ϱ has a generalized BRS-invariance since Q is conserved (7.7). On the other hand this is now just an ordinary global invariance. The requirement that physical amplitudes must be BRS-invariant, is an externally imposed condition on the theory. Inserting the expression (7.13) for $\mathscr{P}^r$ and $\bar{\mathscr{P}}^r$ into Q (7.1) one arrives at

$$Q = G(\eta) + \tfrac{1}{2}\,(\bar{\eta}^u\{\chi_u, C^r{}_{st}\}P_r + C^r{}_{st}\dot{\bar{\eta}}_r)\eta^s \eta^t \tag{7.16}$$

where $G(\eta)$ is given by (4.12). Hence, the part of Q that is linear in η^r is just the generator of infinitesimal gauge transformations with η^r as infinitesimal parameters.

Finally we remark that if $\alpha = 0$ in (7.11) we would not have been able to solve out P_r from the equations of motion (varying P_r we find $\dot{v}_r - \chi_r = 0$). In the functional integral the above procedure corresponds to the integration over the momenta. If one there integrates over P_r the delta function $\delta(\dot{v}_r - \chi_r)$ appears (cf. the Landau gauge formulation in Yang-Mills).

8. *BRS-quantization*

The equivalence between the canonical and the covariant quantization is far from obvious. The formal equivalence proofs within the path integral quantization does not dispel the puzzling differences between the two formulations. In one formulation we have a singular Lagrangian with local gauge invariances and in the other we have a regular Lagrangian with only global invariances. However, within the operator quantization one may give a simple algebraic equivalence proof which also explains the differences between the two formulations [25].

If we within the covariant formulation require physical states to be invariant under generalized BRS-transformations, then we must require within the operator quantization

$$\hat{Q}|\text{phys}\rangle = 0. \tag{8.1}$$

Now since $\hat{Q}^2 = 0$, a physical state will only be defined up to a zero norm state [21]

$$|\chi\rangle \equiv \hat{Q}|\text{unphys}\rangle. \tag{8.2}$$

Obviously

$$\langle\chi_1|\chi_2\rangle = \langle\chi|\text{phys}\rangle = 0. \tag{8.3}$$

Hence, any matrix element between physical states is unchanged by the transformation

$$|\text{phys}\rangle \rightarrow |\text{phys}\rangle + |\chi\rangle. \tag{8.4}$$

Thus, although $|\chi\rangle$ is a physical state according to (8.1) it will not contribute to any observable quantity and is therefore really unphysical. The condition (8.1) subdivides the original state space into *three* different sectors: One sector contains genuine physical states with non-zero (positive) norm, one sector contains physical zero norm states which are orthogonal to the genuine physical states, and finally one sector contains unphysical states.

Corresponding to (8.1) we define a physical operator $\hat{\varphi}$ to be an operator which transforms physical states into physical states. $\hat{\varphi}$ must satisfy

$$[\hat{\varphi}, \hat{Q}]_{\pm} = f_{\varphi}\hat{Q}. \tag{8.5}$$

As there were two types of physical states there are two types of physical operators which we shall call A- and B-type operators. A B-type operator satisfies

$$\hat{B} = [\hat{C}, \hat{Q}]_{\pm} \tag{8.6}$$

where $\hat{C}$ is an unphysical operator and an A-type operator cannot be written as (8.6). By means of Jacobi identities one may easily show that A- and B-type operators satisfy the algebra

$$[\hat{A}_1, \hat{A}_2]_{\pm} = \hat{A}_3 \text{ or } \hat{B},\ [\hat{A}, \hat{B}_1]_{\pm} = \hat{B}_2,\ [\hat{B}_1, \hat{B}_2]_{\pm} = \hat{B}_3. \tag{8.7}$$

On physical states A- and B-type operators yield different results:

$$\hat{A}|\text{phys}\rangle = |\text{phys}\rangle,\ \hat{B}|\text{phys}\rangle = |\chi\rangle. \tag{8.8}$$

