QUASI-CLASSICAL APPROXIMATION BRIDGING THE GAP BETWEEN THE STRONG- AND WEAK-COUPLING REGIONS IN A LATTICE GAUGE THEORY MODEL

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A version of the WKB approximation is recalled, which in a simple model of lattice gauge theory is valid at and in the vicinity of the transition point. In this approach the phase transition reflects a singularity in the classical period of the motion considered as a function of the coupling constant.

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In lattice gauge theories the strong coupling range, where confinement is easy to prove, and the weak coupling range, which is the physically relevant one since it corresponds to the continuum limit, are often separated by singularities (phase transitions). Therefore, validity ranges of the strong coupling and of the weak coupling expansions do not overlap, and other approximations able to bridge the transition region are of interest. Studying from this point of view the one-plaquette gauge model solved by Wadia [1], we noticed that a variant of the WKB approximation is applicable in all the range from weak coupling across the transition region to strong coupling. The inconvenience of using only strong coupling and weak coupling approximations in this problem is clearly seen in Ref. [2], where all the expansions used fail in the most interesting region.

The relevant WKB formula has been derived long ago [3]. In order to make this paper self contained, we give a derivation in the appendix. Consider the one-dimensional Schrödinger equation

$$\psi''+\beta(\varepsilon-V(x))\psi = 0, \tag{1.1}$$

where $2m\beta = \hbar^2$, $\beta\varepsilon = E$ and the potential $V(x)$ is twice differentiable and periodic in x. This is a classical problem in solid state theory, but now it gained new importance, because with β defined in another way it occurs in some models of lattice gauge theory [1], [2].

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Without loss of generality we may assume that the period is π and that the maximum and minimum of $V(x)$ are 0 and 1. We will also assume that there is exactly one maximum and one minimum per period and that the second derivative of $V(x)$ does not vanish at the maximum. The special case considered in Ref. [1] is

$$V(x) = \sin^2 x, \tag{1.2}$$

but the generalization as made here does not complicate the problem. Motivated by gauge theory, we limit our discussion to periodic solutions with period π, but the extension to many other cases is easy. The range $0 \leqslant \varepsilon < 1$ ($\varepsilon > 1$) is known in gauge theory as weak (strong) coupling and in solid state theory as strong (weak) coupling. In the following we use the terminology from gauge theory. At $\varepsilon = 1$ the phase transition occurs.

Under rather general conditions (cf. e.g. Appendix) the approximate eigenvalues of equation (1) in the strong coupling region, in the vicinity of the transition point and sometimes, in particular for large β, even in all the weak coupling region can be obtained from the formula

$$\int_{x_L}^{x_R} \sqrt{\beta(\varepsilon_n^{\pm} - V(x))}\, dx = [2n - \varphi(a) \pm (\delta - \tfrac{1}{2})]\pi. \tag{1.3}$$

Here for $\varepsilon < 1$: x_L and x_R are the adjacent classical turning points; for $\varepsilon \geqslant 1$, x_L is arbitrary and $x_R = x_L + \pi$. The parameter

$$a = \sqrt{\frac{\beta}{2V_2}}(1 - \varepsilon_n^{\pm}), \tag{1.4}$$

V_2 is the absolute value of the second derivative of $V(x)$ at the maximum,

$$\pi\varphi(a) = \arg \Gamma(\tfrac{1}{2} + ia) + a - a \ln |a| \tag{1.5}$$

and δ is the smallest positive root of the equation

$$\sin(\pi\delta) = \frac{1}{\sqrt{1 + e^{2\pi a}}}. \tag{1.6}$$

The phase transition occurs, when $E(\beta)$ has a singularity. For $\varepsilon \approx 1$ the right hand side of relation (1.3) reduces to $(2n \pm \frac{1}{4})\pi$. Substituting this approximation into (1.3) and differentiating, one finds

$$\frac{dE}{d\beta} = \frac{E}{\beta} - \frac{(2n \pm \frac{1}{4})\pi}{\beta T}, \tag{1.7}$$

where

$$T = \int_{x_L}^{x_R} \frac{dx}{\sqrt{E - \beta V(x)}} \tag{1.8}$$

is the classical time (in suitable units) necessary to go from x_L to x_R at energy E. This time becomes logarithmically infinite for $\varepsilon = 1$ and thus generates the singularity.

In Ref. [2] the phase transition has been associated with tunneling. For potential (1.2) the two criteria are equivalent. For a potential given by formula (1.2) for $4|x| \leqslant \pi$ and infinite for $\pi < 4|x| \leqslant 2\pi$, however, there is no tunneling, but T is singular and there is a phase transition. Thus the criterion proposed here generalizes that from Ref. [2]. This is not a purely academic example, because modifications of Wilson's formula for the lattice action, leading in Wadia's problem to potentials coinciding with (1.2) only in the vicinity of the minimum, have been considered [4] and give promissing results (cf. e.g. [5]).

