ON THE SPHERICALLY SYMMETRIC SOLUTIONS OF THE YANG-MILLS EQUATIONS WITH SOURCES

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Spherically symmetric potentials were investigated. It was shown that spherical symmetry excludes nonabelian solutions of Yang-Mills SU(2) equations when certain hypotheses about their sources are supposed.

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1. *Introduction*

The aim of this paper is to study some nonabelian field configurations. The basic problems which are raised below are the following: "What can be said about solutions of Yang-Mills equations with spherically symmetric scalar sources $j^a_\mu = \delta_{\mu 0}\varrho^a(r)$?" In particular, "Are the symmetry properties of a solution in any sense related to its gauge characteristics?" (i.e. "Could the spherically symmetric solution be nonabelian?"). These rather technical questions will find an interesting use.

We will not discuss the relevance of the classical Yang-Mills theory to physical reality, but we may hope that if there is some connection, it can be realized in the simplest possible manner. Because of that higher symmetric configurations seem to be interesting. There is also another reason for exploring them. As is known, physical processes are often associated with the destruction of symmetry of interacting objects. One can ask whether something of the kind could happen in the nonabelian gauge theory. The answer needs the investigation of interacting gauge and matter fields, but even the simplified picture including only self-interacting gauge fields is abounding with curious phenomena. This question is shortly discussed at the end of my article. Now I want only to stress that its rather surprising solution is based on those technical results which are elaborated in Section 2.

The formulation of the spherical symmetry of gauge fields has a relatively long history in physical literature (see, e. g., [1] and references therein). There appear some

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difficulties connected with the gauge degrees of freedom. As is well known, a potential A^a_μ transforms under gauge rotation as follows

$$A^g = gAg^{-1}+e(\nabla g)g^{-1}, \qquad A^g_0 = gA_0g^{-1}+e(\partial_0 g)g^{-1}.$$

Recall that usually the spherical symmetry of geometrical quantities such as scalars or vectors implies $£A_0 = 0$, $£\boldsymbol{A} = \boldsymbol{0}$ where $A_0, \boldsymbol{A}$ are scalars and vectors respectively, and $£$ denotes the Lie derivative. But now it can happen that $A_0, \boldsymbol{A}$ and $\boldsymbol{B}_0, \boldsymbol{B}$ are related by a certain gauge transformation and $£A_0 = 0$, $£\boldsymbol{A} = \boldsymbol{0}$, while $£\boldsymbol{B}_0 \neq 0$, $£\boldsymbol{B} \neq \boldsymbol{0}$. Therefore the usual definition in terms of Lie derivatives demands some modifications.

This is extended in Section 2. It is accomplished by means of Def. 1 and Def. 2. The second definition includes more general configurations than the first. Our main theorem (Corollary) states that for sources such as given above and under some additional requirements, a spherically symmetric solution of the Yang-Mills SU(2) equations must be abelian and it is "naively" symmetric (i.e. in the sense of Def. 1). I should note that theorems 1–4 hold for general semisimple compact gauge groups.

2. *Main results*

At first, we will present two various definitions of spherical symmetry of Yang-Mills potentials. One of them is a straightforward extension of the usual concept of spherical symmetry. It may be called "naive", since it does not include certain known field configurations, which are covered by the second (usually used in later literature) definition. In our notation the upper (isospin) indices change from one to the dimension of a group algebra $\boldsymbol{G}$ and the lower (space) Latin indices change from 1 to 3, while the lower Greek space-time indices take on values from 0 to 3.

Definition "naive" 1. A potential A^a_μ is said to be spherically symmetric if it is gauge equivalent to the one with

$$A^a_0 = f^a(r), \qquad A^a_i = x_i F^a(r).$$

Definition "general" 2. A potential A^a_μ is said to be spherically symmetric if its space rotation could be compensated by a gauge transform, i.e., on the Lie algebra level:

$$£A^a_i = \partial_i\chi^a + ef^a_{bc}A^b_i\chi^c, \qquad £A^a_0 = \partial_0\chi + ef^a_{bc}A^b_c\chi^c$$

where $\chi \in \boldsymbol{G}$, f^a_{bc} — the structure constants of $\boldsymbol{G}$ and $£$ denotes the Lie derivative [2, 3 and references therein].

It is a simple exercise to show that "naively" symmetric configurations satisfy the requirements of the general definition.

We now establish some properties of potentials satisfying these definitions. They are formulated in the sequence of following theorems.

Theorem 1. Suppose that the potential A^a_μ is "naively" spherically symmetric. Then its vector part is a pure gauge one, i.e.

$$A^a_i = e(\partial_i g)g^{-1} \tag{1}$$

for some $g \in \boldsymbol{G}$, and its field strength tensor F^a_{ij} vanishes.

Proof: Let B_i be a gauge copy of A_i having the form $B_i^a(\vec{x}) = x_i B^a(r)$. One easily finds that

$$F_{ij}^a = \partial_i B_j^a - \partial_j B_i^a + e f_{bc}^a B_i^b B_j^c = 0. \tag{2}$$

But this is the integrability condition for the equation

$$B_i = e(\partial_i \boldsymbol{g})\boldsymbol{g}^{-1} \tag{3}$$

hence a solution $\boldsymbol{g}$ of (3) exists. The group valued function g of (1) is now given by $g = g_0 \boldsymbol{g}$, where g_0 transforms A_i^a to B_i^a.

Theorem 2. The potential A_μ^a with nonzero magnetic components F_{ij}^a could not be spherically symmetric in the meaning of Def. 1.

