COULD THE UNIFICATION OF FERMIONIC FLAVOURS AND GENERATIONS PRECEDE THE UNIFICATION OF FLAVOURS AND COLOURS?

By W. Królikowski*

CERN, Geneva

and S. Pokorski

Institute of Theoretical Physics, Warsaw University**

(Received October 8, 1981)

We consider the hypothesis that the "big" unification of fermionic flavours and generations takes over the rôle of the "grand" unification of flavours and colours, possibly being followed by the "extended" unification of flavours, generations and colours. We propose a definite unification scheme of fermion interactions which differs from the conventional one by interchanging the rôles of four colours (l, r, y, b) and four generations (e, μ, τ, ω), thereby implying the existence of the fourth fermion generation (ω) consisting of two leptons (ν_ω, ω^-) and two quarks (h, o).

PACS numbers: 12.20.Hx, 12.30.-s

1. Introduction

In contrast to most of the current work in this field [1, 2], we discuss here the possibility that the "big" unification of fermionic flavours and generations precedes the "grand" unification of flavours and colours [3], the latter unification being in fact replaced then by the "extended" unification of flavours, generations and colours [4]. Speaking more precisely, we consider the hypothesis that the standard electroweak symmetry group $SU(2)\times U(1)$ for flavours is unified first with a "horizontal" symmetry group H for generations and only later, possibly, with the standard strong symmetry group $SU_C(3)$ for colours, while according to the currently popular hypothesis those which are unified first are the two "vertical" symmetry groups $SU(2)\times U(1)$ and $SU_C(3)$ for flavours and colours.

The argument which lay at the roots of our devising this alternative scheme of unification was the affinity between flavours and generations (both displaying broken symmetries)

* On leave from the Institute of Theoretical Physics, Warsaw University, Warsaw, Poland.

** Address: Instytut Fizyki Teoretycznej, Uniwersytet Warszawski, Hoża 69, 00-681 Warszawa, Poland.

and, on the other hand, the different quality of colours (related to exact non-Abelian symmetries).

Another argument in favour of our unification scheme goes as follows. Assuming the fermion contributions to the renormalization group β functions for different couplings to be similar, their Q^2 evolution is determined by the size of the respective gauge groups. It is well known that in the conventional unification scheme the coupling constants g and g_S of the standard SU(2) and $SU_C(3)$ meet at $M_G \sim 10^{15}$ GeV. Here we introduce the horizontal $SU_H(N)$ gauged interactions with some strength g_H and mass scale M_H. Obviously there are several possibilities for the behaviour of the $g_H^2(Q^2)$ on the "coupling plot". First we take $M_H \ll M_G$ and consider the cases:

Case 1: $g^2(M_H^2) < g_H^2(M_H^2) < g_S^2(M_H^2)$ and $N \geqslant 3$. Then g_H and g meet below M_G (see Fig. 1).

Case 2: $g^2(M_H^2) < g_H^2(M_H^2) < g_S(M_H)$ and $N < 3$. Then g_H and g_S meet below M_G.

Case 3: $g_S^2(M_H^2) < g_H^2(M_H^2)$ and $N > 3$. Then the situation depends on the actual values of $g_H^2(M_H^2)$ and N. Note, however, that if these values are such that g_H does not meet g below M_G and if we assume the minimal $SU_G(5)$ unification above M_G then $SU_H(N)$ must be larger than SU(5) in order to get $g_H = g_G$ at some mass scale above M_G.

Case 4: $g_S^2(M_H^2) < g_H^2(M_H^2)$ and $N \leqslant 3$. Then g_H does not meet the other g's.

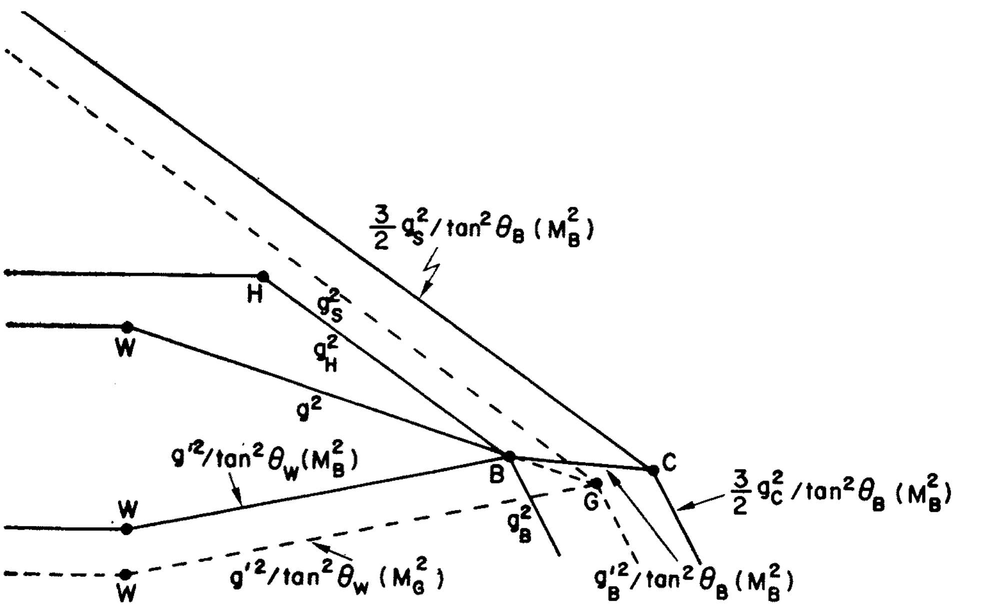

Fig. 1. Ideological coupling plot in Case 1 corresponding to the symmetry-breaking chain $U_{EM}(1) \times SU_C(3) \underset{W}{\subset} SU(2) \times U(1) \times SU_C(3) \underset{H}{\subset} SU(2) \times U(1) \times \widetilde{SU}_H(3) \times SU_C(3) \underset{B}{\subset} SO_B(10) \times U_C(1) \times SU_C(3) \underset{C}{\subset} SO_B(10) \times SU_C(4)$. The dashed lines illustrate the conventional grand unification

