ON BIRKHOFF'S THEOREM IN THE GENERALISED FIELD THEORY

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A study of the field equations of the Generalised Field Theory indicates that genuinely time-dependent spherically symmetric solutions can exist.

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1. Introduction

Birkhoff's theorem, to which I am referring, asserts that there is no genuinely time--dependent, spherically symmetric solution of the general relativistic field equations. A variety of proofs of this theorem is known, differing mainly in the form assumed to start with for the metric. Indeed, the proof is simple (Ref. [1]) when that form is the standard one from which the Schwarzschild solution is usually derived.

We come now to the nonsymmetric unified field theories of gravitation and electromagnetism. The comprehensive field equations are considerably more involved than in the general relativistic case. Even so, there is little reason to doubt that the theorem is valid for the Einstein-Straus theory (Ref. [2]). However, this theory has been superceded by what I call the Generalised Field Theory GFT, a summary account of which will appear shortly (in Ref. [3], where a list of references to the original articles published over a number of years, is given). I have called attention frequently to the profound implications which validity or otherwise of Birkhoff's theorem must have for GFT. In particular, the theory predicts, on the basis of the static, spherically symmetric solution of the field equations, a unique cosmological model. If therefore the theorem holds, the empirical or observational validity of this model represents a crucial test of the theory. Of course, if the theorem is no longer true, it would be more than likely that a hitherto unknown, time-evolving solution may yield a more realistic description of the actual, physical universe in which we live.

The present article is a study aiming at, at least, a partial resolution of Birkhoff's problem. It will be shown that the theorem appears to be valid in a restricted sense. On

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the other hand, a strong argument will be given that it fails in the general case. In fact, using an approximation method which does not appear to restrict generality (time-dependent perturbations of the Schwarzschild metric exist, e.g. Ref. [4]), we shall construct a counter example.

2. *The affine connection*

It is a fundamental assumption of GFT that the connection between geometry and physics is given solely by the field equations (I call this the weak principle of geometrisation) which determine both the nonsymmetric field $g_{\mu\nu}$ (with sixteen components in a four-dimensional theory) and the symmetric metric tensor $a_{\mu\nu}$ of a background space-time. The latter is Riemannian in the usual, relativistic sense (or rather, strictly speaking, pseudo-Riemannian. $a_{\mu\nu}$ itself is given by the "metric hypothesis"

$$\left\{ {\lambda \atop \mu \quad \nu} \right\}_a = \tilde{\Gamma}^{\lambda}_{(\mu\nu)}, \tag{1}$$

where the Christoffel brecket is constructed from $a_{\mu\nu}$, and $\tilde{\Gamma}^{\lambda}_{(\mu\nu)}$ is the symmetric part of the affine connection $\tilde{\Gamma}^{\lambda}_{\mu\nu}$ determined by the well known equation

$$g_{\mu\nu,\lambda} - \tilde{\Gamma}^{\sigma}_{\mu\lambda} g_{\sigma\nu} - \tilde{\Gamma}^{\sigma}_{\lambda\nu} g_{\mu\sigma} = 0 \tag{2}$$

in terms of the field and its first derivatives. It is shown in Ref. [5] that Eq. (1) is not really a hypothesis but a consequence of Einstein's principle of Hermitian symmetry or charge conjugation.

The initial bifurcation of geometry and physics creates a problem which is an unavoidable difficulty of GFT. The field equations can be solved only under stringent symmetry restrictions and because these seem to be geometrical in nature one would expect that they should be imposed a priori on geometry, that is on the metric $a_{\mu\nu}$. But apart from the "hypothesis" Eq. (1), the field equations are the same as in the Einstein-Straus theory and do not seem to be resolvable with only the relation (1) restricted in form. In other words, in order to arrive at a tractable set of equations, we are constrained to impose symmetry requirements on the field $g_{\mu\nu}$ and only a posteriori deduce the corresponding symmetry of the space-time. The above is, of course, only a mathematical difficulty which does not reflect on the coherence of GFT.