Proof: A potential with $F_{ij}^a \neq 0$ cannot be gauged to the form $\boldsymbol{x}f(r)$. This follows directly from Th. 1.

Theorem 3 (An algebraic criterion for "naive" nonsphericality). A continuous potential A_i^a satisfying $x_i A_i^a = 0$ is not "naively" spherically symmetric for compact semisimple gauge groups.

Proof: Suppose the contrary, i.e. let there exist a once differentiable g such that

$$g A_i g^{-1} - e(\partial_i g) g^{-1} = x_i f(r). \tag{4}$$

Multiplying (4) by $\boldsymbol{x}$ one gets

$$-r\left(\frac{\partial}{\partial r} g\right) g^{-1} = r^2 f(r). \tag{5}$$

The general solution of (5) has the form $g = g_1(r) g_0(v, \varphi)$, but the continuity of g at $r = 0$ demands $g_0(v, \varphi) = \text{const}$. Hence g is independent of angles. But if it is so, then multiplying (4) vectorially by $\boldsymbol{x}$ one obtains

$$g \boldsymbol{x} \times \boldsymbol{A} g^{-1} = 0 \tag{6}$$

which contradicts our assumption.

Let us now recall the static version of the Yang-Mills equations with sources j_μ^a:

$$\partial_i F_{i0} + e[A_i, F_{i0}] = j_0$$
$$\partial_i F_{ik} + e[A_0, F_{k0}] = j_k, \tag{7}$$

where $F_{i0}^a = \partial_i A_0^a + e f_{bc}^a A_i^b A_0^c$.

We prove the following, assuming that the gauge group is semisimple.

Theorem 4. Suppose that the sources are purely scalar, $j_\mu^a = \delta_{\mu 0} j^a(r)$, continuous and nonvanishing everywhere (except for isolated points). Then the Yang-Mills twice differentiable static potentials with a nonabelian holonomy group cannot be spherically symmetric in the sense of Def. 1.

Note. The algebra of a holonomy group of a solution is spanned by its strength field tensor $F_{\mu\nu}$ together with the covariant derivatives of $F_{\mu\nu}$ of all orders [1].

A potential is said to be abelian if all commutators of above objects vanish, i.e., the holonomy group is abelian; otherwise it is called nonabelian. A solution of the type

$A^a_\mu = \delta_{\mu 0}\delta_{a1}A(\vec{x})$ is abelian in this sense, of course. It should be remarked that such a determination is gauge invariant.

Proof: We show at first that for statics a nonabelian solution must have nonzero vector components (modulo gauge transformations) with $F^a_{ik} \neq 0$. Indeed, if we suppose the contrary, then the following may be established. From the continuity current equation

$$\partial_\mu j^a_\mu + e f^a_{bc} A^b_\mu j^c_\mu = 0 \tag{8}$$

one concludes that

$$f^a_{bc} A^b_0 j^c_0 = 0 \tag{9}$$

(recall that $j^a_i = 0$). This apparently implies that A_0, j_0 must be parallel (or belong to the Cartan subalgebra of G), because $j^a_0 \neq 0$. Furthermore, their isospin direction must be constant (or lie within the same Cartan subalgebra) throughout all space. (Although Eqs. (9) allow for A_0, j_0 parallel and with isospin directions different at neighbouring points, the regularity assumptions suppress that.) Hence a solution is abelian for vanishing vector components. But on the other hand, we know from Th. 2 that a potential with $F_{ik} \neq 0$ could not be spherically symmetric. Thus we see that spherical symmetry (in the "naive" sense) excludes nonabelian field configurations, and that ends our proof.

We show now that this fact holds also for "generally" symmetric gauge fields. Let the gauge group henceforth be SU(2). Then the "generally" symmetric potentials, satisfying in addition the gauge $x_i A_i = 0$ (we see from Th. 3 that such fields are not included in those covered by Def. 1) are [3]

$$A^a_i = \varepsilon_{aij} x_j V(r, t) + \delta_{ia} S(r, t) + x_i x_a T(r, t)$$

$$S + r^2 T = 0$$

$$A^a_0 = x_a U(r, t) \tag{10}$$

(this is proven by Gu and Hu [3] to be the most general spherical form, but similar ansatzes were used earlier).

Theorem 5. Suppose that the vector part of the sources vanishes, while their time component is $j^a_0(\vec{x}) = j^a(r)$, and vanishes nowhere (except at isolated points). Let the gauge group be SU(2). Then there are no static ("generally") spherically symmetric solutions.

Proof: After inserting (10) into (8) (where $f^a_{bc} = \varepsilon_{abc}$, the Levi-Civita antisymmetric tensor) one gets

$$\varepsilon_{abc} x_b U(r) j^c(r) = 0. \tag{11}$$

Hence $U = 0$, which contradicts our assumption, since then from (7) follows $j^c(r) = 0$.

As a direct consequence of Th. 4 and Th. 5 we obtain

Corollary. Under the hypotheses as in Th. 5, Eqs. (7) (with suitable boundary conditions) can possess only "naively" spherically symmetric solutions, which are abelian; the nonabelian solution, if occurs, must have a lower symmetry.

3. *Application*

It has been found previously that for sufficiently strong sources (or for sufficiently large values of the coupling constant e) from an abelian solution bifurcates another nonabelian [4]. Suppose that the original abelian potential is spherically symmetric (our Corollary asserts that under suitable hypotheses it is "naively" symmetric). Hence we can conclude that bifurcation is always related with symmetry breaking, because:

i) a bifurcating solution must be nonabelian

ii) a nonabelian potential cannot be spherically symmetric, in view of the Corollary.

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