Thus, if we want to include the horizontal interactions with the mass scale $M_H \ll M_G$ into the extended unification programme, the above list suggests that the unification of SU(2)×U(1) and H (Case 1, e.g., H = $SU_H(3)$) or $SU_C(3)$ and H (Case 2, e.g., H = $SU_H(2)$) is likely to precede the unification of SU(2)×U(1) and $SU_C(3)$. In the opposite situation when $M_H \sim M_G$ we get g_H constant (up to fermion effects) below M_G and then, depending on $g_H^2(M_G^2) < g_G^2(M_G^2)$ or $g_G^2(M_G^2) < g_H^2(M_G^2)$, we need the horizontal group smaller or larger than the grand unification group to realize the extended unification programme (unless g_H meets g or g_S just below M_G).

2. *The big unification*

When H is gauged, our scheme of unification leads to a new spontaneously broken gauge group: the electroweak-horizontal symmetry group B ⊃ SU(2)×U(1)×H which describes in a unified way the usual electroweak interactions and the new horizontal interactions. We achieve this by introducing the additional electroweak-horizontal interactions. The first and second of these interactions are mediated by the gauge bosons of SU(2)×U(1) and H respectively, and the third by the rest of the gauge bosons of B [5].

The horizontal symmetry group H, describing the fermion generation pattern, may or may not reflect a preon substructure of leptons and quarks. This group has been extensively discussed in the past, being chosen discrete [1] or continuous [2] in the latter case usually local (gauged). In the present paper, our tentative choice for H is the gauged chiral $\widetilde{SU}_H(3)$ which (as well as the vector-like $SU_H(3)$) is the next-to-minimal H accounting for at least three fermion generations, the minimal one being the popular $\widetilde{SU}_H(2) = \widetilde{SO}_H(3)$ (or $SU_H(2) = SO_H(3)$) [6]. Our choice of the chiral (rather than vector-like) group H, defined in such a way that the left- and right-handed fermions belong to representations conjugate with each other, is motivated by the fact that in this case the usual vector-like colour group $SU_C(3)$ commutes with the big unification group. Otherwise it does not commute and must, therefore, be replaced (at least above the big unification point) by the chiral colour group $\widetilde{SU}_C(3)$. Note that any chiral group $\tilde{G}$ is certainly broken down below the electroweak unification point (where fermion masses appear), but its actual breakdown point may lie much higher if its vector-like counterpart G is broken down at this higher mass scale.

As we shall see, the group $\widetilde{SU}_H(3)$, when unified with SU(2)×U(1) in the minimal way, actually implies the existence of four fermion generations forming a singlet and a triplet under $\widetilde{SU}_H(3)$ in each of four fermion flavour families:

$$\begin{aligned} &\nu_e,\ \nu_\alpha = \nu_\mu, \nu_\tau, \nu_\omega \\ &e^-,\ e_\alpha^- = \mu^-, \tau^-, \omega^- \\ &u,\ u_\alpha = c, t, h \\ &d,\ d_\alpha = s, b, o \qquad (\alpha = \mu, \tau, \omega), \end{aligned} \tag{1}$$

where four fermions of the first generation are assumed as singlets of $\widetilde{SU}_H(3)$, while four new fermions of the fourth generation [7–9] are included in triplets of $\widetilde{SU}_H(3)$. Of course, each of the u, d and u_α, d_α quarks is a triplet of the colour $SU_C(3)$. The symbols h and o used for the up and down quarks of the fourth generation are from the words *harmony* and *odd* which are intended to play the role of charm and strange in the new situation.

In fact, in our case we can repeat the construction of the usual grand unification groups $SU_G(5)$ [10] and $SO_G(10)$ [11], if we do not start from the standard group $SU(2) \times U(1) \times SU_C(3)$ but from the group $SU(2) \times \tilde{U}'(1) \times \widetilde{SU}_H(3)$, where the factors U(1) and $\tilde{U}'(1)$ are generated by Y and $\gamma_5 Y'$ respectively, the latter satisfying the relation $Y-(B-L) = Y'-(B'-L')$. Here, L' and $3B'$ denote the numbers of fermions belonging to the singlet light generation e and to the triplet of heavy generations μ, τ, ω, respectively. From this relation we can see that $\tilde{U}'(1) \times \tilde{U}_H(1) \times U_C(1) \supset U_R(1) \times U_C(1) \supset U(1)$, where the groups $U_C(1)$, $\tilde{U}_H(1)$ and $U_R(1)$ are generated by $B-L$, $\gamma_5(B'-L')$ and $I_R^3 = \frac{1}{2}Y' - \frac{1}{2}(B'-L')$ respectively. The minimal unification of $SU(2) \times U'(1)$ and $\widetilde{SU}_H(3)$ gives the fourth rank group $SU_B(5) \supset SU(2) \times U'(1) \times \widetilde{SU}_H(3)$ having the reducible representation $\underline{5}^* + \underline{10} + \underline{1}$ which may be identified with each of four fermion colour families (l, r, y, b) *if* it contains the members of *four* fermion generations (e, μ, τ, ω) as given in Eq. (1). Moreover, this reducible representation may be considered as the irreducible representation $\underline{16}$ of the fifth rank group $SO_B(10) \supset SU_B(5)$. Under the group $SO_B(10) \times SU_C(3)$, four fermion flavour families (1) can be reorganized into four $\underline{16}$'s, one of them corresponding to leptons and three to quarks of three colours (suppressed in our notation):

