Vol. B13 (1982) *ACTA PHYSICA POLONICA* No 5

ON THE THEORY OF FIELDS IN FINSLER SPACES*

By S. Ikeda**

Department of Mechanical Engineering, Faculty of Science and Technology, Science University of Tokyo***

(Received July 7, 1981)

Some structural features underlying the theory of fields in Finsler spaces are considered by taking into account the intrinsic behaviour of the internal variable (y) associated with each point. The following three themes are considered: A new "parallelism" of y (i.e., δy) representing the intrinsic behaviour of y; The conservation laws for the fields in some special Finsler spaces; The micro-gravitational field as a typical example of the Finslerian field.

PACS numbers: 03.50.Kk, 02,90.+p

1. Introduction

As is well known [1–3], the independent variable of a Finsler space becomes the line-element (x, y), instead of the point (x), so that the y-dependence characterizes essentially the theory of fields in Finsler spaces, where y $(= y^{\lambda};\ \lambda \doteq 1, 2, 3, 4)$ denotes the tangent vector obeying only the linear transformation and playing physically the role of an internal variable associated with each point x $(= x^{\kappa};\ \kappa = 1, 2, 3, 4)$. This y-dependence has been combined, in general, with the concept of "nonlocality" [4] or "anisotropy" [5]. In fact, it has been shown [4] that the so-called nonlocal field theory advanced by Yukawa [6] can be treated by means of Finsler geometry.

Concerning the concept of "nonlocality", it is carried by the internal variable such as y, so that a "nonlocal" field in our sense can be obtained by attaching an internal variable to each point of a "local" (or Riemannian) field. This way of thinking descends from the theory of higher order spaces [7]. From this standpoint, the Finslerian field may be regarded, in a certain sense, as a "nonlocal" gravitational field. These situations will be summarized in Section 2.

---

* Dedicated to Prof. Kazuo Kondo on the occasion of his seventieth birthday.

** Research Institute of Pure and Applied Mechanics.

*** Address: Department of Mechanical Engineering, Faculty of Science and Technology, Science University of Tokyo, Noda, Chiba 278, Japan.

Next, in Section 3, we shall consider a geometrical grasp of the intrinsic behaviour of the internal variable y. Since the vector y shows its own inherent behaviour,its "parallelism" (i.e., δy) may be different, in general, from the ordinary absolute differential of y (i.e., Dy). In this case, the length of y is measured by its own metric tensor $b_{\lambda\kappa}$ (i.e., $y^{\lambda} = b_{\lambda\kappa}y^{\kappa}y^{\lambda}$), while an arbitrary vector, say X, is measured by the ordinary metric tensor $g_{\lambda\kappa}$ (i.e., $X^2 = g_{\lambda\kappa}X^{\kappa}X^{\lambda}$). This introduction of the new metric tensor $b_{\lambda\kappa}$ is inevitable, because two different connections ($\delta \neq D$) cannot be metrical simultaneously for the same one metric tensor ($b_{\lambda\kappa} = g_{\lambda\kappa}$), so that if $\delta y \neq Dy$, then $b_{\lambda\kappa} \neq g_{\lambda\kappa}$. Therefore, it occurs that even if $\delta b_{\lambda\kappa} = 0, \delta g_{\lambda\kappa} \neq 0$ (resp. even if $Dg_{\lambda\kappa} = 0, Db_{\lambda\kappa} \neq 0$). This relation will be used to determine the relation between δy and Dy.

If the conditions that $\delta y \neq Dy$ and $b_{\lambda\kappa} \neq g_{\lambda\kappa}$ are taken into account, then the spatial structure becomes, of course, different from that of Cartan's Finsler space [1]. Therefore, if those conditions are released, then the space reduces to Cartan's Finsler space. By doing so, the y-dependence can be considered in various ways by means of the theory of special Finsler spaces which has recently been developed extensively by Matsumoto [3] and his school. So, in Section 4, we shall consider mainly the conservation laws for the gravitational field in some special Finsler spaces.

As to the physical meaning of y, however, it has not been considered explicitly except the velocity in analytical dynamics [8], and only the physical role of y as the internal variable has been stressed. So, as a typical example, we shall regard y as the space-time fluctuation associated with each point x of the (Riemannian) gravitational field and consider the theory of the micro-gravitational field in Section 5. In this case, the most important problem seems to be the averaging process with respect to y, so that we shall consider this problem from our own standpoint in connection with the concept of entropy.

