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# QUANTIZATION OF GAUGE FIELDS*

By E. Guadagnini

Istituto di Fisica dell' Universita' di Pisa, Piazza Torricelli 2, 56100 Pisa, Italy

and

Istituto Nazionale di Fisica Nucleare, Pisa

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The canonical quantization of the gauge fields and the problems related with gauge invariance are discussed. The elimination of the cyclic variables, to fix the gauge, is worked out in details. The limit of static external charges and the appearance of the parameter $\theta$ in the quantum theory are considered, and the static stationary points of the euclidean action are introduced.

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## 1. *Introduction*

According to our present understanding of particle physics, the interactions between the fundamental constituents of matter are described by gauge theories. The experimental support and the interesting properties, that such theories can have (like charge antiscreening and dimensional transmutation), make quantum gauge-field theories a very attractive matter of investigation. Many problems are still open and need more detailed answers, nevertheless it is remarkable that a locality principle (as specified in the next sections) seems to be at the origin of particle interactions, so that dynamics is strictly connected with geometry.

In these notes we discuss the quantization of gauge fields and some of the problems related with gauge invariance. We follow the basic ideas of canonical quantization, but we use a non-standard method to solve the problem of the gauge fixing.

The main point will be to derive the effective lagrangian for the physical degrees of freedom. We shall eliminate the cyclic variables by constructing the Routh's function. By this method it is possible to obtain, in a simple way, the effective lagrangian for different unitary-gauge choices.

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In particular we show, by explicit construction, that, independently of the gauge--fixing choice, the form (and the meaning) of the effective lagrangian is unique. Unlike the usual Faddeev-Popov trick [1], in this formalism the problem of eliminating the pure gauge degrees of freedom is separated from the problem of the particular choice of the gauge. Therefore the gauge can directly be chosen according to the particular problem (or region of configuration space) one wants to consider.

In Sect. 2 we review some general aspects of field quantization, and local gauge invariance is introduced in Sect. 3.

In Sect. 4 we develop the formalism of cyclic variables and derive the effective classical lagrangian for the physical degrees of freedom. The limit of static external charges, which in the non-abelian case can lead to some troubles with gauge invariance, is clarified.

The quantization of the theory is discussed on Sect. 5. In particular, the gauge $A_0 = 0$ and the Coulomb gauge are considered.

In Sect. 6 the parameter $\theta$ is defined, and in Sect. 7 we discuss the semiclassical limit of the theory. The physically static stationary points of the euclidean action, which correspond to the multi-monopole solutions in the presence of Higgs fields, are introduced.

## 2. *Field quantization. Preliminaries*

Before considering gauge theories, let us review some general aspects of field quantization [2].

Let us assume that the relevant configurations of the system are described by a classical real parameter $\phi(\boldsymbol{x})$. The system is specified by giving the lagrangian density $\mathscr{L}(\dot{\phi}(\boldsymbol{x}), \phi(\boldsymbol{x}))$, and the classical motion is a stationary point of the action.

$$S = \int d^3x dt \mathscr{L} = \int dt L. \tag{2.1}$$

At the quantum level the system is specified by giving also the basic commutation relations between the operators associated with the degrees of freedom; that is, we have to quantize the field. To do this let us consider first the case in which the lagrangian is of the type:

$$L = \int d^3x \tfrac{1}{2} (\dot{\phi}(\boldsymbol{x})^2 - \phi(\boldsymbol{x}) h \phi(\boldsymbol{x})) \tag{2.2}$$

where $h$ is a differential operator (for instance, $h = -\nabla^2 + m^2$).

Let $\{\psi_n(\boldsymbol{x})\}$ be a complete set of real eigenfunctions of $h$ (it is straightford to generalize the equations (2.3)–(2.7) to the case in which $h$ has a continuous spectrum):

$$h\psi_n(\boldsymbol{x}) = \lambda_n \psi_n(\boldsymbol{x}) \tag{2.3}$$

orthonormalized

$$\int d^3x \psi_n(\boldsymbol{x}) \psi_m(\boldsymbol{x}) = \delta_{nm}. \tag{2.4}$$

Any configuration can then be written

$$\begin{aligned} \phi(\boldsymbol{x}) &= \sum_n q_n \psi_n(\boldsymbol{x}) \\ \dot{\phi}(\boldsymbol{x}) &= \sum_n \dot{q}_n \psi_n(\boldsymbol{x}). \end{aligned} \tag{2.5}$$

Using Eqs (2.3)–(2.5), one gets:

$$L = \sum_n \tfrac{1}{2} (\dot{q}_n^2 - \lambda_n q_n^2). \tag{2.6}$$

The lagrangian of the system describes a set of independent harmonical oscillators with frequencies $\omega_n^2 = \lambda_n$, and the quantization can be done in the usual way used in quantum mechanics. To the operators

$$(q_n)_{\text{op}} \quad \text{and} \quad (p_n)_{\text{op}} = \left(\frac{\partial L}{\partial \dot{q}_n}\right)_{\text{op}}$$

(associated with the degrees of freedom) one imposes:

$$[(p_n)_{\text{op}}, (q_m)_{\text{op}}] = -i\delta_{nm},$$
$$[(p_n)_{\text{op}}, (p_m)_{\text{op}}] = 0 = [(q_n)_{\text{op}}, (q_m)_{\text{op}}]. \tag{2.7}$$

From Eq. (2.7) one gets:

$$[\pi(\boldsymbol{x})_{\text{op}}, \phi(\boldsymbol{y})_{\text{op}}] = -i\delta^3(\boldsymbol{x}-\boldsymbol{y}),$$
$$[\pi(\boldsymbol{x})_{\text{op}}, \pi(\boldsymbol{y})_{\text{op}}] = 0 = [\phi(\boldsymbol{x})_{\text{op}}, \phi(\boldsymbol{y})_{\text{op}}], \tag{2.8}$$

for the field operators:

$$\phi(\boldsymbol{x})_{\text{op}} \quad \text{and} \quad \pi(\boldsymbol{x})_{\text{op}} = \left(\frac{\delta L}{\delta \dot{\phi}(\boldsymbol{x})}\right)_{\text{op}}. \tag{2.9}$$

The physics of such a system is obviously related to the quantum mechanical oscillator; for instance, the wave function of the vacuum (ground state) is:

$$\Psi_0[\phi(\boldsymbol{x})] = \prod_n \exp\left(-\frac{\sqrt{\lambda_n}}{2} q_n^2\right) = \exp\left(-\tfrac{1}{2}\sum_n q_n \sqrt{\lambda_n}\, q_n\right)$$
$$= \exp\left(-\tfrac{1}{2}\int d^3x \phi(\boldsymbol{x}) \sqrt{h}\, \phi(\boldsymbol{x})\right). \tag{2.10}$$

It should be noted that Eq. (2.8) defines the field operators and does not depend on the dynamics of the system, in the same sense that, in quantum mechanics, the commutator $[p, q] = -i$ does not depend on the form of the hamiltonian. With a quadratic lagrangian it is easy to derive Eq. (2.8) and Eq. (2.9) (and understand the physics of such a system), but it is clear that, as for the field quantization itself, we can take Eq. (2.8) and Eq. (2.9) as the definition of the canonical quantization procedure, even for more complicated systems.

