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EINSTEIN-CARTAN-MAXWELL-BIANCHI TYPE II SOLUTIONS

By D. Lorenz

Fachbereich für Physik, Universität Konstanz*

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It is shown that solutions of the Einstein-Cartan-Maxwell theory for the Bianchi type II models exist.

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1. Introduction

It has been stated by Tafel (1975), Kuchowicz (1976) and Tsoubelis (1981) that there are no Einstein–Cartan–Bianchi type II solutions. However, this result is due to the special form of the metric and the matter content these authors made use of. The aim of this paper is to show that in the case of a more general metric and in the presence of an electromagnetic field the class of Bianchi type II contains solutions in the Einstein-Cartan theory (ECT) of gravitation (see also Lorenz 1981b).

In a previously published paper (Lorenz 1981a) we have already shown that in the presence of an electromagnetic field ECT-solutions of the Bianchi types VIII and IX exist. This result is of some importance since it has been stated that a homogeneous distribution of polarized spin is incompatible with spatial closure in the case of Bianchi type IX models (Kerlick 1975, 1976; Kuchowicz 1975, 1976). The method, notation and pertinent portions of Ref. (Lorenz 1981a) are utilized.

2. Field equations

In choosing a local orthonormal basis σ^μ, we can put the metric of the Bianchi type II space-time in the form

$$ds^2 = \eta_{\mu\nu}\sigma^\mu\sigma^\nu, \tag{1}$$

where $\eta_{\mu\nu}$ is the Minkowski metric tensor. For a spatially homogeneous model, we take

$$\sigma^0 = \theta^0 = dt, \quad \sigma^i = R_i\theta^i \text{ (no sum)}, \tag{2}$$

* Address: Fachbereich für Physik, Universität Konstanz, Bücklestr. 13, D-7750 Konstanz, West Germany.

where

$$\theta^1 = dx^1 + [2nx^3 + 2m \int R_2(R_1{}^3R_3)^{-1}dt]dx^2,$$

$$\theta^2 = dx^2, \quad \theta^3 = dx^3, \quad n, m = \text{const.} \tag{3}$$

are differential one forms and where, due to homogeneity, the R_i are functions of t only. The exterior derivatives of the orthonormal basis one forms σ^μ are

$$d\sigma^0 = 0,$$

$$d\sigma^1 = H_1\sigma^0 \wedge \sigma^1 + 2n \frac{R_1}{R_2R_3} \sigma^3 \wedge \sigma^2 + \frac{2m}{R_1^2R_3} \sigma^0 \wedge \sigma^2,$$

$$d\sigma^2 = H_2\sigma^0 \wedge \sigma^2, \quad d\sigma^3 = H_3\sigma^0 \wedge \sigma^3, \quad H_i = \dot{R}_i/R_i. \tag{4}$$

Assuming a classical description of spin (see Lorenz 1981a) one finds that the only non-vanishing components of the torsion tensor are:

$$Q^0{}_{12} = -Q^0{}_{21} =: 2s = 2s(t). \tag{5}$$

The connection coefficients are

$$\gamma^0{}_{12} = -\left(s - \frac{m}{R_1^2R_3}\right), \quad \gamma^2{}_{01} = s + \frac{m}{R_1^2R_3},$$

$$\gamma^3{}_{21} = \gamma^3{}_{12} = \gamma^1{}_{23} = -n \frac{R_1}{R_2R_3}, \quad \gamma^i{}_{0i} = H_i. \tag{6}$$

These quantities enter into the formula

$$\sigma^\mu{}_\nu = \gamma^\mu{}_{\nu\lambda}\sigma^2 \tag{7}$$

to provide six connection two-forms $\sigma_{\mu\nu}$. The results are:

$$\sigma^0{}_1 = H_1\sigma^1 + \left(-s + \frac{m}{R_1^2R_3}\right)\sigma^2,$$

$$\sigma^0{}_2 = H_2\sigma^2 + \left(s + \frac{m}{R_1^2R_3}\right)\sigma^1, \quad \sigma^0{}_3 = H_3\sigma^3,$$

$$\sigma^1{}_2 = -\left(s + \frac{m}{R_1^2R_3}\right)\sigma^0 - n \frac{R_1}{R_2R_3}\sigma^3,$$

$$\sigma^1{}_3 = n \frac{R_1}{R_2R_3}\sigma^2, \quad \sigma^2{}_3 = n \frac{R_1}{R_2R_3}\sigma^1. \tag{8}$$