At any rate, in this paper, some structural features of the Finslerian field represented by the y-dependence will be considered concretely.

2. *On the concept of nonlocality*

As has already been mentioned in Section 1, the concept of nonlocality is embodied by the internal variable, say ω ($= \omega^A$; $A = 1, 2, 3, \ldots, N$), annexed to each point x ($= x^{\kappa}$; $\kappa = 1, 2, 3, 4$) of the "local" field. Therefore, under the premise that the local field is Riemannian spanned by the points $\{x\}$, the nonlocal field becomes "non"-Riemannian spanned by the line-elements $\{(x, \omega)\}$. If ω is taken as a vector, say y, the nonlocal field is Finslerian. This way of thinking descends from the theory of higher order spaces of order M ($= 1, 2, 3, \ldots$) [7], in which an arc length along a curve $x^{\kappa} = x^{\kappa}(t)$ (t is a parameter) is given by the integral

$$s = \int F(x, x^{(1)}, x^{(2)}, x^{(3)}, \ldots, x^{(M)})dt, \tag{2.1}$$

where $x^{(\alpha)} = \dfrac{d^{\alpha}x}{dt^{\alpha}}$ ($\alpha = 1, 2, \ldots, M$) and F denotes the fundamental function. It turns out that Finsler space may be regarded as the higher order space of order 1.

Now, as an example, we shall here take up a spinorial nonlocal field $\Phi = \Phi(x, \omega)$. Then, its absolute differential is defined by

$$D\Phi = d\Phi - \Theta_\mu \Phi dx^\mu - G_A \Phi d\omega^A, \tag{2.2}$$

where Θ_μ and G_A mean the spin gauge fields. In order to obtain the covariant derivatives of Φ, it is necessary to introduce the base connection of ω (i.e., $\delta\omega$) [7], which represents physically the intrinsic behaviour of ω and plays geometrically the most important role in the theory of higher order spaces. Of course, $\delta\omega \neq D\omega$, as has been emphasized in Section 1. When we write $\delta\omega$ in the form

$$\delta\omega^A = d\omega^A + \Xi^A_{B\mu}\omega^B dx^\mu + H^A_{BC}\omega^B d\omega^C \equiv P^A_B d\omega^B + Q^A_\mu dx^\mu, \tag{2.3}$$

we can obtain from (2.2) the following covariant derivatives of Φ under the assumption that P^A_B is non-singular:

$$D\Phi = (\Phi_{|\mu})dx^\mu + (\Phi|_A)\delta\omega^A;$$

$$\Phi_{|\mu} = \frac{\overset{*}{\partial}\Phi}{\partial x^\mu} - \overset{*}{\Theta}_\mu \Phi, \tag{2.4}$$

$$\Phi|_A = (P^{-1})^B_A \left(\frac{\partial \Phi}{\partial \omega^B} - G_B \Phi \right),$$

where $P^A_B = \delta^A_B + H^A_{CB}\omega^C$, $Q^A_\mu = \Xi^A_{B\mu}\omega^B$, $\dfrac{\overset{*}{\partial}}{\partial x^\mu} = \dfrac{\partial}{\partial x^\mu} - N^A_\mu \dfrac{\partial}{\partial \omega^A}$, $\overset{*}{\Theta}_\mu = \Theta_\mu - N^A_\mu G_A$ and $N^A_\mu = Q^B_\mu (P^{-1})^A_B$. Then, by constructing the Lagrangian in terms of those gauge invariant quantities, some kinds of field equations, etc. can be considered through the variational principle. From the standpoint of modern gauge field theory [9], $\overset{*}{\Theta}_\mu$ of (2.4) represents a unified gauge field between Θ_μ and G_A, which responds to the "nonlocalization" due to the internal variable ω [10].

By the way, $x^{(\alpha)}$ ($\alpha \geqslant 2$) in (2.1) is not a vector, but an exvector [7]. If $x^{(\alpha)}$ ($\alpha \geqslant 2$) is likened to a spinor [11], then the resulting nonlocal field may be compared, to some extent, to the so-called superfield [9, 10].