For gauge theories, however, one cannot use the canonical quantization procedure straightforward, and the reason will become evident in the next section.

## 3. *Local gauge invariance*

To introduce gauge invariance [3], [4] under local transformations of a group $G$, let us consider first a theory which is symmetric under global transformations of $G$.

Let us assume that the system is described at the classical level by a parameter $\phi(x)$.

At every point $x$, $\phi(x)$ is an element of a vector space $M$, in which a non-trivial representation $R(G)$ of the group $G$ is defined.

The action of $G$ on $M$ will produce a transformation on the field $\phi(x)$:

$$\phi(x) \to R^{-1}(g)\phi(x). \tag{3.1}$$

Since the element $g$ of the group is the same for all points $x$, the transformation is called global, and we assume that it leaves the lagrangian density $\mathscr{L}_0(\phi, \partial_\mu\phi)$ invariant.

Now we introduce locality and assume that to every point $x$ is associated a different space $M_x$. All the spaces $M_x$ are equivalent, up to a unitary transformation, to the same space $M$.

If the spaces $M_x$ were not correlated in some way, the introduction of locality would not be useful. We could not define the derivative of the field $\phi$, because $\phi(x)$ and $\phi(x+\delta x)$ take values in different spaces.

To define the derivative of the field we specify what element $g(x, \delta x)$ of the group $G$ connects $M_{x+\delta x}$ to $M_x$. To first order in $\delta x^\mu$ this element will be:

$$g(x, \delta x) \simeq 1 - i \sum_a A_\mu^a(x)\delta x^\mu I^a + \ldots \tag{3.2}$$

where $I^a$ are the generators of the group $G$, and $\{A_\mu^a(x)\}$ is a set of real vector fields.

To the vector $\phi(x+\delta x)$, which belongs to the space $M_{x+\delta x}$, is associated a (transported) vector $\phi_T(x+\delta x)$ in $M_x$, given by

$$\phi_T(x+\delta x) \simeq \phi(x+\delta x) - i \sum_a A_\mu^a(x)\delta x^\mu R(I^a)\phi(x+\delta x) + \ldots \tag{3.3}$$

and we can take the (covariant) derivative of the field

$$D_\mu(A)\phi = \lim_{\delta x^\mu \to 0} (\phi_T(x+\delta x) - \phi(x)/\delta x^\mu = \partial_\mu\phi(x) - i \sum_a A_\mu^a(x) R(I^a)\phi(x). \tag{3.4}$$

It is useful to define a matrix of vector fields $A_\mu(x)$ by

$$A_\mu(x) = \sum_a A_\mu^a T^a \tag{3.5}$$

where $T^a$ are matrices which represent the generators $I^a$ in the fundamental representation.

The action of $G$ on the spaces $M_x$ will now depend on the point $x$. Let $g(x)$ be the element of $G$ which acts on $M_x$:

$$\phi(x) \to R^{-1}(g(x))\phi(x). \tag{3.6}$$

Then the induced transformation on $A_\mu(x)$ is

$$A_\mu(x) \to U^{-1}(x)A_\mu(x)U(x) + iU^{-1}(x)\partial_\mu U(x) \tag{3.7}$$

where

$$U(x) = \exp(i\Lambda^a(x)T^a) = \exp(i\Lambda(x)) \tag{3.8}$$

represents $g(x)$.

Under the transformations (3.6) and (3.7), gauge transformations, the lagrangian density:

$$\mathscr{L}_0(\phi, D_\mu(A)\phi) \tag{3.9}$$

is invariant, but we have introduced a new set of variables $\{A_\mu^a(x)\}$.

Only two functions of $A_\mu^a$, which we can add to the expression (3.9), are gauge invariant and represent renormalizable interactions:

$$\mathscr{L}_1 = -\frac{1}{4g^2}\sum_a G_{\mu\nu}^a G^{a,\mu\nu} \tag{3.10}$$

$$\mathscr{L}_2 = K\sum_a G_{\mu\nu}^a G_{\tau\sigma}^a \varepsilon^{\mu\nu\tau\sigma} \tag{3.11}$$

where $G_{\mu\nu}^a$ is defined by

$$\sum_a G_{\mu\nu}^a T^a = \partial_\mu A_\nu - \delta_\nu A_\mu - i[A_\mu, A_\nu]. \tag{3.12}$$

$\mathscr{L}_2$ can be written as a total divergence, therefore it does not change the classical equations of motion, but it may have physical effects (related with instantons) at the quantum level [5].

Like in electrodynamics, which is a gauge theory with gauge group U(1), local gauge invariance means that two different field configurations $(\phi(x), A_\mu(x))$ and $(\phi'(x), A'_\mu(x))$, which are related by a gauge transformation Eq. (3.6) and (3.7), describe the same physics.

The reason is the following. Let $\mathcal{O}(x)$ be some variable related to pure gauge degrees of freedom. Since the lagrangian is invariant under gauge transformations with arbitrary space-time dependence, it cannot depend on $\mathcal{O}(x)$ nor on the derivatives of $\mathcal{O}(x)$. Therefore physics does not depend on $\mathcal{O}(x)$.

A very simple example will clarify the situation. Consider a system described by a couple of fields $\phi_1(x)$ and $\phi_2(x)$, and let the lagrangian density $\mathscr{L}$ be invariant under the transformations:

$$\begin{aligned}\phi_1(x) &\to \phi_1(x),\\ \phi_2(x) &\to \phi_2(x)+\lambda(x)\end{aligned} \tag{3.13}$$

with arbitrary $\lambda(x)$. It is obvious, then, that $\mathscr{L}$ depends only on $\phi_1$, $\mathscr{L} = \mathscr{L}(\phi_1, \partial_\mu\phi_1)$, and that two fields configurations $(\phi_1(x), \phi_2(x))$ and $(\phi_1(x), \phi_2(x)+\lambda(x))$ describe the same physics.

In a similar way, in gauge theories the physics is described by gauge invariant degrees of freedom. Unlike the case in which the group $G$ is abelian, it is not easy to explicitly separate pure gauge degrees of freedom from physical ones. However this is precisely what we have to do, in order to understand the physics of gauge theories.

It is also clear that the difficulties one finds, in using canonical quantization procedure [6] ($A_0^a$ have no canonical conjugate momenta), are just a consequence of local gauge invariance.

### 4. *The gauge $A_0 = 0$ and cyclic variables*

Let us consider a gauge theory at the classical level. In the absence of matter fields the lagrangian density is given by the expression (3.10).