$$l = \Big[\underbrace{\begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L, e^-_{\alpha R};}_{\underline{5}^*} \underbrace{e^-_R, \begin{pmatrix} \nu_\alpha \\ e^-_\alpha \end{pmatrix}_L, \nu_{\alpha R};}_{\underline{10}} \underbrace{\nu_R}_{\underline{1}} \; (\alpha = \mu, \tau, \omega)\Big] \tag{2}$$

and

$$q = \Big[\underbrace{\begin{pmatrix} u \\ d \end{pmatrix}_L, d_{\alpha R};}_{\underline{5}^*} \underbrace{d_R, \begin{pmatrix} u_\alpha \\ d_\alpha \end{pmatrix}_L, u_{\alpha R};}_{\underline{10}} \underbrace{u_R}_{\underline{1}} \; (\alpha = \mu, \tau, \omega)\Big]. \tag{3}$$

The lepton $\underline{16}$ denoted by l is a singlet of the usual colour $SU_C(3)$, whereas three quark $\underline{16}$'s comprised into q are a triplet of this group:

$$q = \begin{pmatrix} q_r \\ q_y \\ q_b \end{pmatrix}. \tag{4}$$

Thus, under $SO_B(10) \times SU_C(3) \supset SU_B(5) \times SU_C(3)$

$$l = (\underline{16}, \underline{1}) = (\underline{5}^*, \underline{1}) + (\underline{10}, \underline{1}) + (\underline{1}, \underline{1}) \tag{5}$$

and

$$q = (\underline{16}, \underline{3}) = (\underline{5}^*, \underline{3}) + (\underline{10}, \underline{3}) + (\underline{1}, \underline{3}), \tag{6}$$

where l and q are given explicitly in Eqs. (2) and (3).

At this point, however, the following caution is necessary. Though the fermion multiplets (2) and (3) have the adequate $SU(2)\times\tilde{U}'(1)\times\widetilde{SU}_H(3)$ contents

$$(\underline{2},\underline{1})_{\gamma_5 Y'=-1}+(\underline{1},\underline{3}^*)_{\gamma_5 Y'=2/3}+(\underline{1},\underline{1})_{\gamma_5 Y'=2}+(\underline{2},\underline{3})_{\gamma_5 Y'=1/3}+(\underline{1},\underline{3}^*)_{\gamma_5 Y'=-4/3}+(\underline{1},\underline{1})_{\gamma_5 Y'=0}$$

to constitute the irreducible representations $\underline{16}$ of $SO_B(10)$, they *mix* left and right fermions and lead, therefore, to vanishing of some off-diagonal generators of $SO_B(10)$. In fact, the generators transforming as $(\underline{2},\underline{3})_{\gamma_5 Y'=1/3}$, $(\underline{2},\underline{3}^*)_{\gamma_5 Y'=-1/3}$, $(\underline{2},\underline{3})_{\gamma_5 Y'=-5/2}$ and $(\underline{2},\underline{3}^*)_{\gamma_5 Y'=5/2}$ vanish within these multiplets (while the rest of them, generating the subgroup $SU_L(2)\times SU_R(2)\times\widetilde{SU}_H(4)$, does not). We may call the multiplets (2) and (3) the *selfbreaking representations* $\underline{16}$ of $SO_B(10)$. In spite of the mentioned deficiency they provide, on the one-loop level, the coupling-constant unification for $SU(2)\times\tilde{U}'(1)$ and $\widetilde{SU}_H(3)$ since in the boson sector the group $SO_B(10)$ can be fully represented (and the fermion contributions, if calculated on the one-loop level, are equal for all three factors). So throughout the present paper we shall restrict ourselves to the one-loop-level unification. This "partial" unification, though estetically less satisfying than the grand unification, may be nevertheless a realistic alternative to the latter if there are horizontal interactions corresponding to Case 1 discussed in Introduction.