3. *On the intrinsic behaviour of the internal variable*

As has already been emphasized in the previous Sections, our starting point is that the internal variable y of the Finslerian field shows, from a geometrical viewpoint, its own intrinsic "parallelism" (i.e., δy) different from the ordinary absolute differential of y (i.e., Dy). That is to say, when the ordinary absolute differential of an arbitrary vector, say X, is given by (cf. (2.2)), as usual [1–3],

$$DX^\kappa = dX^\kappa + \Gamma^\kappa_{\mu\lambda} X^\mu dx^\lambda + C^\kappa_{\mu\lambda} X^\mu dy^\lambda, \tag{3.1}$$

the intrinsic behaviour of y cannot be grasped by Dy, i.e.,

$$Dy^\kappa = dy^\kappa + \Gamma^\kappa_{\mu\lambda} y^\mu dx^\lambda + C^\lambda_{\mu\lambda} y^\mu dy^\lambda, \tag{3.2}$$

but by a newly introduced "parallelism" δy (cf. (2.3)), i.e.,

$$\delta y^\kappa = dy^\kappa + \Delta^\kappa_{\mu\lambda} y^\mu dx^\lambda + B^\kappa_{\mu\lambda} y^\mu dy^\lambda \equiv P^\kappa_\lambda dy^\lambda + Q^\kappa_\lambda dx^\lambda, \tag{3.3}$$

where the Finslerian coefficients of connection $\Delta^\kappa_{\mu\lambda}$ and $B^\kappa_{\mu\lambda}$ are different from $\Gamma^\kappa_{\mu\lambda}$ and $C^\kappa_{\mu\lambda}$ respectively, and $P^\kappa_\lambda = \delta^\kappa_\lambda + B^\kappa_{\mu\lambda} y^\mu$, $Q^\kappa_\lambda = \Delta^\kappa_{\mu\lambda} y^\mu$. In (3.2) and (3.3), we do not assume such homogeneity conditions as $C^\kappa_{\mu\lambda} y^\mu = 0$ and $B^\kappa_{\mu\lambda} y^\mu = 0$ from a general standpoint. This is different from Cartan's theory [1] (cf. (4.2)).

When the intrinsic behaviour of y is represented by (3.3), it is reflected in the whole spatial structure as follows: Obtaining dy from (3.3) under the assumption that P^κ_λ is non--singular and then substituting it into (3.1), (3.1) is modified in the form (cf. (2.4))

$$DX^\kappa = dX^\kappa + \overset{*}{\Gamma}{}^\kappa_{\mu\lambda} X^\mu dx^\lambda + \overset{*}{C}{}^\kappa_{\mu\lambda} X^\mu \delta y^\lambda$$

$$(= (X^\kappa{}_{|\lambda}) dx^\lambda + (X^\kappa|_\lambda) \delta y^\lambda), \tag{3.4}$$

where $\overset{*}{\Gamma}{}^\kappa_{\mu\lambda} = \Gamma^\kappa_{\mu\lambda} - N^\nu_\lambda C^\kappa_{\mu\nu}$, $\overset{*}{C}{}^\kappa_{\mu\lambda} = (P^{-1})^\nu_\lambda C^\kappa_{\mu\nu}$ and $N^\nu_\lambda = Q^\kappa_\lambda (P^{-1})^\nu_\kappa$. From (3.4), the covariant derivatives of y with respect to x and y, respectively, are obtained as (cf. (2.4))

$$y^\kappa{}_{|\lambda} = -N^\kappa_\lambda + \overset{*}{\Gamma}{}^\kappa_{0\lambda},$$

$$y^\kappa|_\lambda = \delta^\kappa_\lambda, \tag{3.5}$$

where the symbol 0 means the contraction by y (e.g., $X_0 = X_\mu y^\mu$). The fact that $y^\kappa_{|\lambda} \neq 0$ is different from Cartan's theory [1] and is caused by the prescription that even if $Dy = 0$, $\delta y \neq 0$, and vice versa (cf. (3.6)).

