LETTERS TO THE EDITOR

THE ZERO QUANTIZATION AS AN ORIGIN OF DIRAC SPIN AND FERMIONIC GENERATIONS

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A new "zero quantization" is introduced by taking, as its basic kets, four orthogonal directions $\mu = 1, 2, 3, 4$ in a complex space-time. The "wave function" of the zero quantization $\psi(\mu)$ becomes a complex Fermi "field" of the first quantization. Then it defines 2×15 Hermitian matrices generating the group $SU(2,2)\times SU(4)$, where the first factor is the usual conformal group related to the spin 1/2 and other Dirac degrees of freedom; whereas the second describes new internal degrees of freedom giving 4 fermionic eigenstates possibly interpreted as 4 generations. So, through a quantization procedure, we relate the Dirac spin and fermionic generations to the notion of direction in the complex space-time.

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In this note we turn to the geometrical foundations of the particle physics. Namely, we show how the existence of the Dirac spin and fermionic generations may follow from the fundamental notion of the direction in a complex extension of the physical space-time, *if* the idea of quantization is invoked on a very primordial level.

We conjecture that, in the logical structure of quantum theory, the usual level of the first or particle quantization should be preceded by the level of the *zero quantization* which would be in an analogical relation to the first quantization as the latter is to the second or field quantization. We introduce the zero quantization by the requirement that its state space is spanned on four orthonormal kets $|\mu\rangle$ ($\mu = 1, 2, 3, 4$) interpreted as four arbitrary orthogonal directions in the complex space-time defined as a manifold of points $z_\mu = x_\mu + iy_\mu$ ($\mu = 1, 2, 3, 4$), where x_μ, with $x_4 = it$ are coordinates in the physical space-time. We do not decide at the moment whether y_4 is imaginary or real, while y_1, y_2 and y_3 are taken real. These kets define, of course, the identity operator I of the zero quantization,

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$$I = \sum_{\mu} |\mu\rangle\langle\mu|, \quad \langle\mu \mid \nu\rangle = \delta_{\mu\nu}. \tag{1}$$

Having defined the basis $|\mu\rangle$ ($\mu = 1, 2, 3, 4$), we can speak about the *wave function* in the corresponding representation [1],

$$\psi(\mu) = \langle\mu \mid \psi\rangle, \tag{2}$$

where ψ is the abstract *state vector* (on the level of the zero quantization),

$$\psi = \sum_{\mu} \mid \mu\rangle\psi(\mu), \quad \langle\psi \mid \psi\rangle = \sum_{\mu} \mid \psi(\mu) \mid^2 = 1. \tag{3}$$

The wave function (2) can be interpreted as the probability amplitude of measuring the direction $\mu = 1, 2, 3, 4$ in our complex extension of the physical space-time. Obviously, $\psi_\mu \equiv \psi(\mu)$ ($\mu = 1, 2, 3, 4$) are complex Cartesian components of a versor ψ in the complex space-time. This a priori arbitrary versor becomes determined when a *state equation* (on the level of the zero quantization) and an initial condition are imposed on $\psi = \psi(t)$ in the Schrödinger picture. Evidently, all versors in the complex space-time form the state space of the zero quantization. Note that beside the pure states described by the state vector (3) there may exist also mixed states corresponding to the *density matrix* [2] (on the level of the zero quantization),

$$\varrho = \sum_{\mu} |\mu\rangle\varrho_\mu\langle\mu|, \quad \sum_{\mu} \varrho_\mu = 1, \tag{4}$$

which leads to the statistical weighting of directions in the complex space-time.

Although it is tempting to procede further with the construction of the zero quantization formalism and its probabilistic interpretation, we shall stop at this point in order to avoid a possible false step which is likely to be taken in this new physical situation. Instead, we would like to elaborate to some extent the transition to the first-quantization level where we should meet a more familiar physical content. In this transition we are led by the analogy with the well-known transition from the first to the second quantization level.

So, when passing to the level of the first quantization, the wave function $\psi(\mu)$ of the zero quantization becomes a *quantized "field"* defined on the discrete space of four orthogonal directions $\mu = 1, 2, 3, 4$ in the complex space-time. Then, after the first quantization is carried out, we get

$$\psi(\mu) = a_\mu,$$

where a_μ and $\overset{+}{a}_\mu$ are some annihilation and creation operators obeying in the simplest case either the Bose-Einstein or Fermi-Dirac statistics. Tentatively choosing for $\psi(\mu)$ the Fermi--Dirac statistics, we obtain

$$\{a_\mu, \overset{+}{a}_\nu\} = \delta_{\mu\nu}, \text{ others anticommuting.} \tag{6}$$