We are interested in eliminating the degrees of freedom related to gauge transformations, Eq. (3.7); i.e., we want to fix the gauge. First we take advantage of the arbitrary space-time dependence of gauge transformation for setting:

$$A_0 = 0. \tag{4.1}$$

The lagrangian density then becomes:

$$\mathscr{L} = 2\,\mathrm{Tr}\left\{\frac{1}{2g^2}(\dot{A}_i(x))^2 - \frac{1}{4g^2}G_{ij}^2(A(x))\right\},\quad (i, j = 1, 2, 3) \tag{4.2}$$

where $G_{ij}(A(x)) = \nabla_i A_j(\boldsymbol{x}) - \nabla_j A_i(\boldsymbol{x}) - i[A_i(\boldsymbol{x}), A_j(\boldsymbol{x})]$.

The $T^a$'s satisfy: $\mathrm{Tr}\,(T^a T^b) = \frac{1}{2}\delta^{ab}$, so that $2\,\mathrm{Tr}\,(A_i^2)$ just means $\sum A_i^a A_i^a$, etc.

Condition (4.1) does not fix the gauge completely. The lagrangian density (4.2) is invariant under residual time-independent (but space-dependent) gauge transformations:

$$A_i(x) \to U^{-1}(\boldsymbol{x})A_i(x)U(\boldsymbol{x}) + iU^{-1}(\boldsymbol{x})\nabla_i U(\boldsymbol{x}) \tag{4.3}$$

where

$$U(\boldsymbol{x}) = \exp\,(i\Lambda^a(\boldsymbol{x})T^a) = \exp\,(i\Lambda(\boldsymbol{x})). \tag{4.4}$$

To first order in the parameters $\Lambda(\boldsymbol{x})$, transformations (4.3) are

$$A_i(x) \to A_i(x) - D_i(A)\Lambda(\boldsymbol{x}) \tag{4.5}$$

where

$$D_i(A)\Lambda(\boldsymbol{x}) = \nabla_i \Lambda(\boldsymbol{x}) - i[A_i(x), \Lambda(\boldsymbol{x})]. \tag{4.6}$$

From the invariance of the lagrangian under transformations (4.3), it follows that the quantities:

$$\frac{1}{g^2}D_i(A)\dot{A}_i(\boldsymbol{x}) = \varrho(\boldsymbol{x}) \tag{4.7}$$

are constants of the motion. Eq. (4.7) is the Gauss Law and $\varrho(\boldsymbol{x})$ represents the external charge density.

To eliminate the residual gauge freedom, Eq. (4.3), and to find the form of the lagrangian for the physical variables, we use the method of Routh's function [7].

Suppose we have a system specified by a lagrangian $L(q, \dot{q}; \alpha, \dot{\alpha})$. If $L$ does not depend on the variable $\alpha$

$$\partial L/\partial\alpha = 0$$

then $\alpha$ is called a cyclic variable. It is useful to consider the Routh's function [8]:

$$R(q, \dot{q}; \pi) = \pi\dot{\alpha} - L, \tag{4.8}$$

where $\pi = \partial L/\partial\dot{\alpha}$ is the momentum canonically conjugate to the cyclic variable $\alpha$, and therefore it is a constant of the motion. In this way, for every fixed value $\pi'$ of the momentum $\pi$,

$$L_{\pi'}(q, \dot{q}) = -R(q, \dot{q}; \pi') \tag{4.9}$$

is the effective lagrangian of the remaining variables $q$.

The idea is to proceed in a similar way in a gauge theory: cyclic variables will refer to gauge freedom, whereas non-cyclic variables will describe the gauge invariant content of the theory.

To do this, we change variables [9] and write:

$$A_i(\boldsymbol{x}) = S^{-1}(\boldsymbol{x})B_i(x)S(\boldsymbol{x}) + iS^{-1}(\boldsymbol{x})\nabla_i S(\boldsymbol{x}). \tag{4.10}$$

$S(\boldsymbol{x})$ takes values on the gauge group $G$, whereas $B_i(\boldsymbol{x})$ selects only one configuration for every orbit of the gauge group. In this way, by changing $S$ and $B_i$, we obtain all the possible $A_i$ configurations.

We are interested in the field $A_i$ which are of pure gauge outside some finite region of the space, therefore we can choose $B_i$ (in Eq. (4.10)) to be zero outside that region (actually, we can allow $B_i$ to vanish sufficiently fast when $|\boldsymbol{x}| \to \infty$). This restriction does not fix $B_i$ uniquely; two different possible configuration $B_i$ and $B_i'$ of the same orbit can differ by a gauge transformation (4.3) with parameters $\Omega^a(\boldsymbol{x})$ (in Eq. (4.4)) which vanish sufficiently fast as $|\boldsymbol{x}| \to \infty$. Otherwise the set of values, that $B_i$ can assume, can be chosen in an arbitrary way, for the moment we assume that some choice has been made.

Gauge transformation (4.3) are now given by

$$\begin{aligned} B_i(x) &\to B_i(x) \\ S(x) &\to S(x)U(\boldsymbol{x}). \end{aligned} \tag{4.11}$$

Therefore $B_i(x)$ describe gauge invariant degrees of freedom, whereas $S(x)$ describe pure gauge degrees of freedom. We have

$$\dot{A}_i(\boldsymbol{x}) = S^{-1}(x)\,(\dot{B}_i(\boldsymbol{x}) + iD_i(B)\,(\dot{S}(\boldsymbol{x})S^{-1}(\boldsymbol{x})))S(\boldsymbol{x}) \tag{4.12}$$

and the lagrangian density (4.2) can be written

$$\mathscr{L} = \frac{2}{g^2}\,\mathrm{Tr}\,\{\tfrac{1}{2}\,(\dot{B}_i(\boldsymbol{x}) + iD_i(B)\,(\dot{S}(\boldsymbol{x})S^{-1}(\boldsymbol{x})))^2 - \tfrac{1}{4}\,G_{ij}^2(B)\}. \tag{4.13}$$

In the lagrangian formalism we take $S(\boldsymbol{x})$ and $\dot{S}(\boldsymbol{x})S^{-1}(\boldsymbol{x})$ as independent variables, so that the equation

$$\frac{\partial\mathscr{L}}{\partial S(\boldsymbol{x})} = 0 \tag{4.14}$$

which says that $S(\boldsymbol{x})$ is the cyclic variable, just means that $\mathscr{L}$ is invariant under transformation (4.11).