Now, we shall proceed to the relation between δy and Dy. For that purpose, we shall recall that there exist two different metrical connections ($\delta \neq D$) for two different metric tensors ($b_{\lambda\kappa} \neq g_{\lambda\kappa}$) respectively and that even if $\delta b_{\lambda\kappa} = 0$, $\delta g_{\lambda\kappa} \neq 0$ (resp. even if $Dg_{\lambda\kappa} = 0$, $Db_{\lambda\kappa} \neq 0$). From this, it may be considered that the connection δ is regarded as a metrical connection for $b_{\lambda\kappa}$ (i.e., $\delta b_{\lambda\kappa} = 0$) derived from the non-metrical one D (i.e., $Db_{\lambda\kappa} \neq 0$). Then, by virtue of Kawaguchi's theorem [12], one relation between δy and Dy can be obtained as follows:

$$\delta y^\kappa = D' y^\kappa + M^\kappa_\lambda y^\lambda;$$

$$M^\kappa_\lambda = \tfrac{1}{2} b^{\kappa\nu} D b_{\nu\lambda}. \tag{3.6}$$

Therefore, from (3.3) and (3.2), the following relations can be obtained (cf. (3.1)):

$$\Delta^\kappa_{\mu\lambda} = \Gamma^\kappa_{\mu\lambda} + \tfrac{1}{2} b^{\kappa\nu} \left(\frac{\partial b_{\nu\mu}}{\partial x^\lambda} - \Gamma^\iota_{\nu\lambda} b_{\iota\mu} - \Gamma^\iota_{\mu\lambda} b_{\nu\iota} \right),$$

$$B^\kappa_{\mu\lambda} = C^\kappa_{\mu\lambda} + \tfrac{1}{2} b^{\kappa\nu} \left(\frac{\partial b_{\nu\mu}}{\partial y^\lambda} - C^\iota_{\nu\lambda} b_{\iota\mu} - C^\iota_{\mu\lambda} b_{\nu\iota} \right). \tag{3.7}$$

Here, it must be admitted that this kind of theory is stimulated by the (three-dimensional) Finslerian deformation theory of ferromagnetic substances [13], where the vector

$y\,(=y^i;\,i=1,2,3)$ is regarded as a spin or magnetization vector and its length is normalized as $\delta_{ij}y^iy^j=1$ at each point by adopting suitable units. Namely, δ_{ij} corresponds to our $b_{\lambda\kappa}$. In the case of the magnetization state, each y rotates to become parallel to the direction of the applied magnetic field and neighbouring vectors $\{y\}$ become parallel to each other in a Euclidean sense, not a Finslerian sense. Therefore, in this state, this Euclidean "parallelism" of y, which is nothing but δy in our sense, cannot be grasped by the ordinary Finslerian absolute differential of y (i.e., Dy). That is to say, δy must be introduced in order to preserve the Euclidean length of y (i.e., $\delta_{ij}y^iy^j=1$) invariant under the "parallelism" $\delta y=0$, by which the metric conditions $\delta\delta_{ij}=0$ hold good, but $D\delta_{ij}\neq 0$. Therefore, from (3.6), $\delta y^i = Dy^i+(\frac{1}{2}\,\delta^{ik}D\delta_{kj})y^j$.

4. On the conservation laws in some special Finslerian fields

When the conditions of the previous Section that $\delta y\neq Dy$ and $b_{\lambda\kappa}\neq g_{\lambda\kappa}$ are relaxed and the ordinary relations that $\delta y=Dy$ and $b_{\lambda\kappa}=g_{\lambda\kappa}$ are assumed, the spatial structure reduces to that of Cartan's Finsler space [1]. On the side of Finsler geometry, the so-called theory of special Finsler spaces with Cartan's connection has been developed extensively by Matsumoto [3] and his school, so that in this Section, by using this theory, we shall mainly consider the conservation laws for the gravitational field in some special Finsler spaces [14].

First, the metric tensor $g_{\lambda\kappa}(x,y)$ is given by

$$g_{\lambda\kappa}=\frac{1}{2}\frac{\partial^2L^2}{\partial y^\lambda\partial y^\kappa}, \tag{4.1}$$

where $L(x,y)$ denotes the fundamental function positively homogeneous of degree 1 in y. In this case, since the relations that $C_{\mu\lambda\kappa}=C_{\mu\kappa\lambda}=C_{\lambda\mu\kappa}\left(=\frac{1}{2}\frac{\partial g_{\mu\lambda}}{\partial y^\kappa}\right)$ are assumed and then such homogeneity conditions as

$$C_{\mu\lambda\kappa}y^\kappa = C_{\mu\lambda\kappa}y^\lambda = C_{\mu\lambda\kappa}y^\mu = 0 \tag{4.2}$$

are satisfied, (3.4) is changed into (cf. (3.1))

