RELATIVISTIC CORRECTIONS TO KAON CHARGE RADII IN QUARK MODEL

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Relativistic effects of the order of v^2/c^2 are considered in the wave functions of kaons treated as bound $q\bar{q}$-states. The ratio $\xi = \left|\frac{\langle r^2_{em}\rangle_{K^0}}{\langle r^2_{em}\rangle_{K^+}}\right|$ is calculated for dynamical models with the direct interaction. As is shown, in contrast to results by Greenberg, Nussinov and Sucher, corrections to the nonrelativistic $\xi_{NR} = \frac{m_s^2-m_u^2}{2m_s^2+m_u^2}$ may be positive both in the classical and quantum models of $q\bar{q}$-systems.

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1. Introduction

A nonrelativistic quark model with SU(3)-symmetry breaking by s-quark "weighting" predicts a number of relations between hadron electromagnetic characteristics independent of the qq ($q\bar{q}$)-interaction dynamics. In particular, the ratio of the electromagnetic radii of neutral and charged K-mesons $\xi = \left|\frac{\langle r^2_{em}\rangle_{K^0}}{\langle r^2_{em}\rangle_{K^+}}\right|$, first calculated by Gerasimov [1], is determined in this model by the ratio of masses of s- and u-quarks:

$$\xi_{NR} = \frac{m_s^2-m_u^2}{2m_s^2+m_u^2}. \tag{1}$$

Equality (1) is obtained for a nonrelativistic treatment of the motion of particles with charges (2/3, $\pm 1/3$) in the rest frame of a two-particle bound state.

The problem of corrections to (1) due to relativistic effects has been discussed in papers [2–4]. Using the classical theory and analysing the triangle graph in the Bethe–Salpeter formalism with a constant $Kq\bar{q}$-vertex, Greenberg, Nussinov and Sucher [2] have concluded that the ratio ξ decreases in relativistic models. Calculation of the triangle

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graph in papers [3, 4] with the assumptions from paper [2] support the hypothesis $\xi \leqslant \xi_{NR}$. It should be noted that Nowak and Sucher [4] pointed out a possible breaking of this inequality for certain modifications of the $K q\bar{q}$-vertex, which are artificial within the "relativistic effective-range approximation" proposed in [4].

In this note the problem of relativistic corrections to ξ_{NR} is considered in the framework of the relativistic Hamiltonian formalism [5–7] which is equally valid both for kinematical and dynamical effects of the order of v^2/c^2 in two-particle bound states. The consideration is carried out for classical and quantum theories. The formalism used includes, in particular, models of the interaction of a pointlike particles via scalar and vector fields and allows one to reproduce the results [2–4] of calculation of the triangle graph (with accuracy $\sim v^2/c^2$). It is shown that the sign of the relativistic corrections of the order of v^2/c^2 to ξ_{NR} is model-dependent and defined by $q\bar{q}$-interaction dynamics both in the lowest and the subsequent orders of v^2/c^2. For a wide class of models the inequality $\xi \geqslant \xi_{NR}$ holds in contrast to the one proposed in [2].

2. *Classical treatment*

The relativistic effects in a $q\bar{q}$-system within the classical relativistic theory are described in paper [2] with the use of the c.m.s. relation

$$v_i(t) = \pm \frac{p(t)}{E_i(t)}, \qquad E_i(t) = \sqrt{p^2(t)+m_i^2} \tag{2}$$

where $i = \{\text{s, u}\}$, $v_i = \dfrac{dx_i}{dt}$ are velocities of particles and $\pm p(t)$ their momenta. Relation (2) gives rise to the "relativistic dilution" of the mass asymmetry given by the inequality $\dfrac{\langle x_s^2\rangle}{\langle x_u^2\rangle} \sim \dfrac{\langle E_u^2\rangle}{\langle E_s^2\rangle} \geqslant \dfrac{m_u^2}{m_s^2}$ ($\langle\ldots\rangle$ means averaging over the period of relative motion). As $\xi = \left(\dfrac{\langle x_s^2\rangle}{\langle x_u^2\rangle} - 1\right)\left(2\dfrac{\langle x_s^2\rangle}{\langle x_u^2\rangle} + 1\right)$, from (2) the inequality $\xi \leqslant \xi_{NR}$ follows. Note that the relation (2), valid for free relativistic particles, fails, in general, to describe the motion in the bound state (e.g., in classical electrodynamics). In the relativistic (up to the terms $\sim v^2/c^2$) theory of interacting particles described by the Hamiltonian [5]