The curvature two-forms

$$\Omega^\mu{}_\nu = d\sigma^\mu{}_\nu + \sigma^\mu{}_\lambda \wedge \sigma^\lambda{}_\nu = \tfrac{1}{2} R^\mu{}_{\nu\lambda\kappa}\sigma^\lambda \wedge \sigma^\kappa \tag{9}$$

are given by

$$\Omega^0{}_1 = -\left[\dot{s}+2sH_2+\frac{m}{R_1^2R_3}H_3\right]\sigma^0\wedge\sigma^2+\left[\dot{H}_1+H_1^2-\left(s+\frac{m}{R_1^2R_3}\right)^2\right]\sigma^0\wedge\sigma^1$$

$$+\left[sn\frac{R_1}{R_2R_3}+\frac{nm}{R_1R_2R_3^2}\right]\sigma^1\wedge\sigma^3+n\frac{R_1}{R_2R_3}(H_2-2H_1)\sigma^2\wedge\sigma^3,$$

$$\Omega^0{}_2 = \left[\dot{s}+2sH_1-\frac{m}{R_1^2R_3}H_3\right]\sigma^0\wedge\sigma^1+\left[\dot{H}_2+H_2^2-s^2+2s\frac{m}{R_1^2R_3}+\frac{3m^2}{R_1^4R_3^2}\right]\sigma^0\wedge\sigma^2$$

$$+\left[ns\frac{R_1}{R_2R_3}+\frac{3nm}{R_1R_2R_3^2}\right]\sigma^3\wedge\sigma^2-n\frac{R_1}{R_2R_3}(H_1-H_3)\sigma^1\wedge\sigma^3,$$

$$\Omega^0{}_3 = (\dot{H}_3+H_3^2)\sigma^0\wedge\sigma^3+n\frac{R_1}{R_2R_3}(H_1-H_2)\sigma^1\wedge\sigma^2,$$

$$\Omega^1{}_2 = -n\frac{R_1}{R_2R_3}(H_1-H_2)\sigma^0\wedge\sigma^3-\left[s^2-\frac{m^2}{R_1^4R_3^2}+n^2\left(\frac{R_1}{R_2R_3}\right)^2\right]\sigma^2\wedge\sigma^1$$

$$+H_1H_2\sigma^1\wedge\sigma^2,$$

$$\Omega^1{}_3 = n\frac{R_1}{R_2R_3}(H_1-H_3)\sigma^0\wedge\sigma^2-\left(sH_3-m\frac{H_3}{R_1^2R_3}\right)\sigma^2\wedge\sigma^3$$

$$-\left(sn\frac{R_1}{R_2R_3}+\frac{nm}{R_1R_2R_3^2}\right)\sigma^0\wedge\sigma^1+\left[H_1H_3+n^2\left(\frac{R_1}{R_2R_3}\right)^2\right]\sigma^1\wedge\sigma^3,$$

$$\Omega^2{}_3 = n\frac{R_1}{R_2R_3}(2H_1-H_2-H_3)\sigma^0\wedge\sigma^1+\left(sn\frac{R_1}{R_2R_3}+\frac{3nm}{R_1R_2R_3^2}\right)\sigma^0\wedge\sigma^2$$

$$+\left[H_2H_3-3n^2\left(\frac{R_1}{R_2R_3}\right)^2\right]\sigma^2\wedge\sigma^3-\left(sH_3+m\frac{H_3}{R_1^2R_3}\right)\sigma^1\wedge\sigma^3. \qquad (10)$$

Thus we can easily calculate the Ricci tensor $R^\mu{}_\nu = R^{2\mu}{}_{2\nu}$.

The Einstein-Cartan equations considered here are:

$$-R_{00} = \dot{H}_1+H_1^2+\dot{H}_2+H_2^2+H_3+H_3^2-2s^2+\frac{2m^2}{R_1^4R_3^2} = -\frac{\varepsilon}{2}(3\gamma-2)-E_{00}, \qquad (a)$$

$$R_{11} = \dot{H}_1+H_1^2+H_1H_2+H_1H_3-\frac{2sm}{R_1^2R_2}-\frac{2m^2}{R_1^4R_3^2}+2n^2\left(\frac{R_1}{R_2R_3}\right)^2 = \frac{\varepsilon}{2}(2-\gamma)+E_{11}, \qquad (b)$$

$$R_{22} = \dot{H}_2+H_2^2+H_2H_1+H_2H_3+\frac{2sm}{R_1^2R_3}+\frac{2m^2}{R_1^4R_3^2}-2n^2\left(\frac{R_1}{R_2R_3}\right)^2 = \frac{\varepsilon}{2}(2-\gamma)+E_{22}, \qquad (c)$$

$$R_{33} = \dot{H}_3+H_3^2+H_3H_1+H_3H_2-2n^2\left(\frac{R_1}{R_2R_3}\right)^2 = \frac{\varepsilon}{2}(2-\gamma)+E_{33}, \qquad (d)$$

$$R_{12} = -(\dot{s}+s(2H_2+H_3)) = E_{12}, \tag{e}$$

$$R_{21} = \dot{s}+s(2H_1+H_3) = E_{21}, \tag{f}$$

$$R_{03} = \frac{2nm}{R_1 R_2 R_3^2} = E_{03}, \tag{g}$$

(11)

where the perfect fluid matter inherent in this model is characterized by the equation of state

$$p = (\gamma-1)\varepsilon, \quad 1 \leqslant \gamma \leqslant 2 \tag{12}$$

and $E_{\mu\nu}$ are the components of the electromagnetic stress-energy tensor. In addition we have the following conservation equations:

$$\varepsilon = \frac{\varepsilon_0^2}{(R_1 R_2 R_3)^\gamma}, \quad s = \frac{s_0^2}{R_1 R_2 R_3}, \quad \varepsilon_0^2, s_0^2 = \text{const.} \tag{13}$$