$$\begin{aligned} DX^\kappa &= dX^\kappa+F^\kappa_{\mu\lambda}X^\mu dx^\lambda+C^\kappa_{\mu\lambda}X^\mu Dy^\lambda \\ &= (X^\kappa{}_{|\lambda})dx^\lambda+(X^\kappa|_\lambda)Dy^\lambda, \end{aligned} \tag{4.3}$$

where $F^\kappa_{\mu\lambda}=\Gamma^\kappa_{\mu\lambda}-N^\nu_\lambda C^\kappa_{\mu\nu}$, $N^\nu_\lambda=\Gamma^\nu_{0\mu}$, and the covariant derivatives are defined by

$$X^\kappa{}_{|\lambda}=\frac{\delta X^\kappa}{\delta x^\lambda}+F^\kappa_{\mu\lambda}X^\mu,$$

$$X^\kappa|_\lambda=\frac{\partial X^\kappa}{\partial y^\lambda}+C^\kappa_{\mu\lambda}X^\mu, \tag{4.4}$$

where $\dfrac{\delta}{\delta x^\lambda} = \dfrac{\partial}{\partial x^\lambda} - N^\nu_\lambda \dfrac{\partial}{\partial y^\nu}$. Of course, $y^\kappa{}_{|\lambda} = 0$ and $y^\kappa|_\lambda = \delta^\kappa_\lambda$ (cf. (3.5)). By use of (4.4), there appear in Cartan's Finsler space three kinds of curvature tensors ($S^\kappa_{\nu\lambda\mu}$, $P^\kappa_{\nu\lambda\mu}$, $R^\kappa_{\nu\lambda\mu}$) and two kinds of torsion tensors ($R^\kappa_{\lambda\mu}$, $P^\kappa_{\lambda\mu}$) through the following Ricci-identities [3]:

$$X^\kappa{}_{|\lambda|\mu} - X^\kappa{}_{|\mu|\lambda} = R^\kappa_{\nu\lambda\mu} X^\nu - R^\nu_{\mu\lambda} X^\kappa|_\nu,$$
$$X^\kappa{}_{|\lambda}|_\mu - X^\kappa|_{\mu|\lambda} = P^\kappa_{\nu\lambda\mu} X^\nu - C^\nu_{\lambda\mu} X^\kappa{}_{|\nu} - P^\nu_{\mu\lambda} X^\kappa|_\nu,$$
$$X^\kappa|_\lambda|_\mu - X^\kappa|_\mu|_\lambda = S^\kappa_{\nu\lambda\mu} X^\nu. \tag{4.5}$$

Their definitions are omitted here for the sake of simplicity (cf. [3]). Secondly, in order to consider the Finslerian conservation laws, by analogy with Einstein theory [15], the most Riemann-like curvature tensor $R^\kappa_{\nu\lambda\mu}$ (the third curvature tensor) should be noticed. Then, by contraction of the Bianchi identity

$$\mathfrak{S}_{\mu\lambda\kappa}\{R^\alpha_{\beta\mu\lambda|\kappa} + P_\beta{}^\alpha{}_{\mu\nu} R^\nu_{\lambda\kappa}\} = 0, \tag{4.6}$$

it is found [14] that a conservation law such as

$$(R^\kappa_\mu - \tfrac{1}{2} R\delta^\kappa_\mu)_{|\kappa} = 0 \tag{4.7}$$

cannot be obtained, in general. The symbol $\mathfrak{S}_{\mu\lambda\kappa}$ means cyclic permutation of μ, λ, κ and summation [3]. Therefore, it is necessary to specialize the spatial structure. Here, only one typical example will be described [3, 14]: When the space is of scalar curvature $K(x, y)$, the torsion tensor $R_{\mu\lambda\kappa}$ ($\equiv R^\nu_{\lambda\kappa} g_{\nu\mu}$) is given by

$$R_{\mu\lambda\kappa} = \tfrac{1}{3}\, \mathfrak{A}_{\lambda\kappa}\{(L^2 K|_\lambda + 3Ky_\lambda) h_{\mu\kappa}\}, \tag{4.8}$$

