A MODEL OF RELATIVISTIC QUANTUM MECHANICS

BY A. HERDEGEN

Institute of Physics, Jagellonian University, Cracow*

(Received May 4, 1982)

A model of relativistically invariant quantum mechanics of spin 0 and spin 1/2 particles is proposed. An invariant evolution parameter is employed, whereas both physical time and the mass of the particle become dynamical quantities. The equations of motion of physical observables are obtained, which correspond to classical formulae.

PACS numbers: 11.10.Qr

1. Introduction

The aim of this article is to present a possible relativistic quantum theory of a particle (spinless or with spin 1/2) which is acted upon by an external electromagnetic field. It is often argued, that in relativistic quantum physics any interaction is bound to lead to pair production and therefore the employment of the quantum field theory cannot be avoided. However we point out that there do exist certain experimental situations in which pair production does not take place (or can be neglected) but some relativistic effects are detectable (e.g. in the hydrogen atom). It should be noted here that the usual relativistic wave equations (such as the Klein-Gordon or Dirac equation) though acceptable from the point of view of the relativistic field theory, do not solve the problem of constructing a relativistically invariant quantum mechanics of systems with a finite number of degrees of freedom (such as a particle in an external field). This is because there does not exist a Hilbert space and unitary representation of the Poincare group acting on that space such that the solutions of the wave equation would be defined in that space and observables would act within that space. The quantum mechanical averages of observables do not, in consequence, have proper relativistic transformation properties. This difficulty is removed in the present theory, whereas in other aspects it remains as similar as possible to the usual theory.

In the present article we assume that the lack of relativistic invariance of the usual theory is caused by the inclusion of the evolution parameter (the role of which is played there by physical time) into the transformations of relativity group. Thus following Stue-

* Address: Instytut Fizyki, Uniwersytet Jagielloński, Reymonta 4, 30-059 Kraków, Poland.

ckelberg [1] and many others we introduce an additional evolution parameter τ. Such an invariant parameter was used by many authors, but originally for rather technical purposes [2]. The form of the generator of motion introduced by Stueckelberg was used by Horwitz and Piron [3] to construct their model of the relativistic quantum particle (see also [4]).

Our next assumption is that in the relativistic physics the mass should have dynamical meaning rather than be a fixed constant. Starting with the classical Lagrange formalism and using those two ideas we obtain the classical theory in which the Hamiltonian is the square root of that of Stueckelberg [1] and of Horwitz and Piron [3]. In the classical case the two Hamiltonians generate the same trajectories (with τ, however, only in our theory being always the proper time parameter, irrespectively of the initial conditions), but in the quantum case the equations of motion in the two theories are not equivalent. It seems to us that equations obtained in the present model provide a better physical interpretation, especially in view of the fact that in the quantum case the mass is not defined sharply, so that it should appear in equations of motion as an operator (in [3] and [4] the mass eigenstates are used, but these do not belong to the assumed Hilbert space). We only mention that our model differs also in some other respects from that of Horwitz, Piron and Reuse (especially in the spin 1/2 case). See also Note added in proof.

The general structure of the present article is as follows: in Section 1 we show how furnishing mass with dynamics leads to our Hamiltonian; the canonical quantisation of the classical formulae is performed in Section 2 and in Section 3 we construct the quantum theory more explicitly. It is also shown that the theory is invariant under physical time reflections. In Section 4 the charge conjugation and the free particle solutions are studied. Some other properties of the proposed model will be included in another publication.

2. *Classical particle*

The classical action of a charged particle placed in an external electromagnetic field

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

is usually taken to be[1] [5]:

$$S = -\int \left(mc\sqrt{\dot{z}^2} + \frac{e}{c} A(z)\cdot \dot{z} \right) d\tau. \tag{2.1}$$

Here m, e, z^μ denote the mass, charge, and position of the particle respectively, c stands for the velocity of light, τ is an evolution parameter and the dot represents differentiation with respect to τ. This form of action is a homogeneous functional of $\dot{z}^\mu$ of degree $+1$, which ensures the invariance of the equations of motion with respect to the change of parametrization. However, if we introduce the canonical momentum fourvector

$$p_\mu = -\frac{\delta S}{\delta z^\mu} = mc\frac{\dot{z}_\mu}{\sqrt{\dot{z}^2}} + \frac{e}{c} A_\mu(z),$$

[1] We use the metric tensor $g_{\mu\nu}$ with signature $(+1, -1, -1, -1)$ and denote $g_{\mu\nu}a^\mu b^\nu = a\cdot b$, $g_{\mu\nu}a^\mu a^\nu = a^2$.

its components turn out not to be independent, satisfying the mass-shell relation:

$$\left(p - \frac{e}{c} A(z)\right)^2 = m^2 c^2. \tag{2.2}$$

This constraint prevents the covariant Poisson bracket relations from being applicable on that level. On the other hand m is here a constant parameter rather than a dynamical quantity, as we would prefer it to be. We shall now show, that the constraint on p_μ can be removed if we give the mass dynamical status. For this end let us observe that, with m still being a constant, the proper equations of motion can be obtained with use of the Lagrangian:

$$\mathscr{L}(m, z, \dot{z}) = \tfrac{1}{2} m(\dot{z}^2 - c^2) + \frac{e}{c} A(z) \cdot \dot{z}, \tag{2.3}$$

if after evaluation of the Lagrange equations the additional condition

$$\dot{z}^2 = c^2 \tag{2.4}$$

is imposed, giving the parameter τ the interpretation of proper time. If we now let m be an additional generalised coordinate, then the condition (2.4) becomes the Lagrange-Euler equation with respect to this coordinate. The remaining Lagrange-Euler equations read:

$$\dot{m} = 0,$$

$$m\ddot{z}^\mu = \frac{e}{c} F^{\mu\nu} \dot{z}_\nu, \tag{2.5}$$

where (2.4) and the antisymmetry of $F^{\mu\nu}$ have been used. Lagrangian (2.3) can be now, with help of the additional variables $p_\mu = \dfrac{\partial \mathscr{L}}{\partial \dot{z}^\mu} = m\dot{z}_\mu + \dfrac{e}{c} A_\mu(z)$, reduced to a first order Lagrangian with one solvable constraint (see e.g. [6]):