It means nevertheless that in arriving at a suitable form of $g_{\mu\nu}$ (so erroneously called the "nonsymmetric metric" in many accounts of the theory), we should follow the intuitive methods of, for example, Ref. [6], rather than appeal to say Killing equations whose meaning is purely geometrical.

Using then the reflection invariance arguments of Papapetrou (Ref. [6]), we easily find that the most general, spherically symmetric form of $g_{\mu\nu}$ is given in the coordinate system

$$x^{\mu} = x^0, x^1, x^2, x^3 = (t, r, \theta, \psi), \tag{3}$$

by

$$g_{\mu\nu} = \begin{bmatrix} \gamma & h-w & 0 & 0 \\ h+w & -\alpha & 0 & 0 \\ 0 & 0 & -\beta & u \sin\theta \\ 0 & 0 & -u\sin\theta & -\beta\sin^2\theta \end{bmatrix}, \tag{4}$$

where α, β, γ, h, u and w are functions of

$$x^0 = t \quad \text{and} \quad x^1 = r, \tag{5}$$

only. A coordinate transformation can easily be found to make

$$h = 0, \tag{6}$$

but, and this is an important point, it is by no means certain that the same transformation will (for example) diagonalise the metric tensor $a_{\mu\nu}$. Whether it does or not must be a consequence of the "metric hypothesis" alone.

In the sequel, I shall assume that not only h but also w vanish restricting out attention therefore to the "electric" case of the solution. Denoting by dots, partial derivatives with respect to t, and by dashes those with respect to r, the algebraic equations (2) give, for the nonvanishing components of $\tilde{\Gamma}^{\lambda}_{\mu\nu}$

$$\tilde{\Gamma}^0_{00} = \frac{\dot{\gamma}}{2\gamma}, \quad \tilde{\Gamma}^0_{(01)} = \frac{\gamma'}{2\gamma}, \quad \tilde{\Gamma}^0_{11} = \frac{\dot{\alpha}}{2\gamma}, \quad \tilde{\Gamma}^0_{22} = \tilde{\Gamma}^0_{33}\operatorname{cosec}^2\theta = \frac{(\beta^2-u^2)\dot{\beta}+2u\beta\dot{u}}{2\gamma(\mu^2+\beta^2)},$$

$$\tilde{\Gamma}^0_{[23]} = \frac{(\beta^2-u^2)\dot{u}-2u\beta\dot{\beta}}{2\gamma(u^2+\beta^2)}\sin\theta,$$

$$\tilde{\Gamma}^1_{11} = \frac{\alpha'}{2\alpha}, \quad \tilde{\Gamma}^1_{(01)} = \frac{\dot{\alpha}}{2\alpha}, \quad \tilde{\Gamma}^1_{00} = \frac{\gamma'}{2\alpha},$$

$$\tilde{\Gamma}^1_{22} = \tilde{\Gamma}^1_{33}\operatorname{cosec}^2\theta = -\frac{(\beta^2-u^2)\beta'+2u\beta u'}{2\alpha(u^2+\beta^2)}, \quad \tilde{\Gamma}^1_{[23]} = \frac{(\beta^2-u^2)u'-2u\beta\beta'}{2\alpha(u^2+\beta^2)}\sin\theta, \tag{7}$$

$$\tilde{\Gamma}^2_{(02)} = \tilde{\Gamma}^3_{(03)} = \frac{u\dot{u}+\beta\dot{\beta}}{2(u^2+\beta^2)}, \quad \tilde{\Gamma}^2_{(12)} = \tilde{\Gamma}^3_{(13)} = \frac{uu'+\beta\beta'}{2(u^2+\beta^2)}, \quad \tilde{\Gamma}^2_{33} = -\sin\theta\cos\theta,$$

$$\tilde{\Gamma}^3_{[02]} = \tilde{\Gamma}^2_{[30]}\operatorname{cosec}^2\theta = \frac{\beta\dot{u}-u\dot{\beta}}{2(u^2+\beta^2)}\operatorname{cosec}\theta,$$

$$\tilde{\Gamma}^3_{[12]} = \tilde{\Gamma}^2_{[31]}\operatorname{cosec}^2\theta = \frac{\beta u'-u\beta'}{2(u^2+\beta^2)}\operatorname{cosec}\theta, \quad \tilde{\Gamma}^3_{(23)} = \cot\theta,$$

where $\tilde{\Gamma}^{\lambda}_{[\mu\nu]}$ denotes the skew symmetric part of the affine connection.