The group $SO_B(10)$, since it does not contain U(1), provides no unification group for $SU(2)\times U(1)$ and $\widetilde{SU}_H(3)$. Such a group is provided instead by $SO_B(10)\times U_C(1)$, because

$$\begin{aligned}SO_B(10)\times U_C(1) &\supset SU_L(2)\times\underbrace{SU_R(2)}_{\supset U_R(1)}\times\underbrace{\widetilde{SU}_H(4)}_{\supset\widetilde{SU}_H(3)\times\tilde{U}_H(1)}\times U_C(1)\\ &\supset SU_L(2)\times U_R(1)\times\widetilde{SU}_H(3)\times U_C(1)\supset SU(2)\times U(1)\times\widetilde{SU}_H(3)\end{aligned}\tag{7}$$

since $U_R(1)\times U_C(1)\supset U(1)$. Here $SU_L(2)\equiv SU(2)$. The relation between the generators Y, $\gamma_5 Y'$, $B-L$, $\gamma_5(B'-L')$ and I_R^3 of the groups U(1), $\tilde{U}'(1)$, $U_C(1)$, $\tilde{U}_H(1)$ and $U_R(1)$ respectively, is

$$Y = Y'-(B'-L')+B-L = 2I_R^3+B-L\tag{8}$$

and hence

$$Q = I_L^3+\frac{Y}{2} = I_L^3+I_R^3+\frac{B-L}{2}.\tag{9}$$

Here $I_L^3\equiv I_3$. In terms of the groups $SU_C(4)\supset SU_C(3)\times U_C(1)$ and $SU_H(4)\supset SU_H(3)\times U_H(1)$ we get

$$B-L = \frac{2\sqrt{6}}{3}F_C^{15} = \frac{\sqrt{6}}{3}\lambda_C^{15} = \left\{\begin{matrix}-1&&&\\&\frac{1}{3}&&\\&&\frac{1}{3}&\\&&&\frac{1}{3}\end{matrix}\right\}_C\tag{10}$$

and

$$B'-L' = \frac{2\sqrt{6}}{3} F_{\mathrm{H}}^{15} = \frac{\sqrt{6}}{3} \lambda_{\mathrm{H}}^{15} = \begin{Bmatrix} -1 & & & \\ & \frac{1}{3} & & \\ & & \frac{1}{3} & \\ & & & \frac{1}{3} \end{Bmatrix}_{\mathrm{H}}, \tag{11}$$

where the $\lambda_{\mathrm{C}}^{15}$ and $\lambda_{\mathrm{H}}^{15}$ matrices provide the basic representations in the space of four colours (l, r, y, b) and four generations (e, μ, τ, ω), respectively. The group $\mathrm{SO_B}(10) \times \mathrm{U_C}(1)$ is our tentative choice for the big unification group B.

Summarizing this first step of our unification programme we can say that the big unification group $\mathrm{SO_B}(10) \times \mathrm{U_C}(1)$, being semi-simple (like the electroweak symmetry group $\mathrm{SU}(2) \times \mathrm{U}(1)$), implies two coupling constants g_{B} and g'_{B} which define a new Weinberg-like mixing angle θ_{B}: $\tan \theta_{\mathrm{B}} = g'_{\mathrm{B}}/g_{\mathrm{B}}$. Here, g'_{B} is normalized in such a way that in the coupling it stands in front of the current of the generator $\frac{1}{2}(B-L)$ of $\mathrm{U_C}(1)$. After the electroweak-horizontal symmetry breaking our big unification group transits into $\mathrm{SU}(2) \times \mathrm{U}(1) \times \widetilde{\mathrm{SU}}_{\mathrm{H}}(3)$, where now three coupling constants, g, g' and g_{H}, appear instead of two, g_{B} and g'_{B} (see Fig. 1). Since, due to Eqs. (8) and (9),

$$\frac{1}{e^2} = \frac{1}{g^2} + \frac{1}{g'^2}, \quad \frac{1}{g'^2} = \frac{1}{g_{\mathrm{B}}^2} + \frac{1}{g_{\mathrm{B}}'^2} \tag{12}$$

at any Q^2, we obtain at the big unification point M_{B}^2 where $g^2(M_{\mathrm{B}}^2) = g_{\mathrm{H}}^2(M_{\mathrm{B}}^2) = g_{\mathrm{B}}^2(M_{\mathrm{B}}^2)$, the relation

$$\tan^2 \theta_{\mathrm{B}}(M_{\mathrm{B}}^2) = \frac{\tan^2 \theta_{\mathrm{W}}(M_{\mathrm{B}}^2)}{1-\tan^2 \theta_{\mathrm{W}}(M_{\mathrm{B}}^2)}. \tag{13}$$

As in the case of the grand unification group $\mathrm{SO_G}(10)$, the massless gauge bosons $\mathrm{W_R^3}$ and $\mathrm{B_C}$ of the subgroups $\mathrm{U_R}(1)$ and $\mathrm{U_C}(1)$ of our $\mathrm{SO_B}(10) \times \mathrm{U_C}(1)$ transit into the familiar massless gauge boson B of U(1) and a neutral massive vector boson $\mathrm{Z_R}$:

$$\mathrm{B} = \mathrm{B_C} \cos \theta_{\mathrm{B}} - \mathrm{W_R^3} \sin \theta_{\mathrm{B}},$$

$$\mathrm{Z_R} = \mathrm{W_R^3} \cos \theta_{\mathrm{B}} + \mathrm{B_C} \sin \theta_{\mathrm{B}}. \tag{14}$$

This transition may take place at the big unification point or at the possible lower intermediate unification point corresponding to $\mathrm{SU_L}(2) \times \mathrm{SU_R}(2)$ (see Eq. (7)). Note that our big unification $\mathrm{SO_B}(10) \times \mathrm{U_C}(1)$ (like the electroweak unification $\mathrm{SU}(2) \times \mathrm{U}(1)$) gives us no charge quantization (since Tr Q is not necessarily equal to 0) and still suffers from axial current anomalies.