$$\mathscr{L}_1(m, z, \dot{z}, p) = p \cdot \dot{z} - p \cdot \dot{z}(z, p, m) + \mathscr{L}(m, z, \dot{z}(z, p, m))$$

$$= p \cdot \dot{z} - \frac{1}{2m} (\pi^2 + m^2 c^2) \equiv p \cdot \dot{z} - \mathscr{H}(z, p, m), \tag{2.6}$$

where $\mathscr{H}$ is the Hamiltonian and $\pi_\mu \equiv p_\mu - \dfrac{e}{c} A_\mu$ is the mechanical momentum of the particle. The equations of motion are:

$$\dot{z}^\mu = \frac{\partial \mathscr{H}}{\partial p_\mu} = \frac{\pi^\mu}{m},$$

$$\dot{p}_\mu = -\frac{\partial \mathscr{H}}{\partial z^\mu} = \frac{e}{c} \partial_\mu A_\nu \frac{\pi^\nu}{m}, \tag{2.7}$$

and

$$0 = \frac{\partial \mathscr{H}}{\partial m} = \tfrac{1}{2}\left(c^2 - \frac{\pi^2}{m^2}\right) \tag{2.8}$$

is a primary constraint equation. It is a consequence of the equations (2.7) that π^2 is a constant of motion, on which (2.8) imposes the condition:

$$\pi^2 > 0. \tag{2.9}$$

The constraint equation (2.8) can be solved for m, and inserting $m(z, p)$ in $\mathscr{H}$ leads to:

$$c^2\mathscr{M}(z, p) \equiv \mathscr{H}(z, p, m(z, p)) = c^2 m(z, p) = c\sqrt{\pi^2}, \tag{2.10}$$

where $+1$ overall sign has been chosen. The equations of motion obtained with $c^2\mathscr{M}$ as a Hamiltonian are again (2.7) [6], and can be given the final form:

$$\pi^\mu = m\dot{z}^\mu,$$

$$m\ddot{z}^\mu = \frac{e}{c} F^{\mu\nu}\dot{z}_\nu, \tag{2.11}$$

where $m \equiv \frac{1}{c}\sqrt{\pi^2}$ = constant of motion. The condition (2.9) can be reexpressed as:

$$\mathscr{M}^2 > 0. \tag{2.12}$$

Note that (2.4) is obtained as a consequence, so that τ is the proper time parameter. In view of (2.10) and (2.11) the generator of the evolution $\mathscr{M}$ takes on any trajectory the value of the mass of the particle moving along this trajectory.

The Lagrangian $\mathscr{L}_1$ ensures that the Poisson bracket relations together with the equations of motion have the form:

$$\{z^\mu, p_\nu\} = \delta^\mu_\nu, \quad \{z^\mu, z^\nu\} = \{p^\mu, p^\nu\} = 0,$$

$$\dot{q} = c^2\{q, M\}, \quad (q = z^\mu, p_\nu), \tag{2.13}$$

which allows the canonical quantization procedure to be applied, if only $\mathscr{M}$ can be given quantum meaning.

3. *Canonical quantization*

In the quantum case z^μ, p_ν are replaced by selfadjoint operators acting in some Hilbert space and it does not present any difficulty to define a selfadjoint, at least on the formal level, operator $c^2\mathscr{M}^2 = \left(p - \frac{e}{c}A(z)\right)^2$, in correspondence with the classical case. As explained in the Introduction, it seems to us preferable to use the generator obtained in Section 1, so that the square root of $\mathscr{M}^2$ should be taken. This cannot, however, be accomplished without performing some modifications of the quantum mechanical rules ($\mathscr{M}^2$ is not positive definite, so that using the spectral theorem [7] to this end would lead to a non-self-adjoint operator). Having agreed to any such modifications we prefer to proceed differently.

With z^μ, p_ν retaining for the moment their classical character, we shall linearize $\sqrt{\pi^2}$ in the two simplest ways by adding variables which will, after quantization, represent the charge degree of freedom in one case, and the charge and spin in the other. Namely we put

$$c\mathcal{M}_0 = \frac{1}{\varrho c}\alpha\pi^2 + \varrho c\beta, \qquad c\mathcal{M}_{1/2} = \gamma^\mu\pi_\mu \tag{3.1}$$

and demand

$$c^2\mathcal{M}_s^2 = \pi^2, \qquad (s = 0, 1/2). \tag{3.2}$$

The two values of the subscript s are those of the particle's spin in the two respective cases, as will be seen below[2]. All matrices are dimensionless and ϱ is an arbitrary mass scale needed in the "0" case. The condition (3.2) will be fulfilled if the matrices satisfy:

$$\alpha^2 = \beta^2 = 0, \qquad [\alpha, \beta]_+ = \mathbf{1}, \tag{3.3}$$

$$[\gamma^\mu, \gamma^\nu] = 2g^{\mu\nu}\cdot\mathbf{1}. \tag{3.4}$$

Thus γ's form the algebra of the Dirac matrices, and the linearization method of the "1/2" case is that of Dirac. We choose for γ's, as usual, 4×4 unitary matrices, so that γ^0 is hermitian and γ^i $(i = 1, 2, 3)$ are antihermitian:

$$\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0. \tag{3.5}$$

The method of linearization in the "0" case parallels that of Feshbach and Villars [8] used for reducing the Klein-Gordon equation to the Schrödinger form. Algebra (3.3) can be transformed to another form. Namely, if we define

$$\Gamma^3 = \alpha+\beta, \qquad \Gamma^2 = -i(\alpha-\beta), \qquad \Gamma^1 = \frac{i}{2}[\Gamma^3, \Gamma^2] = [\beta, \alpha] \tag{3.6}$$

then Γ's satisfy

$$\Gamma^i\Gamma^k = \delta^{ik} + i\varepsilon^{ikl}\Gamma^l, \qquad (i, k, l = 1, 2, 3), \tag{3.7}$$

the rules of the Pauli matrices algebra. We choose Γ's as 2×2 hermitian matrices:

$$\Gamma^{i\dagger} = \Gamma^i. \tag{3.8}$$

In terms of α, β (3.8) reads:

$$\beta = \alpha^\dagger. \tag{3.8'}$$

[2] The subscript s will also appear with other symbols, but whenever it is omitted it will always mean that the given formula holds for both spin cases.

