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THE ERNST EQUATION AS A CHIRAL MODEL

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It is shown that the set R_E of solutions of the Ernst equation can be mapped on a subset $R^{\mathscr{M}}_{CH}$ of the set of solutions of the Chiral model on the group G of "unimodular" matrices. The map is a bijection which depends on the choice of a set $\mathscr{M}$ of unimodular matrices; the set plays a role of an external holonomic constraint in the Chiral model. Symmetries of the equations of the Chiral model are investigated. It is shown that $(2k+1)$-soliton solution of this Chiral model generated from $g_0 = \mathbf{1}$ can be mapped on a solution of the Ernst equation.

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1. Introduction

The problem of integrability for the Einstein (in the vacuum case) and Einstein-Maxwell field equations for metrics with two commuting Killing vectors has very intensively been investigated in the last 12 years. The papers by Ernst (1968a, b) and Geroch (1971, 1972) have initiated developing of the integration methods for Einstein equations in this case (Kinnersley 1977, Kinnersley and Chitre 1977, 1978a, b Hoenselaers et al. 1979, Cosgrove 1977, 1980, Harisson 1978, Neugebauer 1979, Hauser and Ernst 1979a, b).

In our paper we shall investigate the problem of finding vacuum axially-symmetric and stationary gravitational fields. We adopt the Zakharov-Mikhailov inverse method (Zakharov and Mikhailov 1978, Belinsky and Zakharov 1978, 1979) for integration of the Ernst equation (Ernst 1968a).

In the complex Ernst formalism describing the problem one introduces two related potentials E and $\xi (E = (\xi-1)/(\xi+1))$. They allow to rewrite the Einstein equations in the case in a short and elegant form of the Ernst equation. In the formalism the norm of the time-like Killing vector K_t is related in a simple way with ξ and E:

$$K_{t\mu}K^{\mu}_{t} = \mathrm{Re}\, E = \frac{|\xi|^2-1}{|\xi+1|^2} = g_{00}. \tag{1}$$

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The relation (1) introduces two regions on the ξ – plane: $\mathscr{X} := \{\xi : |\xi|^2 > 1\}$ and $\mathscr{D} := \{|\xi : |\xi|^2 < 1\}$. On the boundary of $\mathscr{X}$ and $\mathscr{D}(\mathscr{B} := \{\xi : |\xi|^2 = 1\})$ K_t changes its character from time-like (in $\mathscr{X}$) to space-like (in $\mathscr{D}$) and the metric determined by ξ changes its physical properties. The SU(1,1) symmetry group of the Ernst equation (in the homographic representation) acts transitively on $\mathscr{X}$ and on $\mathscr{D}$, i.e. $\mathscr{X}$ and $\mathscr{D}$ are homogeneous spaces of the group. Since SU(1,1) is isomorphic to SO(2, 1, $\boldsymbol{R}$) we can map in one-to-one way $\mathscr{X}$ and $\mathscr{D}$ onto homogeneous spaces of SO(2, 1, $\mathbf{R}$), namely on $\mathscr{H}^{(+)}$ and $\mathscr{H}^{(-)}$:

$$\mathscr{X} \to \mathscr{H}^{(+)} := \{\vec{w} : \eta_{ab} w^a w^b = 1, \quad w^0 > 0\}, \quad \eta_{ab} = \mathrm{diag}\,(1, -1, -1),$$

$$\mathscr{D} \to \mathscr{H}^{(-)} := \{\vec{w} : \eta_{ab} w^a w^b = 1, \quad w^0 < 0\}.$$

The mapping can be done by means of a hyperbolic analogue of the stereographic projection (Hirayama et al. 1978, Mazur private communication):

$$h : \mathscr{H} := \mathscr{H}^{(+)} \cup \mathscr{H}^{(-)} \ni \vec{w} \to \xi = \frac{w^1 + iw^2}{1 - w^0} \in \mathbf{C}. \tag{2}$$

The inverse transformation has the form:

$$h^{-1} : \mathbf{C} \ni \xi \to (|\xi|^2 - 1)^{-1} \begin{pmatrix} |\xi|^2 + 1 \\ -2\,\mathrm{Im}\,\xi \\ -2\,\mathrm{Re}\,\xi \end{pmatrix} \in \mathscr{H}. \tag{3}$$

The map h maps the Ernst Lagrangian density into the Lagrangian density of the nonlinear σ – model:

$$\mathscr{L} = \nabla w^a \nabla w^b \eta_{ab}, \tag{4}$$

$$w^a w^b \eta_{ab} = 1. \tag{5}$$

It is seen that nonlinearity of the Ernst equation is not a result of the dynamics (since the Lagrangian (4) describes a linear model) but arises from the existence of an external holonomic constraint (5).