where $h_{\mu\kappa}$ is the angular metric tensor (i.e., $h_{\mu\kappa} = g_{\mu\kappa} - l_\mu l_\kappa$; $l_\kappa = y_\kappa/L$ (unit vector)), and the symbol $\mathfrak{A}_{\lambda\kappa}$ means interchange of λ, κ and subtraction [3]. Owing to (4.8) and the relation that $P_{\nu\mu\lambda\kappa} - P_{\nu\mu\kappa\lambda} = -S_{\nu\mu\lambda\kappa|0}$, the following equation can be obtained from (4.6) [16]:

$$\{(R^\kappa_\mu - \tfrac{1}{2} R\delta^\kappa_\mu) + B_\nu(S^\nu_\mu - \tfrac{1}{2} S\delta^\nu_\mu) y^\kappa\}_{|\kappa} - B_{\nu|0}(S^\nu_\mu - \tfrac{1}{2} S\delta^\nu_\mu) = 0, \tag{4.9}$$

where $B_\nu = \dfrac{1}{3L} \dfrac{\partial(KL^3)}{\partial y^\nu}$. Therefore, if $S^\nu_\mu - \frac{1}{2} S\delta^\nu_\mu = 0$, then the conservation law (4.7) can be obtained and an Einsteinian field equation

$$R_{\mu\lambda} - \tfrac{1}{2} R g_{\mu\lambda} = T_{\mu\lambda} \qquad (T_\mu{}^\lambda{}_{|\lambda} = 0) \tag{4.10}$$

can be proposed; if $B_{\nu|0} = 0$, then a new conservation law such as

$$\{(R^\kappa_\mu - \tfrac{1}{2} R\delta^\kappa_\mu) + B_\nu(S^\nu_\mu - \tfrac{1}{2} S\delta^\nu_\mu) y^\kappa\}_{|\kappa} = 0 \tag{4.11}$$

can be obtained; if $K =$ constant, that is, the scalar curvature space is reduced to the constant curvature space, then because of $B_\nu = Ky_\nu$, $B_{\nu|0} = 0$ and $y_\nu S^\nu_\mu = 0$, (4.9) becomes

$$(R^\kappa_\mu - \tfrac{1}{2} R\delta^\kappa_\mu - \tfrac{1}{2} KSy_\mu y^\kappa)_{|\kappa} = 0, \tag{4.12}$$

which has often been cited as a typical example of the Finslerian conservation law (cf. [3]).

Thirdly, as to the first curvature tensor $S^{\kappa}_{\nu\lambda\mu}$, there exists the Bianchi identity

$$\mathfrak{S}_{\mu\lambda\kappa}\{S_{\beta}{}^{\alpha}{}_{\mu\lambda}|_{\kappa}\} = 0, \tag{4.13}$$

from which the conservation law such as

$$(S^{\kappa}_{\mu} - \tfrac{1}{2}\, S\delta^{\kappa}_{\mu})|_{\kappa} = 0 \tag{4.14}$$

can be obtained and a field equation corresponding to (4.10) can be proposed as follows [14]:

$$S_{\mu\lambda} - \tfrac{1}{2}\, Sg_{\mu\lambda} = \tau_{\mu\lambda} \quad (\tau_{\mu}{}^{\lambda}|_{\lambda} = 0). \tag{4.15}$$

But no valuable conservation law with respect to the second curvature tensor $P^{\kappa}_{\nu\lambda\mu}$ were obtained, which is still an open problem.

Of course, there exist many other interesting special Finsler spaces, so that in the future, we should apply them to many interesting physical problems.

5. On the micro-gravitational field

Hitherto, any physical meaning of y has not been asked explicitly, so that in this Section, one example will be supplied. Our starting point is quite epistemological: even if the gravitational field in Einstein's sense is Riemannian at the macro-stage, its microscopic structure does not necessarily remain Riemannian. From this standpoint, if we regard y as the so-called space-time fluctuation associated with each point x, then we may consider, at the first stage, that the resulting gravitational field at some micro-stage, that is, the micro-gravitational field in our sense becomes an eight-dimensional Riemann space (R_8), which is a "unified" field between the base space spanned by points $\{x\}$ and the tangent space spanned by vectors $\{y\}$. Of course, this R_8 corresponds to the tangent bundle over the base space [17]. Then, at the second stage, this R_8 can be arranged to become a four--dimensional Finsler space (F_4) with Cartan's connection by means of the theory of tangent bundles (cf. [17]). This transformation process from R_8 to F_4 is very complicated, so that we shall omit it here. Therefore, from our own standpoint, we may consider, with one bound, that the micro-gravitational field is F_4.