The Lagrange equation (related to gauge freedom) is

$$S\frac{\partial \mathcal{L}}{\partial S} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial(\dot{S}S^{-1})} + \left[\frac{\partial \mathcal{L}}{\partial(\dot{S}S^{-1})}, \dot{S}S^{-1}\right] \tag{4.15}$$

and, using Eq. (4.14), one gets

$$\frac{\partial \mathcal{L}}{\partial(\dot{S}S^{-1})}(x) = S(x)K(\boldsymbol{x})S^{-1}(x) \tag{4.16}$$

where $K(\boldsymbol{x})$ is a constant of motion. Using Eq. (4.7), Eq. (4.16) reads

$$D^2(B)\,(\dot{S}(x)S^{-1}(x)) - iD_i(B)\dot{B}_i(x) = -ig^2S(x)\varrho(\boldsymbol{x})S^{-1}(x) \tag{4.17}$$

where

$$D^2(B) = D_i(B)D_i(B). \tag{4.18}$$

Constructing the Routh's function, we obtain as effective lagrangian for the variable $B_i$:

$$\mathcal{L}_e(B_i, \dot{B}_i) = 2\,\mathrm{Tr}\left\{\frac{1}{2g^2}(\dot{B}_i)_\perp^2 - \frac{1}{4g^2}G_{ij}^2(B)\right\}$$

$$-2\,\mathrm{Tr}\left(S\varrho S^{-1}\frac{1}{D^2(B)}D_i(B)\dot{B}_i\right) + 2\,\mathrm{Tr}\left(\frac{g^2}{2}S\varrho S^{-1}\frac{1}{D_2(B)}S\varrho S^{-1}\right) \tag{4.19}$$

where

$$(\dot{B}_i)_\perp = \left(\delta_{ij} - D_i(B)\frac{1}{D^2(B)}D_j(B)\right)\dot{B}_j. \tag{4.20}$$

The projector $\delta_{ij} - D_i(B)\frac{1}{D^2(B)}D_j(B)$ selects the part of $\dot{B}_i$ which is orthogonal to the gauge orbit in the point $B_i$ (see Sect. 5). As for the operator $D^2(B)$, it can be inverted because $B_i$ vanish sufficiently fast as $|\boldsymbol{x}| \to \infty$.

If the gauge group $G$ is $U(1)$, Eq. (4.19) becomes

$$\mathcal{L}_e(B_i, \dot{B}_i) = \frac{1}{2g^2}(\dot{B}_i)_\perp^2 - \frac{1}{4g^2}(\nabla_i B_j - \nabla_j B_i)^2 - \varrho\frac{1}{\nabla^2}\nabla_i\dot{B}_i + \frac{g^2}{2}\varrho\frac{1}{\nabla^2}\varrho \tag{4.21}$$

and Eq. (4.20) says that $(\dot{B}_i)_\perp$ is the transverse part of $\dot{B}_i$

$$(\dot{B}_i)_\perp = \left(\delta_{ij} - \nabla_i\frac{1}{\nabla^2}\nabla_j\right)\dot{B}_j. \tag{4.22}$$

The term $\varrho\frac{1}{\nabla^2}\nabla_i\dot{B}_i$ is a total time-derivative and can be neglected. On the other hand, since in this case gauge transformations (4.3) are just a translation of the longitudinal part of $A_i$, we can choose as $B_i$ the transverse part of $A_i$. The expression (4.21) is then the usual lagrangian density of the electromagnetic field in the Coulomb gauge with an external change distribution $\varrho(\boldsymbol{x})$.

In the non-abelian case, if $\varrho \neq 0$ the expression (4.19) also depends on the variable $S$, and there is no decoupling between the motion of the gauge invariant degrees of freedom (described by $B_i$) and pure gauge variables. Gauge invariance is broken.

The reason is that if we set $\varrho \neq 0$ in Eq. (4.7), we destroy gauge invariance, since $D_i(A)\dot{A}_i$ is not gauge invariant.

Unlike the abelian case, only the value $\varrho = 0$ is compatible with gauge invariance, and the case in which we set $\varrho \neq 0$ cannot represent any physical situation obtained from a gauge invariant theory. In particular, we cannot identify $\varrho(x)$ with the static charge distribution of massive charged particles. In this case, indeed, $\varrho(x)$ should be considered a dynamical variable and it would transform under gauge transformations (4.3) according to:

$$\varrho(x) \to U^{-1}(x)\varrho(x)U(x).$$

When fermions are present, the effective lagrangian in the limit of static massive quarks can be obtained from Eq. (4.27) (discussed below), where gauge invariance is not broken, as it has to be.

It is also clear that the high wave number screening effect, pointed out in Ref. [10], has no physical relevance. The reason why the energy can be arbitrarily lowered, even if $\varrho(x) \neq 0$, is the explicit breaking of gauge invariance. However, if we take the correct gauge invariant lagrangian (4.27), this effect is not present.

Setting $\varrho = 0$, the expression (4.19) can also be written:

$$\mathscr{L}(B_i, \dot{B}_i) = \frac{2}{g^2}\,\mathrm{Tr}\,(\tfrac{1}{2}\dot{B}_i^2 - \tfrac{1}{4}G_{ij}^2(B)) + \frac{2}{g^2}\,\mathrm{Tr}\left(\tfrac{1}{2}D_i(B)\dot{B}_i \frac{1}{D^2(B)} D_j(B)\dot{B}_j\right). \quad (4.23)$$

The effect of eliminating the pure gauge degrees of freedom has been the introduction of the last term in the lagrangian density (4.23). This term brings new interactions, and we may ask if some choice for the variable $B_i$ exists in such a way that this term automatically vanishes (like in the abelian case). Unfortunately the constraint

$$D_i(B)\delta B_i = 0 \quad (4.24)$$

which has to be satisfied, cannot be easily solved because of its non linear nature, and the solution is not known.

In the next section we shall see how to choose the variable $B_i$ when we consider a small-fluctuation perturbative expansion.

We conclude by deriving the effective lagrangian for the physical variables when fermions are present [11]. In this case, to the lagrangian density (4.2) we must add

$$\mathscr{L}_M = \sum_{\alpha\beta} \bar{\psi}_\alpha(i\gamma^0\delta_{\alpha\beta}\partial_0 + i\gamma^j(\delta_{\alpha\beta}\partial_j - i\sum_a A_j^a(T^a)_{\alpha\beta}) - \delta_{\alpha\beta}m)\psi_\beta. \quad (4.25)$$

The fermions transform according to the fundamental representation of $G$.

In changing the variables, together with Eq. (4.10), we set:

$$\psi(x) = S^{-1}(x)\psi_0(x),$$
$$\bar{\psi}(x) = \bar{\psi}_0(x)S(x). \quad (4.26)$$

As before, we construct the Routh's function (with $\varrho = 0$), and the effective lagrangian density for physical variables $B_i$ and $\psi_0$ becomes:

$$\mathscr{L} = \frac{2}{g^2}\,\mathrm{Tr}\,(\tfrac{1}{2}\dot{B}_i^2 - \tfrac{1}{4}G_{ij}^2(B)) + \frac{2}{g^2}\,\mathrm{Tr}\left\{\tfrac{1}{2}(D_i(B)\dot{B}_i + g^2 j_0)\frac{1}{D^2(B)}(D_j(B)\dot{B}_j + g^2 j_0)\right\}$$

$$+\sum_{\alpha\beta}\bar{\psi}_{0\alpha}(i\gamma^0\delta_{\alpha\beta}\partial_0 + i\gamma^j(\delta_{\alpha\beta}\partial_j - i\sum_a B_j^a(T^a)_{\alpha\beta}) - \delta_{\alpha\beta}m)\psi_{0\beta}, \tag{4.27}$$

where

$$j_0(x) = \sum_a j_0^a(x)T^a \tag{4.28}$$

and

$$j_0^a(x) = \sum_{\alpha\beta}\bar{\psi}_{0\alpha}(x)\gamma^0(T^a)_{\alpha\beta}\psi_{0\beta}(x). \tag{4.29}$$

## 5. *Quantum gauge fields*

In this section we consider a quantum gauge theory. First we discuss the quantization in the gauge $A_0 = 0$, then we show how to quantize directly the physical degrees of freedom by using the effective lagrangian derived in the previous section.