On now applying the heuristic rules of canonical quantization to (2.13), the following formal commutator relations and Heisenberg picture equations of motion are obtained:

$$[p_\mu, z^\nu] = i\hbar\delta^\nu_\mu, \quad [z^\mu, z^\nu] = [p^\mu, p^\nu] = 0,$$
$$[z^\mu, \Gamma^i] = [p^\mu, \Gamma^i] = 0, \quad [z^\mu, \gamma^\nu] = [p^\mu, \gamma^\nu] = 0, \tag{3.9}$$
$$\dot{q} = \frac{ic^2}{\hbar}[q, \mathcal{M}],$$

where q is any of the observables in the Heisenberg picture (here the substitution $\{,\} \to \frac{i}{\hbar}[,]$ has been performed). The equations of motion are formally solved by:

$$q(\tau) = e^{-\frac{ic^2}{\hbar}\mathcal{M}\tau} q(0) e^{+\frac{ic^2}{\hbar}\mathcal{M}\tau}, \tag{3.10}$$

so that in the Schrödinger picture we have the following equations for trajectories in Hilbert space:

$$-\frac{i\hbar}{c^2}\frac{\partial}{\partial\tau}\Phi(\tau) = \mathcal{M}\Phi(\tau). \tag{3.11}$$

(The analog of the condition (2.12) we leave for later consideration — see next section.)

However, in the Hilbert spaces defined in the next section (see (4.5)) neither of $\mathcal{M}_s$ is selfadjoint, which compels us to introduce the so called "indefinite metric" [9].

4. *Quantum rules, Poincaré invariance and dynamics*

We shall now construct the quantum theory in the Schrödinger picture more explicitly. z^μ will now be the operators of multiplication by x^μ in $L^2(\mathbf{R}^4, d^4x)$. In this Hilbert space acts the unitary representation of the proper Poincaré group $\mathcal{P}$, defined by

$$[U(P)\Phi](x) = \Phi(\Lambda^{-1}(x-a)), \quad (P \equiv (\Lambda, a) \in \mathcal{P}). \tag{4.1}$$

The selfadjoint generators of this representation will be denoted by $P^\mu, L^{\mu\nu}$ and the conventions concerning signs and factors are fixed by:

$$U(\Lambda, a) = e^{\frac{i}{\hbar}a_\mu P^\mu - \frac{i}{2\hbar}\varepsilon_{\mu\nu}L^{\mu\nu}} \tag{4.2}$$

for infinitesimal Poincaré transformation, where $\Lambda^\mu{}_\nu = \delta^\mu_\nu + \varepsilon^\mu{}_{.\nu}$, $\varepsilon_{\mu\nu} = -\varepsilon_{\nu\mu}$. These generators act on differentiable functions from L^2 according to:

$$P^\mu = i\hbar\partial^\mu, \quad L^{\mu\nu} = z^\mu P^\nu - z^\nu P^\mu. \tag{4.3}$$

For the canonical momentum p_μ we set $p_\mu = P_\mu$, in agreement with (3.9). The transformation properties of z^μ, p^ν are:

$$U^{-1}(\Lambda, a) z^\mu U(\Lambda, a) = \Lambda^\mu{}_\nu z^\nu + a^\mu,$$
$$U^{-1}(\Lambda, a) p^\mu U(\Lambda, a) = \Lambda^\mu{}_\nu p^\nu. \tag{4.4}$$

For the physical Hilbert spaces we now take:

$$H_0 = \mathbf{C}^2 \otimes L^2(\mathbf{R}^4, d^4x), \qquad H_{1/2} = \mathbf{C}^4 \otimes L^2(\mathbf{R}^4, d^4x). \tag{4.5}$$

In the Schrödinger picture all operators Γ^i, γ^μ are matrix operators acting in $\mathbf{C}^2$ and $\mathbf{C}^4$ respectively[3] and satisfying (3.3)–(3.8′).

Our definition of the observable and of quantum mechanical average will be implicated by the properties of the representations of the proper Poincaré group which act in H_s and by a specific form of generators $\mathcal{M}_s$, thus we turn now to these questions. In H_s we represent the proper Poincaré group by:

$$V_0(P) = \mathbf{1} \otimes U(P),$$
$$V_{1/2}(P) = S(\Lambda) \otimes U(P), \tag{4.6}$$

where $S(\Lambda)$ is the bispinor representation of the proper Lorentz group. With the definition of generators:

$$S^{\mu\nu} = \frac{i\hbar}{2} \gamma^{[\mu}\gamma^{\nu]} = \frac{i\hbar}{4} [\gamma^\mu, \gamma^\nu], \tag{4.7}$$

there is

$$S(\Lambda) = e^{-\frac{i}{2\hbar}\varepsilon_{\mu\nu}S^{\mu\nu}} \tag{4.8}$$

for an infinitesimal transformation as in (4.2). $S(\Lambda)$ has the following well-known properties:

$$S^{-1}(\Lambda)\gamma^\mu S(\Lambda) = \Lambda^\mu{}_\nu \gamma^\nu, \tag{4.9}$$

$$S(\Lambda)^\dagger \gamma^0 = \gamma^0 S^{-1}(\Lambda). \tag{4.10}$$

(The dagger denotes the matrix conjugation. For conjugation with respect to the scalar product in Hilbert space we reserve the sign*.) The representations V_s will be interpreted in a passive way: if $\Phi_\Sigma \in H$ describes some physical state as viewed from the inertial frame Σ, then

$$\Phi_{P^{-1}\Sigma} = V(P)\Phi_\Sigma \tag{4.11}$$

describes the same state as viewed from the inertial frame $P^{-1}\Sigma$. Dynamics is introduced by the Schrödinger-type equations (3.11) which admit the following continuity equations: $s = 0$:

$$\frac{\partial}{\partial\tau}(\Phi^\dagger \Gamma^3 \Phi) = -\frac{1}{\varrho}\partial_\mu[\Phi^\dagger\alpha^\dagger\alpha\pi^\mu\Phi + (\pi^\mu\Phi)^\dagger\alpha^\dagger\alpha\Phi], \tag{4.12}$$