In this theory, the most interesting problem seems to be the averaging process with respect to y. This is formally given by, e.g.,

$$\bar{\phi}(x) = \int \phi(x, y) f(y)\, (dy)^4, \tag{5.1}$$

where $f(y)$ means a distribution function of $\{y\}$. From quite a geometrical viewpoint, (5.1) is likened to the "reduction" of F_4 to some "non"-Riemannian space ($\bar{R}_4$). As an example, if this reduction from F_4 to $\bar{R}_4$ is realized by the condition that y is given by a function of x through $dy^{\kappa} = Z^{\kappa}_{\lambda}(x)dx^{\lambda}$, then the spatial structure of the resulting $\bar{R}_4$ is stipulated as [18], from (3.1) or (4.3),

$$DX^{\kappa} = dX^{\kappa} + \bar{\Gamma}^{\kappa}_{\mu\lambda} X^{\mu} dx^{\lambda} = (X^{\kappa}{}_{\|\lambda}) dx^{\lambda};$$

$$X^{\kappa}{}_{\|\lambda} = \frac{\partial X^{\kappa}}{\partial x^{\lambda}} + \bar{\Gamma}^{\kappa}_{\mu\lambda} X^{\mu}, \tag{5.2}$$

where $\bar{\Gamma}^{\kappa}_{\mu\lambda} = \Gamma^{\kappa}_{\mu\lambda}+Z^{\nu}_{\lambda}C^{\kappa}_{\mu\nu}$ and $\dfrac{\bar{\partial}}{\partial x^{\lambda}} = \dfrac{\partial}{\partial x^{\lambda}} + Z^{\nu}_{\lambda}\dfrac{\partial}{\partial y^{\nu}}$. The metric tensor is also reduced to $\bar{g}_{\lambda\kappa}(x) \equiv g_{\lambda\kappa}(x, y(x))$. From (5.2), in a manner similar to (4.5), the following curvature tensor ($\bar{R}^{\kappa}_{\nu\lambda\mu}$) and the torsion tensor ($\bar{S}^{\kappa}_{\lambda\mu}$) can be introduced in $\bar{R}_4$:

$$X^{\kappa}{}_{\|\lambda\|\mu}-X^{\kappa}_{\mu\|\lambda\|} = \bar{R}^{\kappa}_{\nu\lambda\mu}X^{\nu}-\bar{S}^{\nu}_{\lambda\mu}X^{\kappa}{}_{\|\nu}. \tag{5.3}$$

In this case, $\bar{\Gamma}^{\kappa}_{\mu\lambda}$ is given by

$$\bar{\Gamma}^{\kappa}_{\mu\lambda} = \overline{\begin{Bmatrix}\kappa\\ \mu\lambda\end{Bmatrix}} + W^{\kappa}_{\mu\lambda} \equiv \begin{Bmatrix}\kappa\\ \mu\lambda\end{Bmatrix} + V^{\kappa}_{\mu\lambda}, \tag{5.4}$$

where $\overline{\begin{Bmatrix}\kappa\\ \mu\lambda\end{Bmatrix}}$ $\left(\text{resp. } \begin{Bmatrix}\kappa\\ \mu\lambda\end{Bmatrix}\right)$ is the Christoffel three-index symbol derived from $\left(\dfrac{\bar{\partial}}{\partial x^{\lambda}}, \bar{g}_{\lambda\kappa}\right)$ $\left(\text{resp. } \left(\dfrac{\partial}{\partial x^{\lambda}}, \bar{g}_{\lambda\kappa}\right)\right)$, and $W^{\kappa}_{\mu\lambda}$ and $V^{\kappa}_{\mu\lambda}$ are defined as the rests constructed essentially by the torsion $\bar{S}^{\kappa}_{\mu\lambda}$. Therefore, it may be said that by the averaging process with respect to y, F_4 is reduced to $\bar{R}_4$, in which the torsion $\bar{S}^{\kappa}_{\mu\lambda}$ appears in general. This kind of torsion has been combined with the concept of entropy (production) from a thermodynamical viewpoint [19] so that the torsion $\bar{S}^{\kappa}_{\mu\lambda}$ in our case can also be related to the concept of "irreversibility" associated with the "coarse graining" of the averaging process.