We shall consider a theory with gauge group SU(2). In the gauge $A_0 = 0$ one can use the canonical quantization procedure. To the operators $E_i^a(\boldsymbol{x})$ and $A_j^b(\boldsymbol{y})$ ($a, b = 1, 2, 3$), associated with the degrees of freedom, one imposes:

$$[E_i^a(\boldsymbol{x}), A_j^b(\boldsymbol{y})] = -i\delta_{ij}\delta^{ab}\delta^3(\boldsymbol{x}-\boldsymbol{y})$$

$$[E_i^a(\boldsymbol{x}), E_j^b(\boldsymbol{y})] = 0 = [A_i^a(\boldsymbol{x}), A_j^b(\boldsymbol{y})]. \tag{5.1}$$

$E_i^a$ are the momenta canonically conjugate to the $A_i^a$'s and, using the expression (4.2), one gets

$$E_i^a(\boldsymbol{x}) = \frac{1}{g^2}\dot{A}_i^a(\boldsymbol{x}). \tag{5.2}$$

The hamiltonian density is:

$$\mathscr{H}(\boldsymbol{x}) = \tfrac{1}{2}g^2(E_i^a(\boldsymbol{x}))^2 + \frac{1}{4g^2}(G_{ij}(A(\boldsymbol{x})))^2. \tag{5.3}$$

$\mathscr{H}(\boldsymbol{x})$ is invariant under time-independent gauge transformations (4.3). If the parameters $\Omega^a(\boldsymbol{x})$ of the transformation vanish sufficiently fast when $|\boldsymbol{x}| \to \infty$, then these transformations are unitarily implemented[1] and the generators are

$$G(\Omega) = \int d^3x\,\Omega^a(\boldsymbol{x})G^a(\boldsymbol{x}), \tag{5.4}$$

[1] If $\Omega^a(\boldsymbol{x})$ do not vanish when $|\boldsymbol{x}| \to \infty$, transformations (4.3) may no longer be implemented by a unitary operator. We could have a spontaneous symmetry breaking or, in any case, we have to take into account the effects of non-vanishing surface terms.

where

$$G^a(x) = \nabla_i E_i^a(x) + \varepsilon^{abc} A_i^b(x) E_i^c(x). \tag{5.5}$$

Since the $G^a(x)$'s are generators of transformations which are invariances of the system, they are constants of motion and can be simultaneously diagonalized with the Hamiltonian.

The state vectors $|\psi\rangle$, on which:

$$G(\Omega)\,|\psi\rangle = 0, \tag{5.6}$$

correspond to physical states. The space $H$ of physical states is invariant under temporal evolution. In $H$, $G^a(x) = 0$, or

$$\nabla_i E_i^a(x) = \varepsilon^{abc} E_i^b(x) A_i^c(x) \tag{5.7}$$

which is the Gauss Law, since $\varepsilon^{abc} E_i^b A_i^c$ is the charge density operator.

Note that physical states are not normalizable because their wave functions are constants on all the configurations related by a gauge transformation generated by $G(\Omega)$. This non-normalizability is related to the residual gauge freedom (4.3), and factorizes out of physical amplitudes, which depend in a non-trivial way only on the variables which describe gauge invariant degrees of freedom [12].

Therefore in the gauge $A_0 = 0$ the physical content of the theory is not explicitly produced, and to quantize directly the physical degrees of freedom, we can use the effective lagrangian (4.23).

The choice of the variable $B_i$ is suggested by the form of the lagrangian density (4.23) and depends on the point of the configuration space around which one considers the (small-fluctuations) perturbative expansion.

Let $\tilde{B}_i(x)$ be a configuration around which we consider the small fluctuations

$$B_i = \tilde{B}_i + B_i^q. \tag{5.8}$$

Any fluctuation $B_i^q$ can be decomposed with respect to the orthogonal and tangent planes to the gauge orbit (obtained with time-independent gauge transformations) at the point $\tilde{B}_i$.

$$B_i^q = (B_i^q)_\perp + (B_i^q)_{\text{tan}}. \tag{5.9}$$

Using Eq. (4.5), one gets

$$(B_i^q)_\perp = \left(\delta_{ij} - D_i(\tilde{B}) \frac{1}{D^2(\tilde{B})} D_j(\tilde{B})\right) B_j^q, \tag{5.10}$$

$$(B_i^q)_{\text{tan}} = D_i(\tilde{B}) \frac{1}{D^2(\tilde{B})} D_j(\tilde{B}) B_j^q \tag{5.11}$$

and from Eq. (4.19) and Eq. (4.20) one finds that the lagrangian density (to lowest orders in the fluctuations) depends only on $(B_i^q)_\perp$.

Therefore, we can define $B_i$, Eq. (5.8), by imposing, on its possible configurations $\tilde{B}_i + B_i^q$, the constraint:

$$D_i(\tilde{B}) B_i^q = 0. \tag{5.12}$$

In particular, let us consider the case in which $\tilde{B}_i = 0$. Eq. (5.12) then becomes:

$$\nabla_i B_i^q = 0$$

and, from (5.8), we obtain that $B_i$ is a transverse field (Coulomb gauge). In quantizing the field we set

$$[\pi_i^a(x), B_j^b(y)] = -i\delta^{ab}\left(\delta_{ij} - \nabla_i \frac{1}{\nabla^2} \nabla_j\right)\delta^3(x-y)$$

$$[\pi_i^a(x), \pi_j^b(y)] = 0 = [B_i^a(x), B_j^b(y)], \tag{5.13}$$

where $\pi_i^a$ are the momenta conjugate to $B_i^a$. Using the lagrangian density (4.23), one gets

$$\pi_i = \sum_a \pi_i^a T^a = \frac{1}{g^2}(\dot{B}_i)_\perp + \frac{1}{g^2}\nabla_i \frac{1}{\nabla^2}\nabla_j D_j(B)\frac{1}{D^2(B)} D_k(B)\dot{B}_k \tag{5.14}$$

and the hamiltonian density is[2]:

$$\mathscr{H} = 2\mathrm{Tr}\left(\frac{g^2}{2}\pi_i^2 + \frac{1}{4g^2}G_{ij}^2(B)\right)$$

$$-2\mathrm{Tr}\left(\frac{g^2}{2}D_k(B)\pi_k \frac{1}{D_i(B)\nabla_i \frac{1}{\nabla^2}\nabla_j D_j(B)} D_e(B)\pi_e\right). \tag{5.15}$$