Let us finally note that performing the classical limit in lattice theories requires some care. For instance, in the lattice version of the harmonic oscillator problem [6] our parameter β is

$$4x = \frac{\hbar^2}{mkd^2}, \tag{1.9}$$

where k is a constant and d denotes the lattice spacing. Here the continuum limit $d \to 0$ and the usual classical limit $\hbar \to 0$ cannot be taken simultaneously without further assumptions. The correct continuum limit can be obtained only if the parameter (1.9) tends to infinity. Reliability of the WKB solution may then be secured by choosing for analysis levels with sufficiently high n.

APPENDIX

In the vicinity of each maximum we can approximate the Schrödinger equation (1.1) by

$$\psi'' + (\tfrac{1}{4} z^2 - a)\psi = 0, \tag{A1}$$

where

$$z = \sqrt[4]{2\beta V_2}\,(x - x_0), \tag{A2}$$

x_0 is the coordinate of the maximum nearest to x, and a is defined by formula (1.4). Equation (A1) is standard. Its general solution [7] is

$$\psi(x, a) = C_1(a)E(z, a) + C_2(a)E^*(z, a), \tag{A3}$$

where E is the parabolic cylinder function and E^* its complex conjugate. The coefficients C_1, C_2 do not depend on x_0, because of the assumed periodicity of ψ in x. For $x_L < x < x_R$ at a safe distance from x_L and x_R we use the WKB approximation

$$\psi(x, a) = \left(\sqrt{\frac{\beta}{2V_2}}(\varepsilon - V(x))\right)^{-1/4} \left[C_+(a) \exp\left\{i \int_{x_L}^{x_R} \sqrt{\beta(\varepsilon - V(x))}\, dx\right\}\right.$$
$$\left. + C_-(a) \exp\left\{-i \int_{x_L}^{x_R} \sqrt{\beta(\varepsilon - V(x))}\, dx\right\}\right]. \tag{A4}$$

Again the independence of C_+ and C_- on x_L follows from periodicity of ψ.

Let us assume that the validity regions of the quadratic approximation (A3) and the WKB approximation (A4) overlap. This is certainly true, when x_L and x_R are sufficiently close to the maxima of $V(x)$, i.e. for ε sufficiently large. If the quadratic approximation holds for $|z| < z_1$, the condition is

$$a \ll \tfrac{1}{4} z_1^2. \tag{A5}$$

We show later that for large β this constraint may be removed. For x in the overlap region, expression (A3) may be simplified by using the asymptotic forms of the parabolic cylinder functions, and expression (A4) by using the quadratic approximation for the potential in the region between x and the nearest turning point (x_L or x_R).

To the right of x_L the two expressions are

$$\psi(x, a) = \sqrt{\frac{2}{|z|}}\,[C_1(a)e^{if(z,a)} + C_2(a)e^{-if(z,a)}], \tag{A6}$$

$$\psi(x, a) = \sqrt{\frac{2}{|z|}}\,[C_+(a)e^{ig(z,a)} + C_-(a)e^{-ig(z,a)}], \tag{A7}$$

where

$$f(z, a) = g(z, a) + \frac{\pi}{2}\,\varphi(a) + \frac{\pi}{4}, \tag{A8}$$

$$g(z, a) = \tfrac{1}{4} z^2 - a \ln z + \tfrac{1}{2} a \ln |a| - \tfrac{1}{2} a. \tag{A9}$$

The corresponding formulae to the left of x_R are

$$\psi(x, a) = \sqrt{\frac{2}{|z|}}\, i[\{\sqrt{1+e^{2\pi a}}\, C_1(a) + e^{\pi a} C_2(a)\}e^{-if(z,a)}$$

$$-\{\sqrt{1+e^{2\pi a}}\, C_2(a) + e^{\pi a} C_1(a)\}e^{if(z,a)}], \tag{A10}$$

$$\psi(x, a) = \sqrt{\frac{2}{|z|}}\,[C_+(a)e^{i(S(\varepsilon)-g)} + C_-(a)e^{-i(S(\varepsilon)-g)}], \tag{A11}$$

where g is given by (A9) and S denotes the integral on the left hand side of formula (1.3). Identifying (A6) with (A7) and (A10) with (A11), we find a non-zero solution only if condition (1.3) is satisfied.

Since for $a \to \infty$ formula (1.3) yields the correct limit (the Bohr–Sommerfeld formula), constraint (A5) may be dropped, if a becomes sufficiently large in the validity range of (A5). The quadratic approximation (A1) is valid in some range of x depending on the shape of $V(x)$. Consequently, for given $V(x): z_1^2 \sim \sqrt{\beta}$ and constraint (A5) may be skipped for β sufficiently large.

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