The main purpose of this paper is to show integrability of the Ernst equation by means of the Zakharov-Mikhailov inverse method (1978). In Section 2 we establish a 1 : 1 connection between the constraint (5) of the σ – model and a certain set $\mathscr{M}$ of unimodular matrices. The condition $g \in \mathscr{M}$ is compatible with the equations of motion of the Chiral model on the group G of "unimodular" matrices. Therefore, there exists a 1 : 1 relationship between the set R_{E} of solutions of the Ernst equation and some subsets $R_{\mathrm{CH}}^{\mathscr{M}}$ of the set of solutions R_{CH} of the Chiral model on G.

In Section 3 we construct a special set $\mathscr{M}_0$. $\mathscr{M}_0$ is connected with other choices of $\mathscr{M}$ by a set of transformations. A subset of these transformations exhibits the H symmetry of the Ernst equation (Section 4). In Section 5 we investigate $(2k+1)$ – soliton solutions of the Chiral model on $\mathscr{M}_0$ generated from $g_0 = \mathbf{1}$. It turns out that one can interpret them as solutions of the Ernst equation.

2. *Integrability of the Ernst equation*

We formulate the problem of solving the nonlinear σ – model (4), (5) as an integrability problem of a Chiral model on the group G of complex "unimodular" matrices:

$$G := \{g \in \mathrm{GL}\,(2, \mathbf{C}), \quad \det g = \varepsilon, \quad \varepsilon = \pm 1\} \tag{6}$$

with an additional holonomic constraint $g \in \mathscr{M}$. For this purpose we define a subset $\mathscr{M} \subset G$ as a common part of G and of a real vector space $\mathscr{I} : \mathscr{M} := G \cap \mathscr{I}$. $\mathscr{I}$ is a vector space over $\mathbf{R}$ generated by a system of three linearly independent matrices σ_a:

$$\mathscr{I} := \{A \in \mathrm{Mat}_{2\times 2}(\mathbf{C}) : A = \sigma_a w^a; \quad w^a \in \mathbf{R}; \quad a = 0, 1, 2\}. \tag{7}$$

We assume that the matrices σ_a are chosen in such a way that:
(a) $\mathscr{M} \neq \phi$,
(b) mapping $s : \mathscr{H} \ni \vec{w} \to s(\vec{w}) = \sigma_a w^a \in \mathscr{M}$ is a bijection ($\mathscr{H} = \mathscr{H}^{(+)} \cup \mathscr{H}^{(-)}$).
It results from (a) that for $g = \sigma_a w^a \in \mathscr{M}$, the matrices σ_a and a vector $\vec{w}$ satisfy:

$$T_{ab} w^a w^b = \mathbf{1}, \tag{8}$$

where the object T_{ab} is a system of 9 matrices made up of

$$\varepsilon := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } \sigma_a:$$

$$T_{ab} := \varepsilon \sigma_a \varepsilon^T \sigma_b^T \varepsilon. \tag{9}$$

If s maps $\mathscr{H}$ into $\mathscr{M}$ then for

$$T_{(ab)} = \mathbf{1}\eta_{ab} \tag{10}$$

the restriction (8) holds for any $\vec{w} \in \mathscr{H}$.

It follows from (10) that $s(\mathscr{H}) = \mathscr{M}$ and the generators σ_a are linearly independent. A mapping $s^{\mathrm{R}} : \mathbf{R}^3 \ni \vec{w} \to \sigma_a w^a \in \mathscr{I}$ is linear, thus it is a bijection. A restriction of s^{R} to a subset $\mathscr{A}$ of $\mathbf{R}^3$ is a surjection of $\mathscr{A}$ in $\mathscr{I}$. Since s is a restriction of s^{R} to $\mathscr{H}$ ($s = s^{\mathrm{R}}|_{\mathscr{H}}$) and $s(\mathscr{H}) = \mathscr{M}$, s is a bijection of $\mathscr{H}$ on $\mathscr{M}$.

We have proved the

Theorem 1: If σ_a is a system of three matrices satisfying (10) and $\mathscr{I}$ is the vector space (7) then $\mathscr{M} := G \cap \mathscr{I} \neq \phi$ and the mapping s is a bijection.