Since $\pi_i$ is transverse, one has

$$(D_i(B)\pi_i)^a = \varepsilon^{abc}B_i^b\pi_i^c \tag{5.16}$$

and the last term in Eq. (5.15) is the analog of the instantaneous Coulomb interaction of the abelian case. When fermions are present, Eq. (4.27), the hamiltonian density is

$$\mathscr{H} = 2\mathrm{Tr}\left(\tfrac{1}{2}g^2\pi_i^2 + \frac{1}{4g^2}G_{ij}^2(B)\right) + \sum_{\alpha\beta}\bar{\psi}_{0\alpha}\left(-i\gamma^j\left(\delta_{\alpha\beta}\partial_j - i\sum_a B_j^a(T^a)_{\alpha\beta}\right) + m\delta_{\alpha\beta}\right)\psi_{0\beta}$$

$$-2\mathrm{Tr}\left\{\tfrac{1}{2}g^2(D_i(B)\pi_i + j_0)\frac{1}{D_k\nabla_k \frac{1}{\nabla^2}\nabla_e D_e} D_j(B)\pi_j + j_0)\right\}. \tag{5.17}$$

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[2] Really, in constructing the hamiltonian, we have to take some care in the ordering of the canonically conjugate variables. This may produce a non-trivial effect on the kinetic term $\pi_i^2/2$. The hamiltonian (5.15), obtained with the naive rules from the effective lagrangian (4.23) is correct up to terms of order higher than two in the coupling constant. This is enough for our purposes, but for more details on this point see Ref. [13].

Expanding $\left(D_i(B)\nabla_i \frac{1}{\nabla^2} \nabla_j D_j(B)\right)^{-1}$ in powers of $B_i$ one finds

$$\left(D_i(B)\nabla_i \frac{1}{\nabla^2} \nabla_j D_j(B)\right)^{-1} \simeq \frac{1}{\nabla^2} + \frac{2i}{\nabla^2}\left[B_i, \nabla_i \frac{1}{\nabla^2}\right] - \frac{3}{\nabla^2}\left[B_i, \nabla_i \frac{1}{\nabla^2}\left[B_j, \nabla_j \frac{1}{\nabla^2}\right]\right] + \dots \tag{5.18}$$

Together with the Coulomb potential $\frac{1}{\nabla^2}$ between charges, one gets other interaction terms. Because of these terms, one easily finds that non-abelian charges are not screened by the quantum fluctuations, but there is antiscreening [14]. Indeed let us consider the limit of massive static fermions, in which only the charge variables of fermions are taken into account. In this case $j_0$ acts as an external classical charge distribution in the hamiltonian (5.17), and the potential between the external charges is given (to lowest orders) by:

$$-g^2 \operatorname{Tr}\left(j_0 \frac{1}{\nabla^2} j_0\right) + 4g^2 \sum_n \operatorname{Tr}\left(j_0 \frac{1}{\nabla^2} \langle 0|D_i\pi_i|n\rangle\right) \frac{1}{E_0 - E_n}\left(\operatorname{Tr}\left(\langle n|D_j\pi_j|0\rangle \frac{1}{\nabla^2} j_0\right)\right)$$
$$+ 3g^2 \operatorname{Tr}\left(j_0 \frac{1}{\nabla^2} \langle 0|\left[B_i, \nabla_i \frac{1}{\nabla^2}\left[B_j, \nabla_j \frac{1}{\nabla^2} j_0\right]\right]|0\rangle\right) + \dots \tag{5.19}$$

The second term, being a second order correction to the ground state energy, has always the effect to lower the potential energy between the external charges. This just corresponds to the well known screening effect of the vacuum polarization.

The last term, however, has the opposite sign and it is dominant with respect to the usual screening term, so that summing all the terms one gets for the Fourier transform of the potential

$$V(\boldsymbol{k}) = \frac{g_R^2}{\boldsymbol{k}^2} + \frac{11}{24\pi^2} \frac{g_R^4}{\boldsymbol{k}^2} \log \frac{\mu^2}{\boldsymbol{k}^2} + \dots \tag{5.20}$$

where $g_R^2$ is a renormalized coupling constant defined at $\boldsymbol{k}^2 = \mu^2$.

## 6. *The parameter $\theta$*

Let us consider the theory in the gauge $A_0 = 0$, where physical states are characterized by the condition (5.6). This equation means that the wave functions of physical states are constant on all the configurations $A_i$ related by a gauge transformation (4.3) with parameters which go to zero when $|\boldsymbol{x}| \to \infty$.

We may ask how the wave functions change when we pass from a configuration $A_i$ to another one $A_i^A$, obtained from $A_i$ with a gauge transformation (4.3) with parameters $\Lambda^a(\boldsymbol{x})$ which do not vanish when $|\boldsymbol{x}| \to \infty$.

The question is meaningful if the transition amplitude

$$\langle A_i^A| e^{-HT} |A_i\rangle \tag{6.1}$$

(computed with the constraint (5.6)), is different from zero. Otherwise $A_i$ and $A_i^A$ do not belong to the same representation and, in this case, no correlation can exist between $|A_i\rangle$ and $|A_i^A\rangle$.

If the amplitude (6.1) is different from zero, then both $A_i$ and $A_i^A$ contribute to the structure of the theory, and gauge invariance requires that when we pass from the configuration $A_i$ to $A_i^A$ the wave functions of physical states can at most be multiplied by a phase factor.

The fact that the amplitude $\langle A_i'|e^{-HT}|A_i\rangle$ is different from zero is a transitive property of field configurations. Therefore, we can choose in Eq. (6.1) two configurations $A_i$ and $A_i^A$ gauge equivalent to zero; for instance $A_i = 0$, and

$$A_i^A(\boldsymbol{x}) = iU^{-1}(\Lambda(\boldsymbol{x}))\nabla_i U(\Lambda(\boldsymbol{x})) \tag{6.2}$$

and compute the amplitude (6.1) in the large $T$ limit.

To lowest order in the coupling constant $g$, the amplitude (6.1) is given by [15]:

$$\exp(-S_{\mathrm{E}}[\tilde{A}]) \tag{6.3}$$

where

$$S_{\mathrm{E}}[\tilde{A}] = \frac{1}{g^2}\int d^4x\, 2\,\mathrm{Tr}\,(\tfrac{1}{2}(\dot{\tilde{A}}_i)^2 + \tfrac{1}{4}G_{ij}^2(\tilde{A})) \tag{6.4}$$

and the path $\tilde{A}_i(\boldsymbol{x}, t)$ satisfies the euclidean equations of motion

$$\delta S_{\mathrm{E}}[\tilde{A}]/\delta\tilde{A}_i(x) = 0 \tag{6.5}$$

and the boundary conditions

$$\tilde{A}(\boldsymbol{x}, t) \xrightarrow[t\to\mp\infty]{} \begin{cases} 0 \\ A_i^A(\boldsymbol{x}) \end{cases} \tag{6.6}$$

together with the Gauss Law constraint

$$D_i(\tilde{A})\dot{\tilde{A}}_i(\boldsymbol{x}, t) = 0. \tag{6.7}$$

To this approximation the amplitude is different from zero if $S_{\mathrm{E}}[\tilde{A}]$ is finite. In turn this implies

$$U(\Lambda(\boldsymbol{x})) \xrightarrow[|\boldsymbol{x}|\to\infty]{} \pm 1. \tag{6.8}$$

The set of $U$, which satisfy Eq. (6.8), can be decomposed in equivalence classes, each one characterized by an integer index $n \in \mathbf{Z}$ (topological number [16]),

$$n = \frac{1}{24\pi^2}\int d^3x\varepsilon_{ijk}\,\mathrm{Tr}\,(U^{-1}\nabla_i U \cdot U^{-1}\nabla_j U \cdot U^{-1}\nabla_k U). \tag{6.9}$$

All the elements of the same class can be obtained by multiplication by $U(\Omega(\boldsymbol{x}))$, $(\Omega(\boldsymbol{x}) \xrightarrow[|\boldsymbol{x}|\to\infty]{} 0)$.