3. *The field equations*

In addition to the equations (1) and (2), the field equations of GFT are

$$\tilde{R}_{(\mu\nu)} = 0, \qquad \tilde{R}_{[[\mu\nu],\lambda]} = 0, \tag{8}$$

together with the identities

$$\mathfrak{g}^{[\mu\nu]}{}_{,\nu} = 0. \tag{9}$$

Here, $\tilde{R}_{\mu\nu}$ is the Ricci tensor constructed from the nonsymmetric $\tilde{\Gamma}^{\lambda}_{\mu\nu}$. In the case of the spherically symmetric affine connection (7), equations (8) reduce to

$$\tilde{R}_{00} = \tilde{R}_{(01)} = \tilde{R}_{11} = \tilde{R}_{22} = 0, \quad \tilde{R}_{[23]} = k \sin\theta, \tag{10}$$

where k is a dimensionless constant associated with the electric charge. It is convenient in calculating the components of the Ricci tensor explicitly to use the substitution

$$\gamma = e^{2C}, \quad \alpha = e^{2A}, \quad \beta = e^{2R-v}, \quad u = e^{2R-v}\sqrt{e^{2v}-1} \tag{11}$$

with A, C, R and v, functions of and r only. Equations (10) become

$$e^{2A}\left[\ddot{A}+\dot{A}(\dot{A}-\dot{C})+2\ddot{R}+2\dot{R}(\dot{R}-\dot{C})+\frac{2\dot{v}^2}{e^{2v}-1}\right]-e^{2C}[C''+C'(C'-A')+2C'R'] = 0, \tag{12a}$$

$$\dot{R}'+\dot{R}R'+\frac{\dot{v}v'}{e^{2v}-1}-C'\dot{R}-\dot{A}R' = 0, \tag{12b}$$

$$e^{2A}[\ddot{A}+\dot{A}(\dot{A}-\dot{C})+2\dot{R}\dot{A}]-e^{2C}\left[C''+C'(C'-A')+2R''+2R'(R'-A')+\frac{2v'^2}{e^{2v}-1}\right] = 0, \tag{12c}$$

$$-e^{2A}\left[\ddot{R}+\ddot{v}+2\dot{R}+\dot{v}\left(3\dot{R}-\frac{e^{2v}+1}{e^{2v}-1}\dot{v}\right)+(\dot{R}+\dot{v})(\dot{A}-\dot{C})\right]$$
$$+e^{2C}\left[R''+v''+2R'^2+v'\left(-R'+\frac{3-e^{2v}}{e^{2v}-1}\right)+(R'+v')(C'-A')\right] = e^{[v+2(A+C-R)]} \tag{12d}$$

and

$$e^{2A}\left[\ddot{R}-\frac{\ddot{v}}{e^{2v}-1}+2\dot{R}^2-\frac{3\dot{v}\dot{R}}{e^{2v}-1}+\frac{\dot{v}^2}{(e^{2v}-1)^2}+\left(-\dot{R}+\frac{\dot{v}}{e^{2v}-1}\right)(\dot{C}-\dot{A})\right]$$
$$+e^{2C}\left[R''-\frac{v''}{e^{2v}-1}+2R'^2+\frac{v'R'}{e^{2v}-1}+\frac{(4e^{2v}-3)v'^2}{(e^{2v}-1)^2}+\left(-R'+\frac{v'}{e^{2v}-1}\right)(A'-C')\right]$$
$$= \frac{k}{\sqrt{e^{2v}-1}}e^{[+v+2(A+C-R)]}. \tag{12e}$$