$s = 1/2$:

$$\frac{\partial}{\partial\tau}(\Phi^\dagger\gamma^0\Phi) = -c\partial_\mu(\Phi^\dagger\gamma^0\gamma^\mu\Phi). \tag{4.13}$$

---

[3] We shall not differentiate in notation between operators A and $\mathbf{1}\otimes A$, and between B and $B\otimes\mathbf{1}$, whenever it does not lead to confusion.

These equations, after integration over x^μ variables, yield:

$$\frac{d}{d\tau}(\Phi, \eta\Phi) = 0, \tag{4.14}$$

where $\eta_0 \equiv \Gamma^3$, $\eta_{1/2} \equiv \gamma^0$, (.,.) denotes the scalar product in H. The operators η_s have two important properties:

$$\eta^\dagger = \eta^* = \eta, \quad \eta^2 = \mathbf{1}. \tag{4.15}$$

On introducing a sesquilinear form

$$\langle\Phi|\Psi\rangle = (\Phi, \eta\Psi) \tag{4.16}$$

we can state our quantum mechanical

Postulates (cf. [9, 8]).

I. Physical states are described by rays in the Hilbert spaces H_s, such that $|\langle\Phi|\Phi\rangle| = 1$ (though not all such rays are physically realizable — see IVth postulate below).

II. An observable is any operator q in the Hilbert space H, such that ηq is selfadjoint.

III. The quantum mechanical average of the observable q in the state Φ is defined by:

$$\langle q\rangle_\Phi = \frac{\langle\Phi|q\Phi\rangle}{\langle\Phi|\Phi\rangle}. \tag{4.17}$$

The following important consequences can be easily verified:

1. $\langle\Phi|\Psi\rangle$ is invariant under both the τ-evolution and the proper Poincaré transformations.

2. $z^\mu, p^\mu, L^{\mu\nu}, \gamma^\mu, S^{\mu\nu}$ are observables, whose quantum mechanical averages transform under the proper Poincaré group as their classical analogues (for γ^μ, $S^{\mu\nu}$ read: as vector and tensor). These transformations leave $\alpha, \alpha^\dagger$ unchanged.

3. Exactly as in the classical case an external field is not just one function (or 4 functions for vector field) but a set of functions transforming under a relativity group into each other in the prescribed manner, so that in the quantum case it is a set of operators with the transformation law:

$$\langle\Phi_{P^{-1}\Sigma}|A^\mu_{P^{-1}\Sigma}\Phi_{P^{-1}\Sigma}\rangle = \Lambda^\mu{}_\nu\langle\Phi_\Sigma|A^\nu_\Sigma\Phi_\Sigma\rangle, \tag{4.18}$$

so that if A^μ_Σ is the operator of multiplication by $A^\mu_\Sigma(x)$, then $A^\mu_{P^{-1}\Sigma}$ is the operator of multiplication by:

$$A^\mu_{P^{-1}\Sigma}(x) = \Lambda^\mu{}_\nu A^\nu_\Sigma(\Lambda^{-1}(x-a)). \tag{4.19}$$

4. Transformation law of $\mathcal{M}_\Sigma \equiv \mathcal{M}(A_\Sigma)$ which is an observable, is:

$$V^{-1}(P)\mathcal{M}_{P^{-1}\Sigma}V(P) = \mathcal{M}_\Sigma. \tag{4.20}$$

We conclude these remarks by stating the invariance of the theory under proper Poincaré transformations.

The specification of the physical states will be completed by the addition of the following

Postulate IV (dynamical). Physically realizable Schrödinger states are such solutions of (3.11) that $\Phi(\tau) \in H$ for all finite τ and that $\Phi(\tau, x)$ is a regular distribution on $\mathbf{C}^n \times S(\mathbf{R}^5)$ $(n = 2, 4)$.

The first part of this postulate is needed in view of the fact that $\mathcal{M}_s$ are not selfadjoint. The second part plays the role of the condition (2.12) in the quantum case. This is justified by the following considerations. In the quantum theory the classical formula $c^2\mathcal{M}^2 = \pi^2$ is replaced by:

$$c^2\mathcal{M}_0^2 = \pi^2, \quad c^2\mathcal{M}_{1/2}^2 = \pi^2 - \frac{e}{c} S^{\mu\nu}F_{\mu\nu}, \tag{4.21}$$

thus not only are $\mathcal{M}_s^2$ not positive definite, but $\mathcal{M}_{1/2}^2$ is also not (though $\mathcal{M}_0^2$ happens to be) selfadjoint. Moreover, there are no solutions in Hilbert spaces H_s to the equations

$$\mathcal{M}_s\Phi = m\Phi. \tag{4.22}$$

It can be demanded, nevertheless, that physically realizable states be some superpositions of functions from another space satisfying (4.22). Heuristically, if $\Phi \in H$ then $\Psi_m = \int e^{-\frac{ic^2}{\hbar}m\tau} e^{\frac{ic^2}{\hbar}\mathcal{M}\tau} \Phi d\tau$ is such a solution, if any meaning can be given to this expression. The latter, however, means that there exists the Fourier transform of $e^{\frac{ic^2}{\hbar}\mathcal{M}\tau}\Phi$ which in turn is identified as a solution of (3.11). Under these conditions $\Phi(\tau)$ is the Fourier transform, hence a superposition, of Ψ_m. Thus the meaning of the IVth postulate is clarified.