In the case of the complex Fermi "field" $\psi(\mu)$, Eq. (5), we can form 8 Hermitian operators

$$\gamma_\mu = a_\mu + \overset{+}{a}_\mu, \quad \eta_\mu = \frac{1}{i}(a_\mu - \overset{+}{a}_\mu) \tag{7}$$

satisfying, due to Eq. (6), the extended Dirac-like anticommutation relations:

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}, \quad \{\eta_\mu, \eta_\nu\} = 2\delta_{\mu\nu}, \text{ others anticommuting.} \tag{8}$$

Obviously $a_\mu = \frac{1}{2}(\gamma_\mu + i\eta_\mu)$. The operators γ_μ and η_μ can be represented *minimally* by 16×16 matrices. E.g. we can write

$$\gamma_\mu = \gamma_\mu^{\mathrm{D}} \times \mathbf{1}^{\mathrm{D}} = \begin{cases} \sigma_i \times \sigma_2 \times \mathbf{1} \times \mathbf{1} & \mu = i = 1, 2, 3 \\ \mathbf{1} \times \sigma_3 \times \mathbf{1} \times \mathbf{1} & \mu = 4 \end{cases},$$

$$\eta_\mu = \gamma_5^{\mathrm{D}} \times i\gamma_5^{\mathrm{D}}\gamma_\mu^{\mathrm{D}} = \begin{cases} -\mathbf{1} \times \sigma_1 \times \sigma_i \times \sigma_3 & \mu = i = 1, 2, 3 \\ \mathbf{1} \times \sigma_1 \times \mathbf{1} \times \sigma_2 & \mu = 4 \end{cases}, \tag{9}$$

where γ_μ^{D}, $\mathbf{1}^{\mathrm{D}}$ and $\gamma_5^{\mathrm{D}} \equiv -\gamma_1^{\mathrm{D}}\gamma_2^{\mathrm{D}}\gamma_3^{\mathrm{D}}\gamma_4^{\mathrm{D}}$ are 4×4 Dirac matrices ($\gamma_i^{\mathrm{D}} \equiv -i\beta^{\mathrm{D}}\alpha_i^{\mathrm{D}}, \gamma_4^{\mathrm{D}} \equiv \beta^{\mathrm{D}}$), while σ_i and $\mathbf{1}$ denote 2×2 Pauli matrices. Introducing the matrices

$$\varrho_k = \varrho_k^{\mathrm{D}} \times \mathbf{1}^{\mathrm{D}} = \mathbf{1} \times \sigma_k \times \mathbf{1} \times \mathbf{1},$$

$$\omega_k = \mathbf{1}^{\mathrm{D}} \times \varrho_k^{\mathrm{D}} = \mathbf{1} \times \mathbf{1} \times \mathbf{1} \times \sigma_k, \tag{10}$$

we can see that $\gamma_i = \varrho_2 \Sigma_i$, $\gamma_4 \equiv \beta = \varrho_3$, $\eta_i = -\varrho_1\omega_3 T_i$ and $\eta_4 = \varrho_1\omega_2$, where

$$\Sigma_i = \Sigma_i^{\mathrm{D}} \times \mathbf{1}^{\mathrm{D}} = \sigma_i \times \mathbf{1} \times \mathbf{1} \times \mathbf{1},$$

$$T_i = \mathbf{1}^{\mathrm{D}} \times \Sigma_i^{\mathrm{D}} = \mathbf{1} \times \mathbf{1} \times \sigma_i \times \mathbf{1}. \tag{11}$$

Eqs. (11) and (10) give us four independent (i.e. commuting) sets of three spin-1/2-like matrices.

Note that two sets of 15 matrices

$$\Sigma_i, \quad \varrho_k, \quad \Sigma_i\varrho_k, \quad T_i, \quad \omega_k, \quad T_i\omega_k \tag{12}$$

(each supplemented by the 16×16 unit matrix) define two independent (i.e. commuting) Dirac algebras. Identifying the matrices of the first set (12) with the corresponding Dirac matrices [3] logically related to the physical space-time we generate the usual conformal group SU(2,2) containing the Lorentz group SO(3,1) generated by the *particle spin* $\frac{1}{2}\vec{\Sigma}$ and *particle velocity* $\vec{\alpha} = \varrho_1\vec{\Sigma}$ (times the imaginary unit). Then the second set (12) is logically related to the manifold of points y_μ ($\mu = 1, 2, 3, 4$) and describes *new* internal degrees of freedom of the particle. Choosing in this case the compact option (corresponding to real y_4) we generate a new internal group SU(4) implying the existence of 4 new internal eigenstates for a spin-1/2 particle. It is tempting to interpret these 4 eigenstates as the fermionic generations and, therefore, to identify the new internal group as $\mathrm{SU_H}(4)$, a horizontal group of 4 generations which may be denoted as e, μ, τ, ω.[1]

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[1] Choosing for the horizontal group the non-compact option SU(2,2) = SO(4,2) (corresponding to imaginary y_4) we lose finite-dimensional unitary representations, implying thereby the existence of an *infinite* number of fermionic generations described by an infinite-dimensional unitary representation. Finite-dimensional non-unitary representations appearing in this case can be used only in collaboration with the translational degrees of freedom y_μ and their conjugate momenta q_μ in order to construct unitary representations.

