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AN ALTERNATIVE RELATIVISTIC γ-LAW EQUATION OF STATE

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A new form of phenomenological relativistic γ- law equation of state

$$p = k(g_{00})^{-1/2}\varrho^{\gamma}$$

is proposed which seems to be more plausible than the classical one. This equation of state leads to an instructive example of interior Schwarzschild metric, namely the Buchdahl solution. Some numerical deviations from that of classical γ-law equation of state are also given.

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The one-parametric γ-law equation of state has been widely used in the general relativistic literature. Although the actual form of such a polytropic equation of state has been subject to various investigations (Harrison et al. 1965, Tooper 1965) there has been as yet no thorough discussion of the thermodynamic problems involved in it. We know already that in a gravitating system energy as a thermodynamic potential is not extensive, so the relation between the local and the global thermodynamics not obvious. Furthermore it is not the local temperature $T_{\rm loc}$ but the global redshifted "Tolman" temperature $T_{\rm glob} = T\sqrt{g_{00}}$ (Tolman 1934) (g_{00} is the 00-component of the metric in a stationary gravitational field) which acts as an equilibrium parameter. We have also learnt in the last years that the usual γ-law equation of state imposes a somehow strong restriction on the solutions of Einstein field equations (Mansouri 1977, 1980, Mashhoon and Partovi 1980). Trying to soften this restriction, we propose here an alternative γ-law equation of state which seems also to incorporate at least the condition of Tolman temperature as an equilib-

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rium parameter. Besides this, it leads to an instructive example of interior Schwarzschild solution.

Consider a spherical star represented by the Schwarzschild solution for which the equation of hydrostatic equilibrium is given by the TOV equation. Assuming the metric in the form

$$ds^2 = e^{\nu(r)}dt^2 - e^{\mu(r)}dr^2 - r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2)$$

we make the following ansatz for our γ-law equation of state

$$p = ke^{-\nu/2}\varrho^\gamma, \tag{1}$$

where $\gamma = 1 + 1/\kappa$ and n is the "polytropic index". k is a constant. Note that in the case of isothermal gas $n = \infty$ or $\gamma = 1$, the factor $ke^{-\nu/2}$ is proportional to the local temperature $T_{loc} = T_{glob}/\sqrt{g_{00}} = T_{glob}e^{-\nu/2}$ which is more plausible than the usual choice k = const. It is easily seen that for this equation of state the TOV equation

$$-\frac{dp}{dr} = \tfrac{1}{2}(p+\varrho)\frac{d\nu}{dr},$$

can be solved to get ν as a function of ϱ. We then have

$$e^{\nu/2} = k\left[K - \frac{\gamma}{\gamma-1}\varrho^{\gamma-1}\right], \quad \gamma \neq 1$$

$$e^{\nu/2} = k[K - \ln\varrho], \qquad \gamma = 1 \tag{2}$$

K is an integrating constant. Therefore for the equation of state (1) we get

$$p = \varrho^\gamma\left(K - \frac{\gamma}{\gamma-1}\varrho^{\gamma-1}\right)^{-1}, \quad \gamma \neq 1$$

$$p = \varrho(K - \ln\varrho)^{-1}, \qquad \gamma = 1. \tag{3}$$

It is reasonable to express the constant K in terms of the central values of pressure p_c and density ϱ_c. From the equations (1) and (2) we get,

$$K = \frac{\varrho_c^\gamma}{p_c} + \frac{\gamma}{\gamma-1}\varrho_c^{\gamma-1}, \quad \gamma \neq 1$$

$$K = \frac{\varrho_c}{p_c} + \ln\varrho_c, \qquad \gamma = 1.$$

With these values for K, we can rewrite the equations (3) in the following form

$$\bar{p} = \bar{\varrho}^\gamma\left(1 + \frac{\gamma}{3(\gamma-1)}q_c(1-\bar{\varrho}^{\gamma-1})\right)^{-1}, \quad \gamma \neq 1$$

$$\bar{p} = \bar{\varrho}(1 - \tfrac{1}{3}q_c\ln\bar{\varrho})^{-1}, \qquad \gamma = 1$$

where we have used following notations

$$\bar{p} = \frac{p}{p_c}, \quad \bar{\varrho} = \frac{\varrho}{\varrho_c}, \quad q_c = \frac{3p_c}{\varrho_c}.$$

In the case of $n = 5$ or $\gamma = 6/5$, we get

$$\bar{p} = \bar{\varrho}^{6/5}(1+2q_c(1-\bar{\varrho}^{1/5}))^{-1}$$

which is the equation of state proposed by Buchdahl (Buchdahl 1964) for a relativistic polytrope of index $n = 5$. As Buchdahl has shown the Einstein equations for this case are exactly solvable and the solution represents a sphere of infinite radius but finite mass.

Note that when $1/3\, q_c$, the ratio of the central pressure to the central energy density is much less than 1/3, the equation of state of the fluid reduces to the classical form of the γ-law equation of state. For a numerical discussion we restrict ourselves to $q_c = 1$, where the most deviation from the usual polytropic equation of state is to be expected. In the case of $\gamma = 4/3$ ($n = 3$) we get for a central density $\varrho_c = 0.3 \cdot 10^{14}$ a total mass of $0.36\, M_\odot$ (= solar mass), whereas the classical polytrope leads to a total mass of $0.24 \cdot 10^2\, M_\odot$. As a second example we note that for $\gamma = 1.5$ ($n = 2$) and $\varrho_c = 0.1 \cdot 10^{10}$ we get a total mass of $0.3 \cdot 10^1\, M_\odot$ which is to be compared with the classical result of $0.3 \cdot 10^4\, M_\odot$. We therefore see that this relativistic γ-law equation of state may lead to considerable differences to the classical polytropic equation of state.

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