The quantum theory can be (at least formally) cast into the Heisenberg picture form (3.9) with help of the transformation (3.10). Quantum variables in that picture satisfy the following equations[4]:
in both spin cases:

$$\pi^\mu = \mathrm{Sym}\,(\mathcal{M}, \dot{z}^\mu),$$

$$\dot{\pi}^\mu = \frac{e}{c}\,\mathrm{Sym}\,(F^{\mu\nu}, \dot{z}_\nu),$$

$$\mathrm{Sym}\,(\mathcal{M}, \ddot{z}^\mu) = \frac{e}{c}\,\mathrm{Sym}\,(F^{\mu\nu}, \dot{z}_\nu), \tag{4.23}$$

in the case $s = 0$

$$\dot{z}^\mu = \frac{2}{\varrho}\alpha\pi^\mu,$$

$$\mathrm{Sym}\,(\mathcal{M}, \dot{\pi}^\mu) = \frac{e}{c}\,\mathrm{Sym}\,(F^{\mu\nu}, \pi_\nu), \tag{4.24}$$

---

[4] The operation of symmetrization Sym is defined by: $\mathrm{Sym}\,(A_1, \ldots, A_n) = \frac{1}{n!} \sum_{\substack{\text{permutations} \\ (i_1, \ldots, i_n)}} A_{i_1} \ldots A_{i_n}$, hence $\mathrm{Sym}\,(A, B) = \frac{1}{2}[A, B]_+$.

in the case $s = 1/2$

$$\dot{z}^\mu = c\gamma^\mu,$$

$$\mathrm{Sym}\,(\mathcal{M}, \dot{\pi}^\mu) = \frac{e}{c}\,\mathrm{Sym}\,(F^{\mu\nu}, \pi_\nu) + \frac{e}{2c}\, S^{\lambda\kappa}\partial^\mu F_{\lambda\kappa},$$

$$\mathrm{Sym}\,(\mathcal{M}, \dot{S}^{\mu\nu}) = \frac{e}{c}\,(F^{\mu}_{\cdot\lambda}S^{\lambda\nu} - F^{\nu}_{\cdot\lambda}S^{\lambda\mu}). \tag{4.25}$$

These equations of motion confirm the interpretation of $\mathcal{M}$ as the mass operator (in agreement with the classical case). They correspond, moreover, to the classical equations of motion and together with the transformation properties of states justify the interpretation of the subscript s as the spin value. In these equations τ is not, of course, the physical time, but a nonobservable parameter which in the classical limit will become the proper time of the particle. Physical time itself is an observable. If we denote $\frac{1}{c}\langle z^0\rangle(\tau) = f(\tau)$ then the physical time dependence of quantum mechanical averages will be given by[5] $\langle q\rangle(f^{-1}(t))$.

We end this section with the discussion of the space and time reflections. Representation (4.1) can be extended to include the reflections (R_S — space reflection and R_T — time reflection) also. In $H_{1/2}$ reflections are represented by:

$$V_{1/2}(R_S) = S_{1/2} \otimes U(R_S), \qquad V_{1/2}(R_T) = T_{1/2} \otimes U(R_T) \tag{4.26}$$

where $S_{1/2}$, $T_{1/2}$ are determined up to a factor[6] $b = \pm 1, \pm i$ [10] and $S_{1/2} = b_S\gamma^0$ satisfies (4.9), (4.10), whereas $T_{1/2} = b_T\gamma^0\gamma^5$ satisfies (4.9) and

$$T^\dagger_{1/2}\eta_{1/2} = -\eta_{1/2}T^{-1}_{1/2}. \tag{4.27}$$

In H_0 reflections are represented by $V_0(R_{S,T}) = \mathbf{1}\otimes U(R_{S,T})$. Discussion of space reflections does not introduce anything new, but in order to perform realistic physical time reflection it is necessary to employ the antilinear operators $V_s(R_T)K_s$ where K_s are the operators of charge conjugation (see next section). These operators induce the transformations:

$$(V_{1/2}K_{1/2})^{-1}\gamma^\mu V_{1/2}K_{1/2} = R^{\mu}_{T\nu}\gamma^\nu,$$

$$(V_0K_0)^{-1}\alpha V_0K_0 = -\alpha,$$

$$(VK)^{-1}p^\mu VK = -R^{\mu}_{T\nu}p^\nu, \tag{4.28}$$

---

[5] Having stated this, we do not need to assume any specific interpretation of extending the usual space $L^2(\mathbf{R}^3, d^3x)$ to $L^2(\mathbf{R}^4, d^4x)$.

[6] The requirement that the charge conjugation (see next Section) should not change the transformation properties of states leaves only two possible values of this factor: $b = \pm 1$.

and if we define $A_T{}^\mu$:

$$(VK)^{-1}A_T{}^\mu VK = -R^\mu_{T\nu}A^\nu, \tag{4.29}$$

then

$$(VK)^{-1}\mathcal{M}(A_T)VK = -\mathcal{M}(A). \tag{4.30}$$

If Φ is a solution of equation (3.11), then $\Phi_T = VK\Phi$ satisfies:

$$-\frac{i\hbar}{c^2}\frac{\partial}{\partial\tau}\Phi_T(\tau) = \mathcal{M}(A_T)\Phi_T(\tau). \tag{4.31}$$

The transformation properties of physical averages are

$$\langle\pi^0(A_T)\rangle_{\Phi_T} = \langle\pi^0(A)\rangle_\Phi, \quad \langle\vec{\pi}(A_T)\rangle_{\Phi_T} = -\langle\vec{\pi}(A)\rangle_\Phi \tag{4.32}$$

and

$$\langle\dot{z}^0\rangle_{\Phi_T} = -\langle\dot{z}^0\rangle_\Phi, \quad \langle\dot{\vec{z}}\rangle_{\Phi_T} = \langle\dot{\vec{z}}\rangle_\Phi,$$

$$\langle\mathcal{M}(A_T)\rangle_{\Phi_T} = -\langle\mathcal{M}(A)\rangle_\Phi. \tag{4.33}$$

The formulas (4.33) are equivalent to:

$$\frac{d}{d(-\tau)}\langle z^0\rangle_{\Phi_T} = \frac{d}{d\tau}\langle z^0\rangle_\Phi, \quad \frac{d}{d(-\tau)}\langle\vec{z}\rangle_{\Phi_T} = -\frac{d}{d\tau}\langle\vec{z}\rangle_\Phi,$$

$$\langle-\mathcal{M}(A_T)\rangle_{\Phi_T} = \langle\mathcal{M}(A)\rangle_\Phi, \tag{4.34}$$

so that the simultaneous change of signs of τ and $\mathcal{M}(A_T)$ (which does not alter the evolution equation (4.31)) leads to the proper transformation properties.