We must turn our attention next to the conditions imposed on the theory by the "metric hypothesis" (1), or the equivalent equation

$$a_{\mu\nu,\lambda}-\tilde{\Gamma}^{\sigma}_{(\mu\lambda)}a_{\sigma\nu}-\tilde{\Gamma}^{\sigma}_{(\nu\lambda)}a_{\mu\sigma} = 0. \tag{13}$$

It is easy to see that the solution (7) implies that

$$a_{02} = a_{03} = a_{12} = a_{13} = a_{23} = 0, \tag{14}$$

which incidentally guarantees the usual condition of spherical symmetry

$$a_{22} = a_{33}\operatorname{cosec}^2\theta. \tag{15}$$

The remaining equations (13) then become

$$\dot{a}_{00}-2\dot{C}a_{00}-2e^{2(C-A)}C'a_{01} = 0, \tag{16a}$$

$$a'_{00}-2C'a_{00}-2\dot{A}a_{01} = 0, \tag{16b}$$

$$\dot{a}_{11}-2C'a_{01}-2\dot{A}a_{11} = 0, \tag{16c}$$

$$a'_{11}-2A'a_{11}-2e^{2(A-C)}\dot{A}a_{01} = 0, \tag{16d}$$

$$\dot{a}_{01}-(\dot{C}+\dot{A})a_{01}-C'a_{00}-e^{2(C-A)}C'a_{11} = 0, \tag{16e}$$

$$\dot{a}_{01}-(A'+C')a_{01}-e^{2(A-C)}\dot{A}a_{00}-\dot{A}a_{11} = 0, \tag{16f}$$

together with

$$a_{22} = -e^{2R}, \tag{17}$$

and

$$\dot{R}-a_{00}(\dot{R}+\dot{v})e^{-v-2C}+a_{01}(R'+v')e^{-v-2A} = 0, \tag{18a}$$

$$R'-a_{11}(R'+v')e^{-v-2A}-a_{01}(\dot{R}+\dot{v})e^{-v-2C} = 0. \tag{18b}$$

It is now interesting to observe (as is shown by a straightforward calculation) that the equations (16) are integrable under the single condition

$$e^{2A}[\ddot{A}+\dot{A}(\dot{A}-\dot{C})]-e^{2C}[C''+C'(C'-A')] = 0. \tag{19}$$

Equations (12a), (12b) and (12c) therefore reduce to

$$\ddot{R}+\dot{R}(\dot{R}-\dot{C})+\frac{\dot{v}^2}{e^{2v}-1}-e^{C-A}C'R' = 0, \tag{20a}$$

$$\dot{R}'+\dot{R}R'+\frac{\dot{v}v'}{e^{2v}-1}-C'\dot{R}-\dot{A}R' = 0, \tag{20b}$$

$$R''+R'(R'-C')+\frac{v'^2}{e^{2v}-1}-e^{A-C}\dot{A}\dot{R} = 0. \tag{20c}$$

Equations (19) and (20) now suggest a wave like solution with all dependent variables functions of

$$t-r$$

and

$$A = C. \tag{21}$$

Equations (20) then collapse into a single equation

$$R''+R'^2+\frac{v'^2}{e^{2v}-1}-2C'R' = 0 \tag{22}$$

which, together with the equations (12d) and (12e) determines R as a function of v through a first order equation

$$(1-x(1+x^2)U)U'+x(1+x^2)U^3+(1+2kx-x^2)U^2+\frac{2x^4+x^2-2kx-2}{x(1+x^2)^2}U$$

$$-\frac{1}{(1+x^2)^3} = 0, \tag{23}$$

where we have put

$$U = \frac{dR}{dx}, \quad x = \sqrt{e^{2v}-1}.$$

On the other hand, equations (16) now imply a constant (Minkowski) $a_{\mu\nu}$ (which is possible for a nontrivial field, Ref. [7]) but equations (18) are incompatible with (23). Hence Birkhoff's theorem holds if $a_{01} \neq 0$.

In the next section however, I shall show that we can nevertheless obtain a time--dependent, approximate solution of the field equations.