3. The extended unification

Turning to the second step of our unification programme we can see from Eqs. (2) and (3) that the fermion electric charge Q is not traceless in any of the four $\underline{16}$'s. The triangular axial current anomalies appear in any of these four $\underline{16}$'s (also the triplet $(\underline{16}, \underline{3})$ as a whole is not free of them). Fortunately, the reducible representation $(\underline{16}, \underline{1})+(\underline{16}, \underline{3})$, comprising all leptons and quarks of the four flavour families (1), becomes charge traceless and anomaly free. It might suggest that this reducible representation could be regarded as the irreducible representation $\underline{64}$ of an extended unification group $E \supset SO_B(10)\times U_C(1) \times SU_C(3)$, under which

$$f = l+q = \underline{64} = (\underline{16}, \underline{1})+(\underline{16}, \underline{3}). \tag{15}$$

In this case the unification of $SO_B(10)\times U_C(1)$ nad $SU_C(3)$ would be minimal, giving an eight rank group E (similarly the minimal unification of $SU(2)\times U(1)$ and $\widetilde{SU}_H(3)$ gave us the fourth rank group $SU_B(5) \subset SO_B(10)$).

Unfortunately, the seventh rank simple group SO(14), whose complex lowest spinor representation is $\underline{64}$, does not contain the subgroup $SO_B(10)\times U_C(1)\times SU_C(3)$ (it would contain $SO_B(10)\times SO_C(4)$ with some $SO_C(4)$). Instead, the irreducible representation $\underline{64}$ with the content (15) is provided by the eighth rank semi-simple group $SO_B(10)\times SU_C(4) \supset SO_B(10)\times U_C(1)\times SU_C(3)$ since then $\underline{64} = (\underline{16}, \underline{4}) = (\underline{16}, \underline{1})+(\underline{16}, \underline{3})$ [1].

The minimal unification of $SO_B(10)\times SU_C(4)$ leading to a simple group is the eighth rank group $SO(16) \supset SO_B(10)\times SU_C(4)$ because $SU_C(4) = SO_C(6)$. Thus SO(16), if not the semi-simple $SO_B(10)\times SU_C(4)$, is our tip for E, though unlike $SO_B(10)\times SU_C(4)$ it is disfavoured by having only real representations [12]. A related necessary consequence of this group is that it implies the existence of extra four $\underline{16}^*$'s of fermions, forming together with the previous four $\underline{16}$'s its real lowest spinor representation $\underline{128}$ which decomposes under $SO_B(10)\times SU_C(4)$ as $\underline{128} = \underline{64}+\underline{64}^*$. Here $\underline{64}^* = (\underline{16}^*, \underline{4}^*)$ represents a $\underline{16}^*$ of new leptons and a $\underline{16}^*$ of new (colour) quarks, both including four generations (compare Eqs. (2) and (3) for analogous $\underline{16}$'s of "old" leptons and quarks). All of these new leptons and quarks should be heavy but unified with the "old" at the extended unification point. The mass difference between the "old" $\underline{16}$'s and new $\underline{16}^*$'s (forming together the real $\underline{128} = (\underline{16}, \underline{4}) + (\underline{16}^*, \underline{4}^*)$ at the extended unification mass scale) requires presumably an "ugly" SO(16) symmetry-breaking bare mass term for new $\underline{16}^*$'s. In any case, this mass difference is unavoidable because otherwise the new right-handed and "old" left-handed fermions could mix, even at low energies. Such a mixing would redefine the latter (and former) fermions and lead, therefore, to an admixture of right-handed chiral interactions for "old" fermions, which are certainly not observed experimentally at low energies.

Also in the conventional scheme of unification, the group SO(16) [13] can be considered as a disfavoured candidate for the rôle of the extended unification group for the grand unification group $SO_G(10)$ and the horizontal group $\widetilde{SU}_H(4)$ (note that $\widetilde{SO}_H(4)$ would lead to SO(14) [4, 9]). In this case the chain of spontaneous symmetry breaking has the form

$$SO(16) \supset SO_G(10)\times \widetilde{SU}_H(4) \supset SO_G(10)\times \tilde{U}_H(1)\times \widetilde{SU}_H(3)$$

$$\supset SU(2)\times U(1)\times SU_C(3)\times \widetilde{SU}_H(3) \supset U_{EM}(1)\times SU_C(3). \tag{16}$$

[1] The group $SO_B(10)\times SU_C(4)$ contains the subgroup $SU_L(2)\times SU_R(2)\times SU_C(4)$ [11]. However, the inclusion of the big unification as a first step allows, if it is desirable, to avoid the latter group as an intermediate unification.