$$H = \sum_{i=1,2}\left(\frac{\vec{p}_i^{\,2}}{2m_i} - \frac{\vec{p}_i^{\,4}}{8m_i^3c^2}\right) + V(r) - \frac{1}{4m_1m_2c^2}\left[\left(\vec{p}_1 V \vec{p}_2 - \vec{p}_1\vec{r}\,\frac{1}{r}\,\frac{dV}{dr}\,\vec{r}\vec{p}_2\right) + (1 \rightleftarrows 2)\right]$$

$$(\vec{r} = \vec{x}_1 - \vec{x}_2,\ r = |\vec{r}|) \tag{3}$$

relation (2) is certainly invalid for $V(r) \neq 0$. In the c.m.s. $(\vec{p}_1 + \vec{p}_2 = 0)$, $\vec{R} = \dfrac{c^2}{E} \times \left[m_1\left(1 + \dfrac{\vec{p}_1^{\,2}}{2m_1^2c^2}\right)\vec{x}_1 + m_2\left(1 + \dfrac{\vec{p}_2^{\,2}}{2m_2^2c^2}\right)\vec{x}_2 + \dfrac{(\vec{x}_1 + \vec{x}_2)V(r)}{2c^2}\right]$ (E is the total energy) is an integral of motion. Choosing the origin of coordinates so as to fulfil the condition $\vec{R} = 0$, we obtain

$$\vec{x}_1 = \frac{\vec{r}\tilde{m}_2}{\tilde{m}_1+\tilde{m}_2}, \qquad \vec{x}_2 = -\frac{\vec{r}\tilde{m}_1}{\tilde{m}_1+\tilde{m}_2}$$

where

$$\tilde{m}_i = m_i + \left[\frac{\vec{p}^2}{2m_i} + \frac{V(r)}{2}\right]\frac{1}{c^2}; \qquad \vec{p} = \vec{p}_1 = -\vec{p}_2.$$

Within the same accuracy $\sim v^2/c^2$ we have

$$\frac{\langle \vec{x}_1^2 \rangle}{\langle \vec{x}_2^2 \rangle} = \frac{m_2^2}{m_1^2}\left[1 + \frac{m_1-m_2}{m_1 m_2 \langle r^2 \rangle c^2}\left\langle r^2\left(\frac{\vec{p}^2}{\mu} + V(r)\right)\right]\right\rangle, \qquad \mu = \frac{m_1 m_2}{m_1+m_2}. \tag{4}$$

Setting $m_1 = m_s$ and $m_2 = m_u$ it can be seen that the relativistic correction to ξ_{NR} is opposite in sign to the quantity

$$\left\langle r^2\left(\frac{p^2}{\mu} + V(r)\right)\right\rangle = \varepsilon\langle r^2 \rangle + \frac{\langle r^2 p^2 \rangle}{2\mu} \tag{5}$$

(where ε is the binding energy). For the motion in the region of $V(r) < 0$, $|\langle r^2 V(r) \rangle| > \frac{\langle r^2 p^2 \rangle}{\mu}$ the sign of the classical correction to ξ_{NR} is positive, in contrast to the result of Ref. [2].

Equality (4) holds for the interaction of particles via scalar, vector, and tensor fields in the limit of weak coupling. For the general relativistic invariant (within $\sim v^2/c^2$ accuracy) Hamiltonian theory [6] the expression for the centre-of-mass coordinate contains an arbitrary function $\Omega^{(1)}(r; m_1, m_2)$ antisymmetric relative to the transposition $1 \rightleftarrows 2$:

$$\vec{R} = \frac{1}{m}\left[m_1\vec{x}_1 + m_2\vec{x}_2 + \frac{1}{2c^2}\vec{r}\left(\mu\left(\frac{p_1^2}{m_1^2} - \frac{p_2^2}{m_2^2}\right) - \Omega^{(1)}(r; m_1, m_2)\right)\right], \qquad m = m_1 + m_2.$$

In the c.m.s.

$$\frac{\langle \vec{x}_1^2 \rangle}{\langle \vec{x}_2^2 \rangle} = \frac{m_2^2}{m_1^2}\left[1 + \frac{m_1-m_2}{m_1 m_2 \langle r^2 \rangle c^2}\left\langle r^2\left(\frac{p^2}{\mu} + \frac{m}{m_1-m_2}\Omega^{(1)}(r; m_1, m_2)\right)\right\rangle\right]. \tag{6}$$

At $\Omega^{(1)} = \frac{m_1-m_2}{m}V(r)$, expression (6) turns into (4).