It can be shown that for any $n$ the euclidean action $S_E$, Eq. (6.4), is indeed finite [16] and it is given by

$$S_E = \frac{8\pi^2}{g^2}|n|. \tag{6.10}$$

We can then introduce for physical states an angle $\theta$, by the requirement that when we pass from a configuration $A_i$ to $A_i^A$, obtained with a gauge transformation which has topological number $n = 1$, the wave functions are multiplied by a phase factor $e^{-i\theta}$.

Instead of imposing different conditions on the behaviour of the wave functions, we can take into account the effects of having different values of $\theta$, by adding another term $\mathscr{L}_\theta$ to the lagrangian density (4.2). To find $\mathscr{L}_\theta$, let us see how the transition amplitude (6.1), between $A_i$ and $A_i^A$ depends on $\theta$. One has

$$\langle A_i^A|e^{-HT}|A_i\rangle|_\theta = \langle A_i^A|e^{-HT}|A_i\rangle|e^{in\theta}_{\theta=0} \tag{6.11}$$

where $n$ is the topological number (6.9) of the transformation $U(A(\boldsymbol{x}))$. Therefore, $\mathscr{L}_\theta$ must satisfy

$$\langle A_i^A|\exp(i\textstyle\int d^4x\mathscr{L}_\theta)|A_i\rangle = \langle A_i^A|A_i\rangle e^{in\theta}. \tag{6.12}$$

That is, $\mathscr{L}_\theta$ has no effect on the classical equations of motion, since it depends only on the initial and final configurations. This last property suggests that $\mathscr{L}_\theta$ can be written as a total time derivative. Let us now verify that

$$\mathscr{L}_\theta = \frac{\theta}{16\pi^2}\frac{d}{dt}\varepsilon_{ijk}\,\mathrm{Tr}\,[A_i(G_{jk}(A)+\tfrac{2}{3}iA_jA_k)] \tag{6.13}$$

is the wanted term. If we insert (6.13) in the expression (6.12) we get

$$\begin{aligned}&\langle A_i^A|\exp(i\textstyle\int d^4x\mathscr{L}_\theta)|A_i\rangle\\ &= \langle A_i^A|A_i\rangle\exp\left(\frac{i\theta}{16\pi^2}\int d^3x\varepsilon_{ijk}\,\mathrm{Tr}\,[A_i^AG_{jk}(A^A)+\tfrac{2}{3}iA_i^AA_j^AA_k^A\right.\\ &\qquad\left.-A_iG_{jk}(A)-\tfrac{2}{3}iA_iA_jA_k]\right),\end{aligned} \tag{6.14}$$

where

$$A_i^A = U^{-1}(A)A_iU(A)+iU^{-1}(A)\nabla_iU(A). \tag{6.15}$$

Using the identities

$$\textstyle\int d^3x\varepsilon_{ijk}\,\mathrm{Tr}\,(A_i^AG_{jk}(A^A)) = \int d^3x\varepsilon_{ijk}\,\mathrm{Tr}\,[A_iG_{jk}(A)+i\nabla_iUU^{-1}G_{jk}(A)], \tag{6.16}$$

$$\begin{aligned}&\textstyle\int d^3x\varepsilon_{ijk}\,\mathrm{Tr}\,(A_i^AA_j^AA_k^A) = \int d^3x\varepsilon_{ijk}\,\mathrm{Tr}\,[A_iA_jA_k\\ &-iU^{-1}\nabla_iU\cdot U^{-1}\nabla_jU\cdot U^{-1}\nabla_kU-\tfrac{3}{2}\nabla_iU\cdot U^{-1}G_{jk}(A)],\end{aligned} \tag{6.17}$$

one easily finds that Eq. (6.12) is satisfied.

In order to obtain a manifestly Lorentz-invariant form for $\mathscr{L}_\theta$, we add a surface term to the expression (6.13) and write

$$\mathscr{L}_\theta = \frac{\theta}{16\pi^2}\,\partial_\mu J^\mu = \frac{\theta}{16\pi^2}\,\varepsilon^{\mu\nu\lambda\sigma}\,\mathrm{Tr}\,G_{\mu\nu}(A)G_{\lambda\sigma}(A) \tag{6.18}$$

where

$$J^\mu = \varepsilon^{\mu\nu\lambda\sigma}\,\mathrm{Tr}\,[A_\nu(G_{\lambda\sigma}(A)+\tfrac{2}{3}\,iA_\lambda A_\sigma)]. \tag{6.19}$$

The introduction of the parameter $\theta$, which is not present in the theory at the classical level, is similar to the use of the pseudo-momentum in quantum mechanics for periodic potentials. Different values of $\theta$ label physically superselected sectors because physical variables are gauge invariant.

One of the most important consequences of the existence of the $\theta$ parameter in non--abelian gauge theories is the solution of the problem of the mass splitting between the flavour-singlet and the flavour-non-singlet mesons in QCD [5], [17].

## 7. *Static stationary points*

Since $1/g^2$ multiplies the whole lagrangian, the expansion in the coupling constant $g^2$ is like the $\hbar$-expansion, and the structure of the theory in the small $g^2$ limit just corresponds to the semiclassical approximation.

Consider the hamiltonian density in the gauge $A_0 = 0$, Eq. (5.3). If we set

$$E_i^{a'}(x) = gE_i^a(x)$$

$$A_i^{a'}(x) = \frac{1}{g}\,A_i^a(x) \tag{7.1}$$

(a canonical transformation on the variables) the hamiltonian can be written

$$H = \int d^3x \left[\tfrac{1}{2}\,(E_i^{a'}(x))^2 + \frac{1}{4g^2}\,(G_{ij}^a(gA'(x))^2\right] \tag{7.2}$$

and it is easy to see what happens when $g^2$ goes to zero. Eq. (7.2) describes a system with a potential energy:

$$V_g(A') = \frac{1}{g^2}\,V(gA') \tag{7.3}$$

where

$$V(A') = \tfrac{1}{4}\int d^3x(G_{ij}^a(A'(x)))^2 \tag{7.4}$$

is non-negative. From Eq. (7.3) one finds that when $g^2$ decreases the minima of $V_g$ tend to separate and the potential barrier between them tends to increase. The quadratic part of $V_g$ around the minima is left unchanged, whereas terms of higher degree go to zero. Therefore, as $g^2$ goes to zero, one obtains an ensemble of infinitely far separated "free" theories, each one defined by the quadratic part of the $g^2$ expansion around the minima of $V$.