In contrast, in our scheme of unification we propose the chain

$$\mathrm{SO}(16) \supset \mathrm{SO_B}(10)\times \mathrm{SU_C}(4) \supset \mathrm{SO_B}(10)\times \mathrm{U_C}(1)\times \mathrm{SU_C}(3)$$
$$\supset \mathrm{SU}(2)\times \mathrm{U}(1)\times \widetilde{\mathrm{SU}}_\mathrm{H}(3)\times \mathrm{SU_C}(3) \supset \mathrm{U_{EM}}(1)\times \mathrm{SU_C}(3). \tag{17}$$

In Eqs. (16) and (17) the rôles of four generations (e, μ, τ, ω) and four colours (l, r, y, b) are interchanged (except for the lowest mass scale). We can see that in the chain (17) there appear four unification points corresponding successively to the electroweak unification ($\mathrm{SU}(2)\times\mathrm{U}(1)$), big unification ($\mathrm{SO_B}(10)\times\mathrm{U_C}(1)$), colour unification ($\mathrm{SU_C}(4)$) and, finally, extended unification (SO(16)). There may also exist some intermediate unification points such as that corresponding to chiral unification ($\mathrm{SU_L}(2)\times\mathrm{SU_R}(2)$ with $\mathrm{SU_L}(2)\equiv\mathrm{SU}(2)$) or generation unification ($\widetilde{\mathrm{SU}}_\mathrm{H}(4)$).

Let us remark that, if we call alternatively the big unification group $\mathrm{SO_B}(10)\times\mathrm{U_C}(1)$ (which is unified with $\mathrm{SU_C}(3)$ into SO(16)) the "flavour" group, our scheme of unification fits to the general discussion of colour embeddings in simple unifying groups presented in Ref. [14] (see Case 5 there). Then the discussion of Ref. [14] implies that the embedding of $\mathrm{SU_C}(3)$ in SO(16) is uniquely given by $\mathrm{SO_B}(10)\times\mathrm{U_C}(1)$.

At this point we should emphasize, however, that the realization of the extended unification of $\mathrm{SO_B}(10)\times\mathrm{U_C}(1)$ and $\mathrm{SU_C}(3)$ into a simple group encounters, in the conventional field theory, a serious problem related to the increasing difference of the running coupling constants, $g_S^2-g_B^2$, with the increasing mass scale (see Fig. 1). Thus, unless the too fast decrease of g_B^2 is slowed down by the appearance of some scalars (perhaps sub-elementary scalar constituents) destroying the asymptotic freedom of $\mathrm{SO_B}(10)$, the simple group SO(16) must be abandoned as a candidate for the extended unification group E in favour of the semisimple group $\mathrm{SO_B}(10)\times\mathrm{SU_C}(4)$. In this case the extended unification becomes equivalent to the colour unification ($\mathrm{SU_C}(4)$). However, the extended unification into a larger simple group is still possible if the too slow decrease of g_S^2 is accelerated by an additional unification of the colour with some supercolour into a large strong symmetry group [15].

Summarizing this second step of our unification programme, we can see that if the hypothetical extended unification into a simple group exists, then starting from the extended unification point up the electroweak charges gI_3 and $g'\frac{1}{2}Y$ should have equal norms when evaluated for the full fermion irreducible representation (including leptons and quarks, "old" and new). Since the full democracy of all fermions rules from this point up, these norms become easily calculable, giving us the asymptotic value of the Weinberg angle entirely determined by the electroweak content of the full fermion representation:

$$\tan^2\theta_\mathrm{W} = \frac{g'^2}{g^2} = 4\,\frac{\mathrm{Tr}\, I_3^2}{\mathrm{Tr}\, Y^2}\,. \tag{18}$$

In this way, in the case of the sequential electroweak structure of the fermion representation, we get the popular asymptotic result $\tan^2\theta_\mathrm{W} = 3/5$, which has been previously obtained in the conventional unification scheme [3] by using, in the range from the grand unification point up, the electroweak content of any sequential fermion generation described by $\underline{16} = \underline{5}^* + \underline{10} + \underline{1}$ of $SO_G(10) \supset SU_G(5)$, and neglecting differences between generations (so the extended unification was, in fact, applied to this calculation). A relevant difference between both unification schemes is that in ours there is much more structure in the "grand plateau" between electroweak and extended unification points since there should be placed the thresholds corresponding to gauge bosons of the spontaneously broken big unification group $SO_B(10) \times U_C(1)$ including its horizontal subgroup $\widetilde{SU}_H(3)$. If the kinks at these thresholds were not significant, the $SU_G(5)$ predictions of: (*i*) the correct value for $\sin^2 \theta_W$ in the experimental range and (*ii*) the magnitude of 10^{14}–10^{15} GeV for the grand unification point would be valid in our scheme without major changes (the second prediction being only reinterpreted as concerning the extended unification point). Since these kinks are in fact significant, the discussion is much changed and becomes involved, depending on unknown factors. It may go as follows.

Let us assume the colour unification $SU_C(4)$ (but not necessarily the extended unification into a simple group). Then making use of the renormalization group equations (on the one-loop level) for the groups SU(2), U(1), $SU_C(3)$ and $U_C(1)$ we obtain respectively (see Fig. 1):

$$\frac{1}{g_B^2(M_B^2)} \equiv \frac{1}{g^2(M_B^2)} = \frac{1}{g^2(M_W^2)} + \frac{1}{(2\pi)^2}\{\tfrac{5}{3}, 1\} \ln \frac{M_B}{M_W}, \tag{19}$$

$$\frac{1}{g_B^2(M_B^2)} + \frac{1}{g_B'^2(M_B^2)} \equiv \frac{1}{g'^2(M_B^2)} = \frac{1}{g'^2(M_W^2)} - \tan^{-2}\theta_W(M_B^2)\frac{1}{(2\pi)^2}\{2, \tfrac{8}{3}\} \ln \frac{M_B}{M_W}, \tag{20}$$