So, the classical theory reveals a strong dependence of the relativistic corrections to ξ_{NR}, in sign and magnitude, on the interaction dynamics both in the lowest (nonrelativistic) and in the subsequent orders in v^2/c^2.

3. *Quantum models*

Following [1–4], we shall calculate ξ in quantum theory by expressing $\langle r^2_{\rm em}\rangle$ through the electromagnetic form factor

$$\langle r^2_{\rm em}\rangle = 6\,\frac{\partial F_{\rm em}(q^2)}{\partial q^2}\bigg|_{q^2=0},$$

where q is the 4-dimensional vector of the transferred momentum. We shall consider (cf. [2]) the bound states of spinless particles. In the model with Hamiltonian (3) the wave functions of states with a definite total momentum $\vec{P} = \vec{p}_1+\vec{p}_2$ can be obtained in x-representation through the standard procedure [7]:

$$\Psi_{\vec{P}}(\vec{x}_1, \vec{x}_2) = \exp(i\vec{P}\vec{R}_0)\exp(i\hat{\chi}(\vec{P}))\,[\varphi_0(\vec{r})+\varphi_1(\vec{r})], \tag{7}$$

where

$$\hat{\chi}(\vec{P}) = \hat{\chi}_1(\vec{P})+\hat{\chi}_2(\vec{P}) \tag{8}$$

$$\hat{\chi}_1(\vec{P}) = \frac{1}{4m^2c^2}(\vec{P}\vec{r}\cdot\vec{P}\vec{p}+\vec{P}\vec{p}\cdot\vec{P}\vec{r}) \tag{8a}$$

$$\hat{\chi}_2(\vec{P}) = -\frac{m_1-m_2}{2m_1m_2c^2}\left[\mu(\vec{P}\vec{r})V(r)+\tfrac{1}{2}(\vec{p}^{\,2}(\vec{P}\vec{r})+(\vec{P}\vec{r})\vec{p}^{\,2})\right]$$

$$\vec{R}_0 = m^{-1}(m_1\vec{x}_1+m_2\vec{x}_2), \tag{8b}$$

$\vec{p}$ is an operator canonically conjugate to $\vec{r}$ ($\vec{x}_i$, $\vec{p}_i$ obey the conventional commutation relations). The internal wave function $\varphi(\vec{r}) = \varphi_0(\vec{r})+\varphi_1(\vec{r})$ is a solution of the Schrödinger equation for the internal motion

$$(H_0+H_1)\varphi = \varepsilon\varphi,$$

where $H_0 = \dfrac{\vec{p}^{\,2}}{2\mu}+V(r)$ is the nonrelativistic Hamiltonian,

$$H_1 = -\frac{1}{c^2}\left[\frac{\vec{p}^{\,4}}{8}(1/m_1^3+1/m_2^3)+\frac{1}{2m_1m_2}\left(\vec{p}\vec{r}\,\frac{1}{r}\,\frac{dV}{dr}\,\vec{r}\vec{p}-\vec{p}V\vec{p}\right)\right], \tag{9}$$

and $\varphi_0(\vec{r})$ is an eigenfunction of H_0. The $\sim v^2/c^2$ corrections to the electromagnetic form factor of states described by the wave functions (7) are of three kinds:

i) Corrections due to the difference of the internal wave function from $\varphi_0(\vec{r})$;

ii) Kinematic effects of the "Lorentz contraction" of the wave function [8] described by operator $\hat{\chi}_1$ quadratic in $\vec{P}$. Up to the term of the order v^2/c^2 these effects imply the change $F_{\rm NR}(q^2) \to F_{\rm NR}\left(q^2+\dfrac{(q^2)^2}{4m^2c^2}\right)$;

iii) Corrections due to the operator $\hat{\chi}_2$ linear in $\vec{P}$; they exist only for particle systems with $m_1 \neq m_2$.