5. *Free particle*

We shall now apply the formalism of the preceding Sections to a free quantum particle. In this case physical solutions of equation (3.11) are easily found. Using postulate IV we can write symbolically:

$$\Phi(\tau, x) = \frac{1}{(2\pi)^{5/2}}\int e^{\frac{ic^2}{\hbar}m\tau - \frac{i}{\hbar}x\cdot k}F[\Phi](m, k)dmd^4k$$

$$= \frac{1}{(2\pi)^2}\int e^{-\frac{i}{\hbar}x\cdot k}\tilde{\Phi}(\tau, k)d^4k. \tag{5.1}$$

(The precise meaning of these expressions is given by the theory of distributions [11].) Applying $\left(-\frac{i\hbar}{c}\frac{\partial}{\partial\tau}\right)^2$ to $\Phi(\tau, x)$ and using (3.11) we obtain:

$$(m^2c^2 - k^2)F[\Phi](m, k) = 0. \tag{5.2}$$

It follows that $\theta(-k^2)F[\Phi](m, k) = 0$, which together with (5.1) yields:

$$\theta(-k^2)\tilde{\Phi}(\tau, k) = 0. \tag{5.3}$$

$\tilde{\Phi}(\tau, k)$ should satisfy the equation:

$$-\frac{i\hbar}{c^2}\frac{\partial}{\partial\tau}\tilde{\Phi}(\tau, k) = \mathcal{M}(k)\tilde{\Phi}(\tau, k) \tag{5.4}$$

$\left(\text{here } c\mathcal{M}_0(k) = \dfrac{k^2}{\varrho c}\alpha + \varrho c\alpha^\dagger,\ c\mathcal{M}_{1/2}(k) = \gamma^\mu k_\mu\right)$, which is readily solved by:

$$\begin{aligned}\tilde{\Phi}(\tau, k) &= (e^{\frac{ic}{\hbar}\sqrt{k^2}\,\tau}Q_+ + e^{-\frac{ic}{\hbar}\sqrt{k^2}\,\tau}Q_-)\tilde{\Phi}(0, k)\\ &\equiv \tilde{G}(\tau, k)\tilde{\Phi}(0, k),\end{aligned} \tag{5.5}$$

where

$$Q_\pm = \tfrac{1}{2}\left(\mathbf{1} \pm \frac{c\mathcal{M}(k)}{\sqrt{k^2}}\right). \tag{5.6}$$

We note the following properties of operators $Q_\pm$:

$$\begin{aligned}&Q_\pm Q_\pm = Q_\pm, \quad Q_+Q_- = Q_-Q_+ = 0\\ &Q_+ + Q_- = \mathbf{1}, \quad (\eta Q_\pm)^* = \eta Q_\pm,\\ &V^{-1}(P)Q_\pm V(P) = Q_\pm,\end{aligned} \tag{5.7}$$

where P is any orthochronous Poincaré transformation. In order to check the first part of postulate IV we display $\tilde{G}(\tau, k)^\dagger\tilde{G}(\tau, k)$ in a somewhat evaluated form:

$$\begin{aligned}\tilde{G}_0(\tau, k)^\dagger\tilde{G}_0(\tau, k) &= \tfrac{1}{2}\left(\mathbf{1} + \frac{k^2}{(\varrho c)^2}\alpha^\dagger\alpha\right)\\ &+ \tfrac{1}{2}\left(\mathbf{1} - \frac{k^2}{(\varrho c)^2}\alpha^\dagger\alpha\right)\cos\frac{2c}{\hbar}\sqrt{k^2}\,\tau + \frac{(\varrho c)^2}{2}\alpha\alpha^\dagger\,\frac{1-\cos\frac{2c}{\hbar}\sqrt{k^2}\,\tau}{k^2}\\ &+ i\frac{\varrho c}{2}(\alpha^\dagger - \alpha)\frac{\sin\frac{2c}{\hbar}\sqrt{k^2}\,\tau}{\sqrt{k^2}} + i\frac{\sqrt{k^2}}{\varrho c}(\alpha - \alpha^\dagger)\sin\frac{2c}{\hbar}\sqrt{k^2}\,\tau,\\ \tilde{G}_{1/2}(\tau, k)^\dagger\tilde{G}_{1/2}(\tau, k) &= \mathbf{1}\cos\frac{2c}{\hbar}\sqrt{k^2}\,\tau\\ &+ k_0\gamma^0 k_\mu\gamma^\mu\,\frac{1-\cos\frac{2c}{\hbar}\sqrt{k^2}\,\tau}{k^2} + ik_l\gamma^l\,\frac{\sin\frac{2c}{\hbar}\sqrt{k^2}\,\tau}{\sqrt{k^2}}.\end{aligned} \tag{5.8}$$

Corollary. Physically realizable free particle Heisenberg states are those vectors $\Phi \in H$ (with nonvanishing form $\langle\Phi|\Phi\rangle$) which lie in the range of operators p_μ and in "momentum representation" have their support within the light cone (we denote this physical subspace by H^f). For these states equation (5.5) gives the evolution in the Schrödinger picture.