$$\tfrac{3}{2}\frac{1}{g_B'^2(M_C^2)} \equiv \frac{1}{g_S^2(M_C^2)} = \frac{1}{g_S^2(M_W^2)} + \frac{1}{(2\pi)^2}\{\tfrac{7}{2}, \tfrac{17}{6}\} \ln \frac{M_C}{M_W}, \tag{21}$$

$$\tfrac{2}{3}\frac{1}{g_S^2(M_C^2)} \equiv \frac{1}{g_B'^2(M_C^2)} = \frac{1}{g_B'^2(M_B^2)} - \tfrac{2}{3}\frac{1}{(2\pi)^2}\{2, \tfrac{8}{3}\} \ln \frac{M_C}{M_B}, \tag{22}$$

where M_W, M_B and M_C are the electroweak, big and colour unification mass scales (no other intermediate unification being allowed in this calculation). Here, the numbers in parentheses { , } refer to the case of three and four fermion generations respectively. Eliminating $g_B^2(M_B^2)$, $g_B'^2(M_B^2)$, $g_B'^2(M_C^2)$ and $g_S^2(M_C^2)$ from Eqs. (19)–(22) and using the first Eq. (12) applied at point M_W^2 we get the compact relation

$$\frac{4\pi}{e^2(M_W^2)} - 2\frac{4\pi}{g^2(M_W^2)} - \tfrac{2}{3}\frac{4\pi}{g_S^2(M_W^2)}$$

$$= \frac{11}{3\pi} \ln \frac{M_B M_C}{M_W^2} + \frac{2}{3\pi}\{3, 4\}\left[\tan^{-2}\theta_W(M_B^2) - \tfrac{5}{3}\right] \ln \frac{M_B}{M_W}, \tag{23}$$

where $\tan^2\theta_W(M_B^2) \lesssim 3/5$. Hence the formula for $\sin^2\theta_W = e^2/g^2$:

$$\sin^2\theta_W(M_W^2) \lesssim \tfrac{1}{2} - \tfrac{1}{3}\frac{e^2(M_W^2)}{g_S^2(M_W^2)} - \frac{11}{6\pi}\frac{e^2(M_W^2)}{4\pi} \ln \frac{M_B M_C}{M_W^2}. \tag{24}$$

Putting $\alpha(M_W^2) = 1/137$, $\alpha_S(M_W^2) = 0.2$ and $\sin^2\theta_W(M_W^2) = 0.23$, we obtain from Eq. (24)

$$M_B M_C \lesssim 10^{26} M_W^2 \sim 10^{30}\ \mathrm{GeV}^2. \tag{25}$$

So, if the colour unification $SU_C(4)$ follows closely the big unification, we have

$$M_B \lesssim 10^{15}\ \mathrm{GeV}, \quad M_C \sim 10^{15}\ \mathrm{GeV}, \tag{26}$$

while if the colour unification approaches the Planck mass we get

$$M_B \lesssim 10^{11}\ \mathrm{GeV}, \quad M_C \sim 10^{19}\ \mathrm{GeV}. \tag{27}$$

Obviously, $M_H < M_B$ may be much below M_B. An analogous formula to (25) has been derived in Ref. [16] in the framework of the conventional unification scheme involving the chiral and colour unifications, where our M_B is replaced by the chiral unification mass scale.

4. Conclusion

In conclusion, we would like to emphasize the characteristic features of our unification scheme, where the "big" unification of flavours and generations precedes the over-all "extended" unification of flavours, generations and colours (and possibly also some supercolours). A general feature is the existence of neutral horizontal vector bosons W_H related to the spontaneously broken group H with the mass scale M_H lying in between the familiar electroweak unification mass scale M_W and a new big unification mass scale M_B which in turn precedes an extended unification mass scale M_E corresponding to either a semisimple or simple group. The mass scales M_B and M_E are connected with the additional vector bosons related to the spontaneously broken groups B and E. A more specific feature of the model, where $H = \widetilde{SU}_H(3)$, $B = SO_B(10) \times U_C(1)$ and $E = SO_B(10) \times SU_C(4)$ or $E = SO(16)$ (or eventually E could be a larger simple group including some supercolour), is the existence of four fermion generations (e, μ, τ, ω), each in four colours (l, r, y, b), forming a singlet (say e) and a triplet (say μ, τ, ω) under $\widetilde{SU}_H(3)$, both of them decomposed into a singlet (l) and a triplet (r, y, b) under $SU_C(3)$. If the singlet under $\widetilde{SU}_H(3)$ is the generation e, then the horizontal octet of vector mesons W_H can be coupled only to the horizontal triplet of generations μ, τ, ω leaving the generation e uncoupled. In this case the W_H mediated decays $b \to s\tau^-\mu^+$ and $t \to c\tau^-\mu^+$ are allowed, while $s \to d\mu^-e^+$ and $c \to u\mu^-e^+$ are forbidden, implying the decays $B^- \to K^-\tau^-\mu^+$ and $T^+ \to D^+\tau^-\mu^+$ and prohibiting $K^- \to \pi^-\tau^-e^+$ and $D^+ \to \pi^+\tau^-e^+$. Similarly, the W_H mediated scattering $\mu^+ s \to \tau^+ b$ (and $\mu^+ c \to \tau^+ t$) is permitted, whilst $e^+ d \to \mu^+ s$ (and $e^+ u \to \mu^+ c$) is prevented, the former giving an associate τ+bottom (very weak) production in the deep inelastic μ scattering on the quark-antiquark sea in the nucleon. The W_H mediated annihilation $s\bar{b} \to \mu^-\tau^+$ (and $c\bar{t} \to \mu^-\tau^+$) is also possible, while $d\bar{s} \to e^-\mu^+$ (and $u\bar{c} \to e^-\mu^+$) is excluded, the former implying the decay of the strange bottom mesons into $\mu^\mp\tau^\pm$.