4. *An approximation*

An alternative procedure to the above is to put

$$a_{01} = 0 \tag{24}$$

when the equations (16) become directly integrable (with no condition implied) and give without loss of generality

$$a_{00} = e^{2C}, \quad a_{11} = -e^{2A}. \tag{25}$$

Also, equation (18) now yield

$$R' = \frac{v'}{e^v-1}, \quad \dot{R} = \frac{\dot{v}}{e^v-1},$$

so that

$$e^{-R} = \frac{Ke^v}{e^v-1}, \tag{26}$$

where K is a constant. This result leaves us with just the field equations (12) which become after some rearranging

$$e^{2A}\left[A_{00}+A_0(A_0-C_0)-\frac{2v_0C_0}{e^v-1}+\frac{2v_{00}}{e^v-1}-\frac{2e^v v_0^2}{e^{2v}-1}\right]$$

$$-e^{2C}\left[C_{11}+C_1(C_1-A_1)+\frac{2v_1C_1}{e^v-1}\right]=0, \tag{27a}$$

$$e^{2A}\left[v_{00}-\frac{e^v v_0^2}{e^v+1}-v_0(A_0+C_0)\right]+e^{2C}\left[v_{11}-\frac{e^v v_1^2}{e^v+1}-v_1(A_1+C_1)\right]=0, \tag{27b}$$

$$C_1v_0+A_0v_1=v_{10}-\frac{e^v v_1 v_0}{e^v+1}, \tag{27c}$$

$$e^{2C}\left[2v_{11}+\frac{-2e^{2v}+3e^v+8}{e^{2v}-1}v_1^2+2v_1(C_1-A_1)\right]-\frac{3e^v v_0^2 e^{2A}}{e^{2v}-1}$$

$$=e^{2(A+C-R)}(e^v-1)\left(k\sqrt{\frac{e^v+1}{e^v-1}}+1\right), \tag{27d}$$

$$e^{2A}\left[2v_{00}+\frac{-2e^{2v}+3e^v}{e^{2v}-1}v_0^2+2v_0(A_0-C_0)\right]+\frac{5e^v v_1^2 e^{2C}}{e^{2v}-1}$$

$$=e^{2(A+C-R)}(e^v-1)\left(k\sqrt{\frac{e^v+1}{e^v-1}}-1\right), \tag{27e}$$

where now $v_0=\dfrac{\partial v}{\partial t}$, $v_1=\dfrac{\partial v}{\partial r}$ etc. (the dot/dash notation was used in the previous section to avoid possible confusion with tensor indices). Equations (27b), (27d) can be solved immediately for A_0, A_1, C_0 and C_1 whose integrability conditions

$$A_{01}=A_{10} \quad \text{and} \quad C_{01}=C_{10}, \tag{28}$$

give two equations in e^{2A} and e^{2C}. A third equation in the latter is then obtainable from Eq. (27a) by eliminating the derivatives of A and C. Hence e^{2A} and e^{2C} can be in principle eliminated between these three equations yielding a third order partial differential equation in v alone. Thus, again in principle, there should exist a time dependent solution of the field equations.

To derive a more explicit expression for v, A and C however, we must resort to an asymptotic method. Let us first change the first variable again, writing

$$z=\sqrt{\frac{e^v+1}{e^v-1}} \tag{29}$$

equivalent to putting

$$\frac{u}{\beta}=\frac{2z}{z^2-1} \quad \text{or} \quad uz=\beta+\sqrt{\beta^2+u^2}\,.$$

Equations (27) become

$$e^{2A}\left[A_{00}+A_0(A_0-C_0)+\frac{4zzC_0}{z^2+1}+\frac{4}{z^2+1}\left(-zz_0+\frac{3z^2+1}{z^2+1}z_0^2\right)\right]$$
$$-e^{2C}\left[C_{11}+C_1(C_1-A_1)-\frac{4zz_1C_1}{z^2+1}\right]=0, \tag{30a}$$

$$e^{2A}\left[z_{00}-\frac{3z^2+1}{z^2+1}z_0^2-zz_0(A_0+C_0)\right]+e^{2C}\left[zz_{11}-\frac{3z^2+1}{z^2+1}z_1^2-zz_1(A_1+C_1)\right]=0, \tag{30b}$$