Hence only physical operators of A-type are genuine physical operators. B-type operators on the other hand may be viewed as generators of a new type of gauge transformations. Notice that a physical operator $\hat{\phi}$ satisfies

$$[\hat{B}_i, \hat{\phi}]_\pm = f^{\varphi}_{ij}\hat{B}_j \forall_i. \tag{8.9}$$

The properties (8.7–9) of the B-type operators establish these as gauge generators. The difference between the canonical and the covariant quantizations is that $\hat{B}|\text{phys}\rangle = 0$ is replaced by $\hat{B}|\text{phys}\rangle = |\chi\rangle$ with identical results for physical amplitudes. Are the gauge generators $\hat{B}$ the same in the two formulations? From the form (7.1) of the generalized BRS-charge, i.e.

$$\hat{Q} = \hat{P}_r\hat{\overline{\mathscr{P}}}^r + \hat{\varphi}_r\hat{\eta}^r - \tfrac{1}{2}\hat{C}^r{}_{st}\hat{\mathscr{P}}_r\hat{\eta}^s\hat{\eta}^t \tag{8.10}$$

we find the following B-type operators

$$\begin{aligned}
\hat{B}_r &= i[\hat{Q}, \hat{\bar{\eta}}_r]_+ = \hat{P}_r \\
\hat{B}'_r &= i[\hat{Q}, \hat{v}_r]_- = \hat{\overline{\mathscr{P}}}_r \\
\hat{B}''_r &= i[\hat{Q}, \hat{\mathscr{P}}_r]_+ = \hat{\varphi}_r - \hat{C}^s{}_{tr}\hat{\mathscr{P}}_r\hat{\eta}^t \\
\hat{B}'''_r &= i[\hat{Q}, \hat{\chi}_r]_- = \hat{M}'_{rs}\hat{\eta}^s
\end{aligned} \tag{8.11}$$

where $\hat{M}'_{rs} = \hat{M}_{rs} + \frac{1}{2}[\hat{C}^t{}_{us}, \hat{\chi}_r]\,\hat{P}_t\hat{\eta}^u$, $\hat{M}_{rs} = [\hat{\psi}_s, \hat{\chi}_r]_-$. $\hat{B}_r$ we identify as the canonical primary constraints. $\hat{B}'_r$ tells us that physical states do not depend on $\hat{\bar{\eta}}^r$. Provided χ^r are consistent gauge choices to ψ_r, $\hat{B}'''_r$ tells us that physical states do not depend on P_r and $\hat{\eta}^s$ has zero eigenvalue. From the form of $\hat{B}''_r$ this means that also $\hat{\psi}_r$ generate gauge transformations. The equivalence between canonical and covariant quantization is thereby established.

Ex. In the case of a Yang-Mills theory we have the following expression for the BRS--charge

$$Q = \int d^3x(G_a(x)\eta_a(x) + E^0_a(x)\overline{\mathscr{P}}_a(x) - \tfrac{1}{2}gC_{abc}\mathscr{P}_a\eta_b\eta_c(x)). \tag{8.12}$$

The above construction yields that G_a, η_a, E^0_a and $\overline{P}_a$ are gauge generators. However, notice that $E^0_a = -\frac{1}{2}\partial_\mu A^\mu_a$ since the Lagrangian is given by (6.3). The expression (6.3) is obtained by means of the gauge fixing variable

$$\varrho(x) = \mathscr{P}_a A^0_a + \bar{\eta}_a(\partial_i A^i_a + \tfrac{1}{2}\alpha E^0_a). \tag{8.13}$$

Since physical states satisfy

$$\hat{E}^0_a|\text{phys}\rangle = |\chi\rangle \tag{8.14}$$

this may be written as

$$\partial_\mu \hat{A}^\mu_a|\text{phys}\rangle = |\chi\rangle. \tag{8.15}$$

This shows the similarity with the Gupta-Bleuler formalism. One may consider the BRS--quantization as the consistent form of Gupta-Bleuler quantization.

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