We now introduce operators $E_\pm = \theta(\pm k_0)$ and observe, that $E_\pm$ satisfy (5.7) and $[E, Q] = 0$. With the use of the analog of Pryce-Foldy-Wouthuysen transformation [12] (see Appendix) it is evident, that in each of the four subspaces QEH^f the forms $\langle\Phi|\Phi\rangle$, $\langle\Phi|p_0\Phi\rangle$, $\langle\Phi|\dot{z}^0\Phi\rangle$, $\langle\Phi|\mathscr{M}\Phi\rangle$ are definite (positive or negative) and the following table of signs holds:

Type	P_+	A_+	A_-	P_-
subspace	$Q_+E_+H^f$	$Q_-E_-H^f$	$Q_+E_-H^f$	$Q_-E_+H^f$
$\langle p_0\rangle$	+	−	−	+
$\langle \dot{z}_0\rangle$	+	+	−	−
$\langle \mathscr{M}\rangle$	+	−	+	−
$\langle \mathscr{M}^2\rangle$	+	+	+	+

(5.9)

Before turning to the interpretation of these results let us take into consideration the charge conjugation. It is represented by antilinear operators $K_s = B_sC$, where C is the operator of the complex conjugation in "configuration representation" and B_s are unitary matrices (unique up to a phase factor) such that $\overline{\gamma^\mu} = B_{1/2}^\dagger\gamma^\mu B_{1/2}$, $\bar{\alpha} = -B_0^\dagger\alpha B_0$. The operators K_s induce the following transformations:

$$K_{1/2}^{-1}\gamma^\mu K_{1/2} = \gamma^\mu, \quad K_0^{-1}\alpha K_0 = -\alpha,$$
$$K^{-1}p_\mu K = -p_\mu, \quad K^{-1}A_\mu K = A_\mu,$$
$$K^{-1}\mathscr{M}(e)K = -\mathscr{M}(-e), \quad K^{-1}iK = -i, \tag{5.10}$$

thus if Φ satisfies (3.11), then $\Phi_C = K\Phi$ is a solution of the equation:

$$-\frac{i\hbar}{c^2}\frac{\partial}{\partial\tau}\Phi_C(\tau) = \mathscr{M}(-e)\Phi_C(\tau). \tag{5.11}$$

For a free particle we have:

$$\langle p_\mu\rangle_{\Phi_C} = -\langle p_\mu\rangle_\Phi, \quad \langle\dot{z}^\mu\rangle_{\Phi_C} = \langle\dot{z}^\mu\rangle_\Phi,$$
$$\langle\mathscr{M}\rangle_{\Phi_C} = -\langle\mathscr{M}\rangle_\Phi, \tag{5.12}$$

and

$$K^{-1}Q_\pm K = Q_\mp, \quad K^{-1}E_\pm K = E_\mp. \tag{5.13}$$

Therefore type $A_\pm$ physical states are charge conjugated states of type $P_\pm$. This enables us to interpret the positive energy states (type P_+ and P_-) as those of a particle (with charge $+e$), whereas the negative energy states (type A_+ and A_-) as charge conjugations of antiparticle positive energy states. The two types P_+, P_- (and similarly A_+, A_-) are physical time reflections of each other.

We note that as long as there is no interference between the different types of states, the simultaneous change of signs of τ and $\mathcal{M}$ can be performed, so as to have all $\langle \mathcal{M} \rangle$ positive. We see then that, for the particle's states, the physical time t increases with τ, whereas for antiparticle states it decreases. This result is in agreement with the interpretation of an antiparticle as a particle moving backwards in proper time [1, 13].

We end this section by solving the equation of motion for z^μ. The equation

$$\dot{z}^\mu = \mathcal{M}^{-1}(\tfrac{1}{2}[\mathcal{M}, \dot{z}^\mu]_+ + \tfrac{1}{2}[\mathcal{M}, \dot{z}^\mu]_-) = \frac{p^\mu}{\mathcal{M}} + \frac{i\hbar}{2\mathcal{M}c_2}\ddot{z}^\mu,$$

i.e.

$$\frac{d}{d\tau}\left(\dot{z}^\mu - \frac{p^\mu}{\mathcal{M}}\right) = -i\frac{2\mathcal{M}c^2}{\hbar}\left(\dot{z}^\mu - \frac{p^\mu}{\mathcal{M}}\right) \tag{5.14}$$

is easily solved (both $\mathcal{M}$ and p^μ are constant):

$$z^\mu(\tau) = \frac{p^\mu}{\mathcal{M}}\tau + \frac{i\hbar}{2\mathcal{M}c^2}(e^{-i\frac{2\mathcal{M}c^2}{\hbar}} - 1)\left(\dot{z}^\mu(0) - \frac{p^\mu}{\mathcal{M}}\right) + z^\mu(0). \tag{5.15}$$

As $Q_\pm q Q_\pm = \pm\frac{1}{2} Q_\pm \left[\frac{\mathcal{M}c}{\sqrt{k^2}}, q\right]_+$ and $\left[\frac{\mathcal{M}c}{\sqrt{k^2}}, \dot{z}^\mu - \frac{p^\mu}{\mathcal{M}}\right]_+ = 0$ it is clear, that if Φ is any pure type state ($P_\pm, A_\pm$) then

$$\langle \dot{z}^\mu \rangle_\Phi = \left\langle \frac{p^\mu}{\mathcal{M}} \right\rangle_\Phi \tag{5.16}$$

and Zitterbewegung is absent.

Summing up, the given theory is relativistically invariant, while it shares many other properties with the usual one (cf. [8]).

I would like to thank Prof. E. Kapuścik for constant interest and encouragement and Prof. A. Staruszkiewicz for some remarks on the manuscript.