Thus we have found a map of the holonomic constraint (5) onto the set $\mathscr{M}$. Now, if one considers the Chiral model on G with the Lagrangian

$$\mathscr{L}_{\mathrm{CH}} = \tfrac{1}{2} \mathrm{Tr}\,(\nabla g \nabla g^{-1}) \tag{11}$$

(operator ∇ is in the cylindrical co-ordinates ϱ, z) and assumes that $g \in \mathscr{M}$ then one immediately sees that (11) is transformed into the Lagrangian (4) of the σ model. This suggests to describe the integrability problem for the Ernst equation in terms of the Chiral model on G. The suggestion is not obvious at all since the constraint $g \in \mathcal{M}$ does not have to be compatible with the field equations resulting from (11). This is not the case, however, and we can prove the following:

Theorem 2: If ξ is a solution of the Ernst equation, $\vec{w}$ is a three-dimensional vector determined by ξ according to (3), $g = \sigma_a w^a(\xi)$ is an "unimodular" matrix, then g is a solution of the field equations resulting from (11):

$$(\nabla^2 g)g^{-1} = \nabla g \nabla g^{-1} \tag{12}$$

iff $T_{(ab)} = \mathbf{1}\eta_{ab}$. Morever,

Theorem 3: If T_{ab} satisfies (10) and $g(\varrho, z)$ is a solution of equation (12) with the additional property $g \in \mathcal{M}$ then the potential:

$$\xi = (h \circ s^{-1})(g(\varrho, z)) \tag{13}$$

is a solution of the Ernst equation.

One can prove theorems 2 and 3 by a straightforward calculation. The theorems imply that the mapping $h \circ s^{-1}$ establishes a one-to-one relationship between the set of solutions R_E of the Ernst equation and a subset $R_{CH}^{\mathcal{M}}$ of the set of solutions R_{CH} of the Chiral model on G, where $R_{CH}^{\mathcal{M}} = \{g \in R_{CH} : g \in \mathcal{M}\}$. The subset $R_{CH}^{\mathcal{M}}$ is non-empty because the condition $g \in \mathcal{M}$ is compatible with equation (12). Thus, we have shown that due to the integrability of Chiral models the Ernst equation is also integrable by means of the Zakharov-Mikhailov mothod.[1]

3. *A basis of generators σ_a*

The purpose of this Section is to determine all possible representations of the system of generators σ_a. The T_{ab} will play a fundamental role. For convenience, we shall change its definition, making use of the conclusion from (10):

$$\det \sigma_a = \varepsilon \delta_a, \quad \delta_a = \begin{cases} 1 & a = 0 \\ -1 & a = 1,2 \end{cases} \tag{14}$$

in definition (9). We obtain:

$$T_{ab} = \sigma_a \sigma_b^{-1} \delta_b. \tag{15}$$

We shall treat the formula (15) as a new definition of T_{ab}. Then, the conclusion (14) will be an independent property of the σ_a. It will be, besides (10), the second condition which the σ_a have to satisfy.

The conditions (10) and (14) are invariant under the following set of transformations:

$$\mathcal{T} := \{\tau : \tau(\sigma_a) = g_1 \sigma_a g_2,\ g_1, g_2 \in \mathrm{GL}(2, \mathrm{C});\ \det g_1 = \det g_2^{-1}\}. \tag{16}$$

[1] Other proofs of the integrability for the Ernst equation can be found in papers by Maison (1979) and Mazur (1983).

These transformations can change the sign of det σ_a. However, the sign is not significant because G is not connected. It has two connected components:

$$G^{(+)} = \mathrm{SL}(2, \mathbf{C}), \quad G^{(-)}: = \{g \in \mathrm{GL}(2, \mathbf{C}) : \det g = -1\}. \tag{17}$$

The choice of a sign in (14) is determined by a component ($G^{(+)}$ or $G^{(-)}$) which the space $\mathscr{J}$ intersets: $\mathscr{M}^{(\varepsilon)} = G^{(\varepsilon)} \cap \mathscr{J}$. Transformations changing the sign of det σ_a map $\mathscr{M}^{(\varepsilon)}$ on a $\mathscr{M}'^{(-\varepsilon)}$. We do not need the whole $\mathscr{T}$ for our purposes. It is sufficient, if we confine ourselves to subsets $\mathscr{T}_1$, $\mathscr{T}_2$ and $\mathscr{T}_3$ of $\mathscr{T}$ (which are groups):

$$\mathscr{T}_1 : = \{\tau : \tau(\sigma_a) = \sigma_a g, \quad g \in G\}, \tag{18}$$

$$\mathscr{T}_2 : = \{\tau : \tau(\sigma_a) = g\sigma_a, \quad g \in G\}, \tag{19}$$

$$\mathscr{T}_3 : = \{\tau : \tau(\sigma_a) = g\sigma_a g^{-1}, \quad g \in \mathrm{GL}(2, \mathbf{C})\}. \tag{20}$$