Note that the matrices Σ_i and ϱ_k can be represented in some equations by their Dirac 4×4 forms $\Sigma_i^{\mathrm{D}} = \sigma_i \times \mathbf{1}$ and $\varrho_k^{\mathrm{D}} = \mathbf{1}\times\sigma_k$ if the matrices T_i and ω_k are absent from these equations. An example of this situation is the usual Dirac equation.

We can summarize our procedure and results as follows. We start from the zero quantization, where basic kets are four orthogonal directions $\mu = 1, 2, 3, 4$ in the complex space-time. Then we get on the level of the first quantization a complex Fermi "field" defined on the discrete space of $\mu = 1, 2, 3, 4$. This "field" implies the existence of (*i*) 15 Dirac matrices generating the conformal group SU(2,2) (connected with the spin 1/2 and other Dirac degrees of freedom) and (*ii*) additional 15 Dirac-like matrices generating in the compact case a new internal group SU(4) (connected with new internal degrees of freedom). The resulting number 4 of new internal eigenstates may be interpreted as the fermionic generations e, μ, τ, ω. When the translational degrees of freedom y_μ and their conjugate momenta q_μ are *physically* absent (or, alternatively, included into the diagonalized mass matrix), a free fundamental fermion is described on the level of the first quantization by the Dirac equation

$$(i\gamma_\mu p_\mu + m)\psi(x) = 0, \tag{13}$$

where $p_\mu = -i\partial/\partial x_\mu$ and

$$\psi(x) = \langle x \mid \psi\rangle \tag{14}$$

is the wave function in the position representation ($x = (x_\mu)$), ψ being the abstract state vector (on the level of the first quantization. In Eq. (13)

$$\gamma_\mu = \gamma_\mu^{\mathrm{D}}\times\mathbf{1}^{\mathrm{D}} = \begin{pmatrix} \gamma_\mu^{\mathrm{D}} & 0 & 0 & 0 \\ 0 & \gamma_\mu^{\mathrm{D}} & 0 & 0 \\ 0 & 0 & \gamma_\mu^{\mathrm{D}} & 0 \\ 0 & 0 & 0 & \gamma_\mu^{\mathrm{D}} \end{pmatrix}, \quad \psi(x) = \begin{pmatrix} \psi_e(x) \\ \psi_\mu(x) \\ \psi_\tau(x) \\ \psi_\omega(x) \end{pmatrix} \tag{15}$$

and

$$m = \begin{pmatrix} m_e & 0 & 0 & 0 \\ 0 & m_\mu & 0 & 0 \\ 0 & 0 & m_\tau & 0 \\ 0 & 0 & 0 & m_\omega \end{pmatrix}. \tag{16}$$

In the Dirac equation (13), the real part γ_μ of the complex Fermi "field" $\psi(\mu)$ collaborates with the momentum p_μ conjugate to the position x_μ being the "physical" part of the complex Bose "field" $\phi(\mu) = z_\mu \equiv x_\mu + iy_\mu$ defined on the discrete space of $\mu = 1, 2, 3, 4$ (the conjugate momentum to this "field" is $\Pi(\mu) = -i\partial/\partial z_\mu \equiv \frac{1}{2}(p_\mu - iq_\mu)$). An analogy with the level of the second quantization is evident: there a number of Fermi and Bose fields (defined in that case on the physical space-time) collaborate to give a description of a fundamental physical system. The level of the second quantization is, of course, necessary in order to describe particle interactions in a fully relativistic way.

Note that the diagonal mass matrix m in Eq. (13) can be replaced by a *mass operator M* including the matrices T_i and ω_k (and eventually the operators y_μ and q_μ). Then the relation

$$(\vec{\alpha}\cdot\vec{p} + \beta M)^2 = \vec{p}^{\,2} + M^2 \tag{17}$$

still holds. The operator M here need not to be invariant under the whole internal group SU(4), similarly as the Dirac equation is not invariant under the whole conformal group SU(2, 2) unless $M = 0$. This may give mass splitting between fermionic generations.

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