By simple, though lengthy, calculations we obtain the following expression for the form factor of a two-particle system with charges e_1, e_2 (within the terms $\sim q^2$, v^2/c^2):

$$F_{\mathrm{em}}(q^2) \simeq e_1+e_2+\frac{q^2}{6}\left[\langle r^2\rangle \frac{m_2^2 e_1+m_1^2 e_2}{m^2}\right.$$

$$\left. -\frac{(m_1-m_2)}{m^3c^2}(e_2m_1-e_1m_2)\left\langle r^2\left(V(r)+\frac{\vec{p}^2}{2\mu}\right)+\vec{r}\,\frac{\vec{p}^2}{2\mu}\,\vec{r}\right\rangle\right]. \tag{10}$$

The first term in brackets is averaged with the wave function $\varphi = \varphi_0+\varphi_1$ (that results in the correction of class (*i*)); the second term with $1/c^2$ is averaged with the non-relativistic wave function $\varphi_0(\vec{r})$. Note that the corrections of class (*ii*) are of higher order in q^2 and do not contribute to the root-mean-square (rms) charge radius of a bound state.

Calculating the ratio $\dfrac{\langle r^2_{\mathrm{em}}\rangle_{K^0}}{\langle r^2_{\mathrm{em}}\rangle_{K^+}}$ by formula (10) we find that the relativistic correction to ξ_{NR} is opposite in sign to the quantity

$$\eta = \left\langle r^2\left(V(r)+\frac{\vec{p}^2}{2\mu}\right)+\vec{r}\,\frac{\vec{p}^2}{2\mu}\,\vec{r}\right\rangle \tag{11}$$

i.e. the quantum result coincides with the classical one up to the ordering of operators $\vec{r}$, $\vec{p}^2$.

The Hamiltonian of the general relativistic-invariant (up to v^2/c^2) theory contains, besides $V(r)$, 4 arbitrary functions of the variable $r = |\vec{x}_1-\vec{x}_2|$ [6]. Three of them enter into the Hamiltonian of internal motion generalising (9). This arbitrariness only changes $\langle r^2\rangle$ in (10) and does not influence the ratio of the mean square charge radii ξ. The fourth function, nonzero for nonidentical particles only, enters into the operator $\hat{\chi}_2(\vec{P})$ (8b). It can be shown that, like in classical theory, the general expression for the form factor (10) and η (11) can be obtained by the change $V \to \dfrac{m}{m_1-m_2}\,\Omega(r; m_1, m_2)$, and all characteristics of the bound state in the c.m.s. (binding energy, $\langle r^2\rangle$, ...) are independent of $\Omega(r; m_1, m_2)$.

Consider some examples within models of the type (3). For s-states specified by the potential $V(r) = -\dfrac{\mathrm{const}}{r}$ the result of the calculation of $\eta = 2\varepsilon\langle r^2\rangle+\dfrac{3}{2\mu}-\langle r^2V\rangle$ is $\eta \sim 1-n^2$ (n is the principal quantum number); in the ground state ($n = 1$) there is no relativistic correction of the order v^2/c^2 to ξ_{NR}. A more difficult task is the calculation of η for the ground state in the square well potential

$$V(r) = \begin{cases} -V_0, & r < r_0 \\ 0, & r > r_0. \end{cases} \tag{12}$$

From the calculation it follows that the sign of η is determined by the parameter $\alpha = r_0^{-1}(2\mu V_0)^{-1/2}\left(0 < \alpha \leqslant \dfrac{2}{\pi}\right)$:

$$\mathrm{sign}\,\eta = -\mathrm{sign}\,(\xi-\xi_{NR}) = \mathrm{sign}\,(\alpha-\alpha_0), \qquad \alpha_0 \simeq \frac{2}{3\pi}. \tag{13}$$

For $\alpha < \alpha_0$ the sign of the relativistic correction is positive. Note that the potential of zero range forces, for which the wave function of the ground state is defined by the constant $K q\bar{q}$-vertex of the triangle graph (in the nonrelativistic limit), can be derived from (12) as $r_0 \to 0$, $\alpha \to \frac{2}{\pi}$. By formula (13) in this case $\xi < \xi_{NR}$ in conformity with the result of papers [2–4].

So, in quantum models the sign of the correction to ξ_{NR} also depends both on the form of the lowest order potential and on the $q\bar{q}$-interaction dynamics in higher order in v^2/c^2. The consideration performed reveals the possibility of positive or extremely small $(\xi - \xi_{NR})$. In this connection it is interesting to note that the recent experimental data on $\langle r^2_{em} \rangle_{K^-, K^0}$ [9] give the value $\xi = 0.19 \pm 0.1$ in agreement with the nonrelativistic estimate $\xi \simeq 0.2$ [1, 3].

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