Note that, in the case when the W_H mediated generation changing processes involving the lowest generation e are forbidden by the decoupling of e from the higher generations μ, τ, ω, the mass scale M_H may be smaller than its estimate in the case when such decoupling does not appear [5]. In particular, the lower limit of 10^4–10^5 GeV for M_H (see the first Ref. [5]) extracted from the experimental bounds on $K_L^0 \to \mu^\mp e^\pm$ and $K^+ \to \pi^+ e^- \mu^+$, does not hold in the new situation. At any rate, if the big unification is followed by the Pati--Salam colour unification we have $M_W < M_H < M_B < M_C$ where $M_B M_C \sim 10^{26} M_W^2$ $\sim 10^{30}$ GeV^2 (no other intermediate unifications appearing in this case).

Finally, it is worthwhile to point out that our unification scheme *should* be realized in Nature *if* the running coupling constant for horizontal interactions, g_H^2, lies between those for electroweak and strong interactions, g^2 and g_S^2, and the horizontal symmetry group H is not smaller in size than $\widetilde{SU}_H(3)$ (see Case 1 discussed in Introduction). In fact, in this case g_H joins g before g_S will be able to meet g. If H were $\widetilde{SU}_H(2)$, g_H could join g_S before g_S meets g. The alternative "big" unification of generations and colours would then be realized (see Case 2 in Introduction). In the case of big unification of flavours and generations, the extended unification into a simple group may require an extension of the colour group $SU_C(4)$ into a larger strong symmetry group including some supercolour [15], in order for g_S and g_B to meet. (The other possibility: scalars destroy the asymptotic freedom for $SO_B(10)$.)

One of us (W. K.) would like to express his gratitude to the CERN Theoretical Division for the hospitality extended to him in September 1981. He is also indebted to Jacques Prentki for helpful discussions and suggestions.

REFERENCES

[1] Cf., e.g., F. Wilczek, A. Zee, *Phys. Lett.* **70B**, 418 (1977); S. Pakvasa, H. Sugawara, *Phys. Lett.* **78B**, 497 (1978); H. Sato, *Nucl. Phys.* **B148**, 433 (1979); D. Wyler, *Phys. Rev.* **D19**, 3369 (1979).

[2] Cf., e.g., T. Maehara, T. Yanagida, *Progr. Theor. Phys.* **60**, 822 (1978); F. Wilczek, A. Zee, *Phys. Rev. Lett.* **42**, 421 (1979); A. Davidson, K. C. Wali, P. K. Mannheim, *Phys. Rev. Lett.* **45**, 1135 (1980); H. Sato, *Phys. Rev. Lett.* **45**, 1997 (1980).

[3] For a review cf., H. Harari, *Phys. Rep.* **42**, 235 (1978).

[4] For references cf., R. N. Cahn, H. Harari, *Nucl. Phys.* **B176**, 135 (1980); H. Sato, loc. cit. in Ref. [2].

[5] For a discussion of the generation changing interactions and the related notion of the generation number or charge cf., R. N. Cahn, H. Harari, loc. cit. in Ref. [4]; W. Królikowski, *Acta Phys. Pol.* **B12**, 637 (1981); *Acta Phys. Pol.* **B13**, 27 (1982).

[6] F. Wilczek, A. Zee, loc. cit. in Ref. [2].

[7] Cf., e.g., W. Królikowski, *Acta Phys. Pol.* **B10**, 767 (1979); *Acta Phys. Pol.* **B12**, 913 (1981). A recurrence mass formula predicts for ω^- and h, o the masses 28.5 GeV and 250, 62 GeV, respectively.

[8] A. Davidson, K. C. Wali, P. D. Mannheim, loc. cit. in Ref. [2].

[9] H. Sato, loc cit. in Ref. [2].

[10] H. Georgi, S. L. Glashow, *Phys. Rev. Lett.* **32**, 438 (1974).

[11] J. C. Pati, A. Salam, *Phys. Rev.* **D10**, 275 (1974); H. Fritzsch, P. Minkowski, *Ann. Phys.* (N. Y.) **93**, 193 (1975); *Nucl. Phys.* **B103**, 61 (1976).

[12] For arguments in favour of the unifying groups possessing complex representations (as SO($4n+2$) among the orthogonal groups) cf., e.g., H. Georgi, *Nucl. Phys.* **B156**, 126 (1979) A. Zee, *Phys. Lett.* **99B**, 110 (1981).

[13] I. Bars, M. Günaydin, *Phys. Rev. Lett.* **45**, 859 (1980).

[14] M. Gell-Mann, P. Ramond, R. Slansky, *Rev. Mod. Phys.* **50**, 721 (1978).

[15] In this case one must evade a non-go theorem, cf., P. H. Frampton, *Phys. Rev. Lett.* **43**, 1912 (1979).

[16] T. Goldman, D. A. Ross, *Nucl. Phys.* **B162**, 102 (1980).