APPENDIX

The Pryce-Foldy-Wouthuysen transformation [12]

We shall show here that there exist transformations acting in subspaces H^f, which are translation invariant

$$[W, P^\mu] = 0 \tag{A.1}$$

(hence in "momentum representation" W_s are k^μ-dependent matrices), and offer strict analogues to the Pryce-Foldy-Wouthuysen transformation in the sense that they conserve the forms $\langle \Phi | \Psi \rangle$, i.e.

$$W^\dagger \eta W = \eta \tag{A.2}$$

and diagonalize $\mathcal{M}_s(k)$ in the representations in which η_s are diagonal, namely:

$$W_0^{-1}\left(\frac{k^2}{\varrho c}\,\alpha+\varrho c\alpha^\dagger\right)W_0 = \sqrt{k^2}\,(\alpha+\alpha^\dagger) \tag{A.3}$$

$$W_{1/2}^{-1}\gamma^\mu k_\mu W_{1/2} = \operatorname{sgn} k_0\cdot\sqrt{k^2}\,\gamma^0. \tag{A.4}$$

As we do not need the most general form of such transformations, we can make the following

Ansatz

$$W_0 = a\alpha+b\alpha^\dagger,\qquad W_{1/2} = a_\mu\gamma^\mu. \tag{A.5}$$

Note, that a_μ cannot be a vector quantity, as then the l.h.s. of (A.4) would be a scalar operator, while the r.h.s. is not. After transforming (A.3, 4) with help of (A.2) into the more convenient form:

$$\frac{k^2}{\varrho c}\,\alpha\alpha^\dagger+\varrho c\alpha^\dagger\alpha = \sqrt{k^2}\,W_0W_0^\dagger,$$

$$\gamma^\mu\gamma^0 k_\mu = \operatorname{sgn} k_0\cdot\sqrt{k^2}\,W_{1/2}W_{1/2}^\dagger, \tag{A.6}$$

it is a matter of simple algebra to show that:

$$W_0 = \left(\frac{\sqrt{k^2}}{\varrho c}\right)^{1/2}\alpha+\left(\frac{\varrho c}{\sqrt{k^2}}\right)^{1/2}\alpha^\dagger,$$

$$W_{1/2} = \left[2\sqrt{k^2}\,(\sqrt{k^2}+|k_0|)\right]^{-1/2}(\operatorname{sgn} k_0\cdot\sqrt{k^2}\,\gamma^0+k_\mu\gamma^\mu), \tag{A.7}$$

where arbitrary phase factors have been fixed.

The defining properties (A.3, 4) can be stated in another way:

$$W_0^{-1}Q_{0\pm}W_0 = \tfrac{1}{2}\,(\mathbf{1}\pm\eta_0),$$

$$W_{1/2}^{-1}Q_{1/2\pm}W_{1/2} = \theta(k_0)\,\tfrac{1}{2}\,(\mathbf{1}\pm\eta_{1/2})+\theta(-k_0)\,\tfrac{1}{2}\,(\mathbf{1}\pm\eta_{1/2}). \tag{A.8}$$

Finally we note some additional properties of operators chosen in the specific form (A.7):

$$[W_0, L_{\mu\nu}] = 0,\qquad [W_{1/2}, L_{ik}+S_{ik}] = 0,$$

$$[W, V(R_{S,T})] = 0,\qquad [W, K]_+ = 0,$$

$$W^2 = \mathbf{1}. \tag{A.9}$$

Note added in proof. Only after the present article had been accepted for publication did I come upon the works by Johnson [14] which contain many of our results. Our construction is, however, more explicit and sticks consequently to the Hilbert space formalism. There is no general analogue of our postulate IV to be found in [14]. There are also some differences in the interpretation of the results. On the other hand the reference [14] includes more detailed discussion of the algebraic properties and their representations.

The classical equations of motion generated by the Hamiltonian $\sqrt{\pi^2}$ can be found in [15].

REFERENCES

[1] E. C. G. Stueckelberg, *Helv. Phys. Acta* **14**, 322, 588 (1941); **15**, 23 (1942).

[2] V. Fock, *Physik. Z. Sowjetunion* **12**, 404 (1937); Y. Nambu, *Prog. Theor. Phys.* **5**, 82 (1950); R. P. Feynman, *Phys. Rev.* **80**, 440 (1950); J. Schwinger, *Phys. Rev.* **82**, 664 (1951).

[3] L. P. Horwitz, C. Piron, *Helv. Phys. Acta* **46**, 316 (1973).

[4] L. P. Horwitz, C. Piron, F. Reuse, *Helv. Phys. Acta* **48**, 546 (1975); C. Piron, F. Reuse, *Helv. Phys. Acta* **51**, 146 (1978); F. Reuse, *Helv. Phys. Acta* **51**, 157 (1978).

[5] L. D. Landau, E. F. Lifshitz, *Classical Theory of Fields*, Perganion Press, Oxford 1971.

[6] K. Symanzik, *Lectures on Lagrangian Quantum Field Theory*, delivered in January 1968 at the University of Islamabad, Pakistan, revised and edited by H. J. Thun.

[7] M. Reed, B. Simon, *Methods of Modern Mathematical Physics*, vol. I, Academic Press, 1972.

[8] H. Feshbach, F. Villars, *Rev. Mod. Phys.* **30**, 24 (1958).

[9] W. Pauli, *Rev. Mod. Phys.* **15**, 175 (1943).

[10] A. Messiah, *Quantum Mechanics*, vol. II, North-Holland, Amsterdam 1962.

[11] I. M. Gelfand, G. E. Shilov, *Prostranstva osnovnykh i obobshchennykh funktsii*, Moskva 1958.

[12] M. H. L. Pryce, *Proc. Roy. Soc.* **A195**, 62 (1948); L. L. Foldy, S. A. Wouthuysen, *Phys. Rev.* **78**, 29 (1950).

[13] R. P. Feynman, *Phys. Rev.* **74**, 939 (1948).

[14] J. E. Johnson, *Phys. Rev.* **181**, 1755 (1969); **D3**, 1735 (1971); J. E. Johnson, K. K. Chang, *Phys. Rev.* **D10**, 2421 (1974).

[15] C. Garrod, *Rev. Mod. Phys.* **38**, 483 (1966); *Phys. Rev.* **167**, 1143 (1968).