We now turn to the Maxwell equations. The only nonvanishing structure constant for Bianchi type II is $C_{23}{}^1$. Thus the sourceless Maxwell equations become

$$E_1 R_1 - \partial_t(B_1 R_2 R_3) = 0, \quad B_1 R_1 + \partial_t(E_1 R_2 R_3) = 0,$$
$$\partial_t(B_i R_j R_k) = 0, \quad \partial_t(E_i R_j R_k) = 0, \quad i = 2, 3. \tag{14}$$

Owing to homogeneity, the electromagnetic field E_i and the magnetic field B_i depend only on t. Introducing a new time variable τ by $d\tau := R_1/(R_2 R_3) dt$ we obtain the general solutions

$$E := E_1 = (a/R_2 R_3)^2 \cos(\tau+\tau_0), \quad B := B_1 = (a/R_2 R_3)^2 \sin(\tau+\tau_0),$$
$$B_i = (B_i/R_j R_k), \quad E_i = (e_i/R_j R_k), \quad a, b_i, e_i, \tau_0 = \text{const.} \quad i = 2, 3. \tag{15}$$

Thus we can easily calculate the electromagnetic stress-energy tensor

$$E_{\mu\nu} = E_{\mu\alpha} E_\nu{}^\alpha - \tfrac{1}{4}\, \eta_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}, \tag{16}$$

where $F_{i0} = E_i$ and $F_{jk} = B_i \varepsilon_{ijk}$.

3. *Discussion*

We now discuss the field equations in some more details. In the case $E_{11} = E_{22}$, $E_{12} = E_{21} = 0$ we obtain, comparing the Eqs. (11b, c), the constraint equation

$$sm(R_1 R_2)^2 R_3 + m^2 R_2^2 - n^2 R_1^6 = 0. \tag{17}$$

If in addition $E_{03} = 0$ we find from Eq. (11g) and (17) that $R_1 = 0$ for $n \neq 0$, se only in this case we obtain the result that there are no ECT-type II solutions. The above mentioned authors consider only the case $m = 0$. The more general metric considered here with $m \neq 0$ for the Bianchi type II model has been first given by King and Ellis (1973).

In addition, it must be said that Kuchowicz (1976) did not make any calculations about type II (he only quotes the result of Tafel's study). The special choice for the magnetic field made by Tsoubelis gives indeed $E_{03} = 0$. (Note that in Tsoubelis (1980) the erroneous conclusion was made that there are ECT-Bianchi type II solutions for $m = 0$).

However, in general the nonvanishing components of the electromagnetic stress-energy tensor are:

$$E_{00} = \tfrac{1}{2}\left[\left(\frac{a}{R_2R_3}\right)^2 + \frac{e_2^2+b_2^2}{(R_1R_3)^2} + \frac{e_3^2+b_3^2}{(R_1R_2)^2}\right],$$

$$E_{11} = \tfrac{1}{2}\left[-\left(\frac{a}{R_2R_3}\right)^2 + \frac{e_2^2+b_2^2}{(R_1R_3)^2} + \frac{e_3^2+b_3^2}{(R_1R_2)^2}\right],$$

$$E_{22} = \tfrac{1}{2}\left[\left(\frac{a}{R_2R_3}\right)^2 - \frac{e_2^2+b_2^2}{(R_1R_3)^2} + \frac{e_3^2+b_3^2}{(R_1R_2)^2}\right],$$

$$E_{33} = \tfrac{1}{2}\left[\left(\frac{a}{R_2R_3}\right)^2 + \frac{e_2^2+b_2^2}{(R_1R_3)^2} - \frac{e_3^2+b_3^2}{(R_1R_2)^2}\right],$$

$$E_{12} = E_{21} = -\frac{a}{R_1R_2R_3^2}[e_2\cos(\tau+\tau_0)+b_2\sin(\tau+\tau_0)],$$

$$E_{03} = -\frac{a}{R_1R_2R_3^2}[b_2\cos(\tau+\tau_0)-e_2\sin(\tau+\tau_0)]. \tag{18}$$

Thus in general we have $E_{03} \neq 0$ and $R_{11} \neq R_{22}$. Thus we have shown that ECT-solutions of Bianchi type II models exist, contrary to the claims in the literature of the subject.

Editorial note. This article was proofread by the editors only, not by the author.

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