Using $\mathscr{T}_i$, we shall find the simplest representation of σ_a. We shall find the generators σ_a in two steps. First, we shall find all systems T_{ab} satisfying (10) and additional conditions arising from the special form of T_{ab} (15). Next, we shall treat (15) as a definition of σ_a; then, using an arbitrary $\sigma_0 \in G$ we define σ_1, σ_2 as follows:

$$\sigma_1 := T_{10}\sigma_0, \tag{21}$$

$$\sigma_2 := T_{20}\sigma_0. \tag{22}$$

The formula (15) introduces some relations among the generators (for instance $\sigma_1 = -T_{12}\sigma_2$). These relations will be consistent with definitions (21), (22) if T_{ab} satisfies additional conditions (resulting from (10) and (15)):

$$T_{ac} = T_{ab}T_{bc}\delta_b, \tag{23}$$

$$\mathrm{Tr}\, T_{ab} = \mathrm{Tr}\, T_{ba} = 2\eta_{ab}, \tag{24}$$

$$\det T_{ab} = \delta_a\delta_b. \tag{25}$$

Now, in order to determine all T_{ab} it is sufficient to find two matrices T_{10} and T_{20} with properties:

$$T_{10}T_{20} + T_{20}T_{10} = 0, \tag{26}$$

$$\mathrm{Tr}\, T_{10} = \mathrm{Tr}\, T_{20} = 0, \tag{27}$$

$$\det T_{10} = \det T_{20} = -1. \tag{28}$$

The remaining T_{ab}'s can be found from (10) and (23). The formulas (26), (27), (28) result from (23), (24) and (25).

Since we can bring any traceless matrix $A \in \mathrm{Mat}_{2\times 2}(\mathbf{C})$ with a determinant $\det A = -1$ to the form $A = \mathrm{diag}(1, -1)$ by a transformation from $\mathscr{T}_3$ and then any $B \in \mathrm{Mat}_{2\times 2}(\mathbf{C})$

which anticommutes with A by a transformation from $\mathscr{T}_3$ leaving A invariant to the form $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, we can put:

$$T_{10} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad T_{20} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{29}$$

Furthermore, without loss of generality we can put $\sigma_0 = \mathbf{1}$, because of invariance of (10) with respect to transformations from $\mathscr{T}_1$. In this way we have obtained the simplest representation of σ_a:

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{30}$$

Any other representation can be obtained from (30) by a transformation from $\mathscr{T}_1, \mathscr{T}_2$ or $\mathscr{T}_3$. The space $\mathscr{J}$ generated by the basis (30) will be denoted $\mathscr{J}_0$. The constraint $\mathscr{M}_0 := \mathscr{J}_0 \cap G^{(+)}$ is unconnected; it can be easily shown that $\mathscr{M}_0 = \mathscr{M}_0^{(+)} \cup \mathscr{M}_0^{(-)}$ where $\mathscr{M}_0^{(\pm)} = \{A \in \mathscr{M} : \pm \frac{1}{2} \operatorname{Tr} A \geqslant 1\}$.

4. $\mathscr{M}_0^{(\pm)}$ *as a homogeneous space*

The $\mathscr{J}_0$ is a vector space of real and symmetric 2×2 matrices. Let us define an automorphism of $\mathscr{J}_0$:

$$d_g^R : \mathscr{J}_0 \ni A \to g^T A g \in \mathscr{J}_0, \; g \in \mathrm{SL}(2, \boldsymbol{R}). \tag{31}$$

The operation $d_g := d_g^R|_{\mathscr{M}_0^{(\pm)}}$ acts transitively on $\mathscr{M}_0^{(\pm)}$. Therefore, $(\mathscr{M}_0^{(\pm)}, d_g, \mathrm{SL}(2, \mathbf{R}))$ is a transitive group of transformation of $\mathscr{M}_0^{(\pm)}$; $\mathscr{M}_0^{(\pm)}$ is an homogeneous space of $\mathrm{SL}(2, \mathbf{R})$: $\mathscr{M}_0^{(\pm)} = \mathrm{SL}(2, \mathbf{R})/\mathrm{SO}(2)$. The $\mathrm{SL}(2, \mathbf{R})$ is homomorphic to $\mathrm{SU}(1, 1)$. Therefore the transitivity of $\mathrm{SL}(2, \mathbf{R})$ on $\mathscr{M}_0^{(\pm)}$ corresponds to the H symmetry of the Ernst equation. This can be seen from the following argumentation:

Let $\tilde{g}$ be a matrix connected with a solution ξ of the Ernst equation. Then $d_g(\tilde{g}) = \tilde{g}'$ is a solution of the Chiral model since (12) is invariant under the operation $d_g \cdot \tilde{g}' \in \mathscr{M}_0^{(\pm)}$, and therefore ξ' connected with $\tilde{g}'$ is a solution of the Ernst equation. ξ' and ξ are related to each other by a homography, which belongs to the homographic representation of $\mathrm{SU}(1, 1)$.

5. $(2k+1)$-*soliton solutions of the Ernst equation*

The problem of soliton solutions of equation (12) was investigated in 1978 by Belinsky and Zakharov (1978). They showed that only $2k$-soliton solutions generated from physical matrices are axisymmetric and stationary gravitational fields. A similar result was obtained in our paper (Gruczczak 1981). Soliton solutions we have investigated were solutions of the Chiral model on $\mathrm{SL}(2, \mathbf{R})/\mathrm{SO}(2)$. They can be interpreted as solutions of the Ernst equation generated from the Minkowski space time ($E = 1$, $g_0 = \mathbf{1}$ in the basis (30)). $(2k+1)$-soliton solutions were not solutions of the Chiral model at all. In the present paragraph we shall show that a $(2k+1)$-soliton solutions of (12) (from $g_0 = \mathbf{1}$) can also be interpreted as solutions of the Ernst equation. For example consider the one-soliton solution found by Belinsky and Zakharov (1978):

$$\hat{g}(\varrho, z) = \begin{bmatrix} 1-\dfrac{\varrho^2+\mu^2}{\mu^2}\sin^2 u, & -\dfrac{\varrho^2+\mu^2}{\mu^2}\sin u\cos u \\ -\dfrac{\varrho^2+\mu^2}{\mu^2}\sin u\cos u, & 1-\dfrac{\varrho^2+\mu^2}{\mu^2}\cos^2 u \end{bmatrix}, \tag{32}$$

$$\mu = v-z+\varepsilon'\sqrt{(v-z)^2+\varrho^2}, \quad \varepsilon' = \pm 1,$$

where u, v are complex parameters. The matrix $\hat{g}$ has a negative determinant and therefore is unphysical (Belinsky and Zakharov 1978). By means of (32) we can obtain a solution of (12) which belongs to $G^{(-)}$:

$$g(\varrho, z) = (-\det \hat{g})^{-1/2}\hat{g}, \quad \det \hat{g} = -\varrho^2/\mu^2. \tag{33}$$

This means that in order to obtain a solution of the Ernst equation from (33) it is necessary to find a proper basis σ'_a, associated with a cut of $G^{(-)}$. This can be done, for instance, by the transfromation of the basis (30) $\sigma_a \to \sigma'_a = \sigma_a\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$:

$$\sigma'_0 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \quad \sigma'_1 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \quad \sigma'_2 = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}. \tag{34}$$

Now, we restrict u, v in such a way that (33) is decomposable in the basis (34). One-soliton solution of the Ernst equation can be found by means of (13):

$$\xi = \text{ctg}\left(\frac{l+1}{2}\cdot\frac{\pi}{4} - \theta/2\right), \quad l = 1,5, \quad \varrho = R\sin\theta, \quad z = R\cos\theta. \tag{35}$$

This solution is a non asymptotically flat Weyl's solution. Other transformations from $\mathcal{T}_1$ or $\mathcal{T}_2$ changing $\mathcal{M}_0$ on $\mathcal{M}^{(-)}$ also give Weyl's solutions and their H generalizations. $(2k+1)$-soliton solutions of the Ernst equation can be found in a similar way. We find a $(2k+1)$-soliton solution g_{2k+1} of (12). Next, we transform the basis (30) by a transformation transforming $\mathcal{M}_0$ on $\mathcal{M}^{(-)}$ and confine parameters in g_{2k+1} in such a way that the matrix g_{2k+1} is decomposable in the "new" basis. Finally, we find ξ_{2k+1} by means of the mapping (13). The one-soliton solutions (33) of equation (12) satisfy the restriction $g = g^T$. The full set of one-soliton solutions can be found when one investigates the soliton solutions without this restriction.