$$zz_0C_1+zz_1A_0 = zz_{10}-\frac{3z^2+1}{z^2+1}z_1z_0, \tag{30c}$$

$$3z_0^2e^{2A}+e^{2C}\left[2zz_{11}-5\left(\frac{3z^2-1}{z^2+1}\right)z_1^2+2zz_1(C_1-A_1)\right] = -\frac{K^2}{8}(z^2+1)^3(kz+1)e^{2(A+c)}, \tag{30d}$$

$$5z_1^2e^{2C}-e^{2A}\left[2zz_{00}-\frac{7z^2+3}{z^2+1}z_0^2+2zz_0(A_0-C_0)\right]=\frac{K^2}{8}(z^2+1)^3(kz-1)e^{2(A+C)}. \tag{30e}$$

It is interesting to note that the apparent singularity $v = 0$ (corresponding to $z = \infty$) has disappeared. Let us consider what happens when

$$z = 1+\eta, \quad \eta^2 \ll \eta. \tag{31}$$

To the first order in η, the equations become

$$e^{2A}[A_{00}+A_0(A_0-C_0)]-e^{2C}[C_{11}+C_1(C_1-A_1)] = 0, \tag{32a}$$

(our old integrability condition of the equations (16)),

$$e^{2A}[\eta_{00}-\eta_0(A_0+C_0)]+e^{2C}[\eta_{11}-\eta_1(A_1+C_1)] = 0, \tag{32b}$$

$$\eta_0C_1+\eta_1A_0 = \eta_{10}, \tag{32c}$$

$$\eta_{11}+\eta_1(C_1-A_1) = -\frac{K^2}{2}(k+1)e^{2A}, \tag{32d}$$

$$\eta_{00}+\eta_0(A_0-C_0) = -\frac{K^2}{2}(k-1)e^{2C}, \tag{32e}$$

indicating that K^2 must be considered as small for a solution to be possible. A general solution of equations (32) is still too complicated but as far as Birkhoff's theorem is concerned, it is sufficient to demonstrate the existence of a particular solution which appears to be physically meaningful. Thus, let us put

$$\eta = \varepsilon(t-r), \tag{33}$$

where

$$\varepsilon^2 \ll \varepsilon$$

is a constant. Then equations (32b), (32e) are integrable providing

$$k \neq 1,$$

$$e^{2A} = \frac{1-k}{1+k} e^{2C} = q^2 e^{2C}, \quad \text{say.} \tag{34}$$

Putting finally

$$x = 2C_0, \quad y = 2C,$$

we find that x (and y) satisfy a (quasi) wave equation

$$x_{11} = q^2 x_{00}, \tag{35}$$

whence a solution (which will necessarily depend explicitly on time) can be written down.

It follows that in this case Birkoff's theorem is no longer valid.

5. *Conclusions*

We have now demonstrated the assertions made in the introduction. Existence of explicit (time-dependent), approximate solution for any given solution of the wave equation (35) suggest that the general equation for v obtainable in the non-linearised case will also have a non-static solution. Hence we must conclude that Birkhoff's theorem cannot be valid in the Generalised Field Theory. In view of its connection with the problem of empirical validity of GFT, search for such a solution becomes particularly urgent.

REFERENCES

[1] R. C. Tolman, *Relativity, Thermodynamics and Cosmology*, Oxford 1934.
[2] A. Einstein, E. G. Straus, *Ann. Math.* **47**, 731 (1946).
[3] A. H. Klotz, L. J. Gregory, *Gen. Relativ. Gravitation* **13**, 155 (1981).
[4] A. H. Klotz, G. V. Bicknell, *Acta Phys. Pol.* **A39**, 113 (1971).
[5] A. H. Klotz, *Acta Phys. Pol.* **B11**, 501 (1980).
[6] A. Papapetrou, *Proc. R. Irish Acad.* **A52**, 69 (1948).
[7] A. H. Klotz, *Acta Phys. Pol.* **B11**, 779 (1980).