In this limit, the vacuum wave function will be approximated in each region of the configuration space by the ground state wave function of the free theory constructed around the stationary points. For this reason we are interested in the static stationary points of the theory, i.e. time-independent field configurations which are minima of the energy (euclidean lagrangian). Actually, one easily realizes that physically static configurations are not necessarily described by time-independent fields. Any time-dependent field, in which a time translation can be reabsorbed by a gauge transformation, represents a physically static situation. This possibility gives origin to new interesting non-trivial solutions.

Indeed, let us consider the field

$$A_i(x) = e^{-i\phi(x)t}B_i(x)e^{i\phi(x)t} + ie^{-i\phi(x)t}\nabla_i e^{i\phi(x)t}. \tag{7.5}$$

One has

$$\dot{A}_i(x) = e^{-i\phi(x)t}(-D_i(B)\phi(x))e^{i\phi(x)t}. \tag{7.6}$$

The field defined in (7.5) is not static, because it depends explicitly on time, however the time dependence is just of gauge type, and therefore it represents a physically static situation. For this field the euclidean lagrangian is

$$L_E = \frac{2}{g^2} \operatorname{Tr} \int d^3x (\tfrac{1}{2}(D_i(B)\phi)^2 + \tfrac{1}{4} G_{ij}^2(B)) \tag{7.7}$$

and the stationarity conditions are:

$$D^2(B)\phi(x) = 0 \tag{7.8}$$

$$i[D_i(B)\phi, \phi] = D_j(B)G_{ji}(B). \tag{7.9}$$

Eq. (7.8) represents the Gauss Law

$$D_i(A)\dot{A}_i = 0 \tag{7.10}$$

and Eq. (7.9) corresponds to the euclidean equation of motion

$$\ddot{A}_i = -D_j(A)G_{ji}(A) \tag{7.11}$$

for the field $A_i$ given in Eq. (7.5).

Note that Eq. (7.11) differs from the true (or minkowskian) equations of motion by a minus sign. This means that a field (7.5) with non-zero $\phi(x)$, which satisfies the equations (7.8) and (7.9), does not represent a solution of the minkowskian equations of motion[3]. If we require $L_E$, Eq. (7.7), to be finite, the behaviour of $\phi(x)$ when $|x| \to \infty$ is limited by

$$\phi(x) \xrightarrow[|x|\to\infty]{} kn(\theta, \varphi) \tag{7.12}$$

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[3] The reason is that the stationary points of the action are not in general stationary points of the euclidean action. On the other hand, if we are interested on the vacuum structure, the relevant configurations are those which minimize the euclidean action, as one can also see, by considering the expression

$$\langle A_i' \mid e^{-HT} \mid A_i \rangle \xrightarrow[T\to\infty]{} e^{-E_0 T}\Psi_0[A_i']\Psi_0^*[A_i].$$

where $k$ is some dimensional constant and $n^a(\theta, \varphi)$ is a unit vector which can depend on the angular variables $\theta$ and $\varphi$. Then, we can define an integer number $m \in \mathbf{Z}$ [18], [19]:

$$m = \frac{1}{4\pi k} \int d^3x \varepsilon_{ijk} \operatorname{Tr} [D_i(B)\phi G_{jk}(B)] \tag{7.13}$$

which represents the number of times the unit vector $n^a(\theta, \varphi)$ covers the unit sphere in the space of internal numbers $a$ when we moove on the sphere at $|\boldsymbol{x}| = \infty$. The expression (7.7) can be written:

$$L_{\mathrm{E}} = \frac{1}{g^2} \int d^3x \operatorname{Tr} (D_i(B)\phi \mp \tfrac{1}{2} \varepsilon_{ijk} G_{jk}(B)^2 \pm \frac{1}{g^2} \int d^3x \varepsilon_{ijk} \operatorname{Tr} (D_i(B)\phi G_{jk}(B)) \tag{7.13}$$

and, since $L_{\mathrm{E}}$ is positive, we derive, for fixed $k$ and $m$, a lower bound for $L_{\mathrm{E}}$ [19]:

$$L_{\mathrm{E}} = \frac{4\pi}{g^2} k|m|. \tag{7.14}$$

If $G = \mathrm{SU}(2)$, a solution with $m = 1$ is [19]:

$$B_i^a(\boldsymbol{x}) = \varepsilon_{aij} \frac{x_j}{\boldsymbol{x}^2} \left(1 - \frac{k|\boldsymbol{x}|}{\operatorname{sh} k|\boldsymbol{x}|}\right)$$

$$\phi^a(\boldsymbol{x}) = \frac{x^a}{\boldsymbol{x}^2} (1 - k|\boldsymbol{x}| \coth k|\boldsymbol{x}|)$$

and it is called a monopole solution, since it corresponds also to a time independent solution of the minkowskian equations of the motion of a non-abelian gauge theory with a triplet of scalar fields [18].

The physical consequences of the existence of these non-trivial stationary points (and of approximate stationary points corresponding to far separated monopole solutions) are not completely known. We mention the works of S. L. Adler [20] on the possible effects of these solutions in the computation of the static potential between non-abelian charges.

## 8. *Concluding remarks*

The formalism, that we have used to quantize the gauge fields, is not manifestly Lorentz-covariant. In order to produce explicitly the physical content of gauge theories, we have eliminated the superfluous degrees of freedom.

It may be useful to see why it is not possible to give a covariant description of physical states. Let us consider an abelian gauge theory. The one-particle states, which we can describe with a vector field $A_\mu$, are characterized by the momentum value $k_\mu$ and by a polarization vector $\varepsilon_\mu$.

For massless particles we have

$$k_\mu k^\mu = 0 \tag{8.1}$$

and of the four possible polarization states only two correspond to physical states. It is possible to find two invariant subspaces $G_1$ and $G_2$ of the space $G$ of state vectors. $G_1$ is characterized by

$$\varepsilon_\mu \propto k_\mu \tag{8.2}$$

(longitudinal polarizations), and $G_2$ by

$$k_\mu \varepsilon^\mu = 0 \tag{8.3}$$

(longitudinal and transverse polarizations). From Eq. (8.1) one gets that $G_1 \subset G_2$, and physical polarization states belong to $G_2$, but they have no components in $G_1$.

However the representation, even if reducible, is not decomposable. With a Lorentz transformation we can transform a transverse polarization state in a linear combination of a transverse and a longitudinal polarizations. Therefore physical states cannot be selected in a covariant way.

Actually, we can use a manifestly covariant formalism, but then we must introduce a non-definite metric in the space of state vectors.

In covariant gauges (described by L. Bonora in his lectures at this School) one can also prove the renormalizability of the theory, whereas in a physical gauge the renormalizability is not manifest.

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