Let $g(\alpha_1, \ldots, \alpha_q;\ \varrho, z)$ be a solution of equation (12). Suppose that we have found a system of parameters $\alpha_1^0, \ldots, \alpha_q^0$, $q \in N$ that $g(\alpha_1^0, \ldots, \alpha_q^0;\ \varrho, z)$ is decomposable in the "new" basis $\sigma'_a = \sigma_a g_1$ (or $\sigma'_a = g_1\sigma_a$), $g_1 \in G$:

$$g(\alpha_1^0, \ldots, \alpha_q^0; \quad \varrho, z) = \sigma'_a w^a. \tag{36}$$

Since equation (12) is invariant with respect to right (or left) translations ($g \to g\tilde{g}$ (or $g \to \tilde{g}g$), $\tilde{g} \in G$, $\partial_\varrho \tilde{g} = \partial_z \tilde{g} = 0$) the matrix g': $g'(\alpha_1^0, \ldots, \alpha_q^0;\ \varrho, z) = g(\alpha_1^0, \ldots, \alpha_q^0;$

$\varrho, z)g_1^{-1}$ (or $g'(\alpha_1^0, \ldots, \alpha_q^0; \varrho, z) = g_1^{-1}g(\alpha_1^0, \ldots, \alpha_q^0; \varrho, z))$ is a solution of this equation. Matrix g' is decomposable in the basis (30); components of g' in this basis are the same as components of g in the "new" basis. This means that the Ernst potential which we determine from g by the "new" basis is the same as the Ernst potential from g' by means of the "old" basis (30).

Let us assume that g can be obtained from a solution g_0 of (12) by the Zakharov--Mikhailov inverse method. It is easy to show that the solution $g' = g\tilde{g}(g' = \tilde{g}g)$ can be generated from $g'_0 = g_0\tilde{g}(g'_0 = \tilde{g}g_0)$. Therefore, the solution g' associated with (36) is generated from $g'_0 = g_0 g_1^{-1}(g'_0 = g_1^{-1}g_0)$.

In the case of $(2k+1)$-soliton solutions generated from $g_0 = \mathbf{1}$: $g'_0 = g_1^{-1}$. Since det $g_1 = -1$, the $(2+k1)$-soliton solutions of the Ernst equation are generated from unphysical solutions of (12) $(g'_0 = g_1^{-1} \notin \mathscr{M}_0)$. This result is compatible with the conclusions of Belinsky and Zakharov (1978).

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REFERENCES

Belinsky, V. A., Zakharov, V. E., *Zh. Eksp. Teor. Fiz.* **75**, 1953 (1978); *Zh. Eksp. Teor. Fiz.* **77**, 3 (1979).

Cosgrove, Ch. M., *J. Phys. A. Math. Gen.* **10**, 1481 (1977); *J. Math. Phys.* **21**, 2417 (1980).

Ernst, F. J., *Phys. Rev.* **167**, 1175 (1968a); *Phys. Rev.* **168**, 1415 (1968b).

Geroch, R., *J. Math. Phys.* **12**, 918 (1971); *J. Math. Phys.* **13**, 394 (1972).

Gruszczak, J., *J. Phys. A: Math. Gen.* **14**, 3247 (1981).

Harrison, B. K., *Phys. Rev. Lett.* **41**, 1197 (1978).

Hauser, I., Ernst, F. J., *Phys. Rev.* **D20**, 362 (1979a); *Phys. Rev.* **D20**, 1783 (1979b).

Hirayama, M., Chia, Tze H., Ishida, J., Kawabe, T., *Phys. Lett.* **A66**, 352 (1978).

Hoenselaers, C., Kinnersley, W., Xanthopoulos, B. C., *J. Math. Phys.* **20**, 2530 (1979).

Kinnersley, W., *J. Math. Phys.* **18**, 1529 (1977).

Kinnersley, W., Chitre, M., D. *J. Math. Phys.* **18**, 1538 (1977); *J. Math. Phys.* **19**, 1926 (1978a); *J. Math. Phys.* **19**, 2037 (1978b).

Maison, D., *J. Math. Phys.* **20**, 871 (1979).

Mazur, P., to be published in *Acta Phys. Pol.* 1983.

Neugebauer, G., *J. Phys. A.: Math. Gen.* **12**, L67 (1979).

Zakharov, V. E., Mikhailov, A. V., *Zh. Eksp. Teor. Fiz.* **74**, 1953 (1978).