If we rewrite $\bar{R}^{\kappa}_{\nu\lambda\mu}$ with the use of (5.4) in the form

$$\bar{R}^{\kappa}_{\nu\lambda\mu} = \bar{K}^{\kappa}_{\nu\lambda\mu}(\{\ \})+\bar{L}^{\kappa}_{\nu\lambda\mu}(V), \tag{5.5}$$

then under the assumption of teleparallelism, i.e., $\bar{R}^{\kappa}_{\nu\lambda\mu} = 0$, we can obtain an Einsteinian field equarion [18]

$$\bar{K}_{\mu\lambda}-\tfrac{1}{2}\,\bar{K}\bar{g}_{\mu\lambda} = \bar{U}_{\mu\lambda}, \tag{5.6}$$

where $\bar{K}^{\kappa}_{\nu\lambda\mu}$ is the Riemann-Christoffel curvature tensor and the energy-momentum tensor $\bar{U}_{\mu\lambda}$ is constructed properly by $\bar{L}^{\kappa}_{\nu\lambda\mu}$ or the torsion $\bar{S}^{\kappa}_{\lambda\mu}$. In other words, the torsion, therefore the entropy, contributes to the material term at the stage of the field equation, which has often been stated within the thermodynamics of the gravitational field (cf. [18, 20]).

6. *Conclusion*

In this paper, we have considered some structural features of the Finslerian field caused by the y-dependence: the relation between the concepts of nonlocality and y-dependence (Section 2); the intrinsic behaviour of y (i.e., δy) and its relation to Dy (Section 3); the conservation laws for the gravitational field in some special Finsler spaces (Section 4); the micro-gravitational field in which y is regarded as the space-time fluctuation and the averaging process with respect to y (Section 5).

On the side of Finsler geometry, the theory of special Finsler spaces will progress more and more, so that the author would like to appeal again to physicists and geometricians to supply many valuable applications.

REFERENCES

[1] E. Cartan, *Les espaces de Finsler*, Hermann, Paris 1934.
[2] H. Rund, *The Differential Geometry of Finsler Spaces*, Springer, Berlin-New York 1959.
[3] M. Matsumoto, *Foundation of Finsler Geometry and Special Finsler Spaces*, to be published by Springer, Berlin-New York.
[4] J. I. Horváth, *Suppl. Nuovo Cimento* **9**, 444 (1958); Y. Takano, *Prog. Theor. Phys.* **40**, 1159 (1968).
[5] J. I. Horváth, *Acta Phys. Chem. Szeged* **7**, 3 (1961).
[6] H. Yukawa, *Phys. Rev.* **77**, 219 (1950).
[7] A. Kawaguchi, *Rend. Circ. Mate. Palermo* **56**, 246 (1932); M. Kawaguchi, *RAAG Memoirs* **3**, 718 (1962).
[8] J. Klein, *Ann. Inst. Fourier, Grenoble* **12**, 1 (1962).
[9] R. Arnowitt and P. Nath (eds.), *Gauge Theories and Modern Field Theory*, MIT Press, Cambridge 1976; K. Bleuler and A. Reetz (eds.), *The Differential Geometrical Methods in Mathematical Physics*, Springer Berlin-New York 1977.
[10] S. Ikeda, *Found. Phys.* **10**, 281 (1980).
[11] K. Kondo, M. Kawaguchi, *RAAG Memoirs* **4**, 351 (1968).
[12] A. Kawaguchi, *Akad. Wetensch. Amsterdam Proc.* **40**, 596 (1937).
[13] S. Amari, *RAAG Memoirs* **3**, 193 (1962); G. A. Maugin, *C. R. Acad. Sci. Paris* **A273**, 474 (1971).
[14] Y. Takano, *Lett. Nuovo Cimento* **10**, 747 (1974); Proc. Int. Symp. Relativity and Unified Field Theory, Calcutta 1975–76, pp. 17–26; S. Ikeda, *J. Math. Phys.* **22**, 1211, 1215 (1981).
[15] A. Einstein, *Ann. der Phys.* **49**, 769 (1916); *The Meaning of Relativity*, Princeton Univ. Press, Princeton 1955.
[16] S. Numata, *Tensor* **32**, 27 (1978); C. Shibata, *Tensor* **32**, 311 (1978); H. Shimada, *J. Korean Math. Soc.* **14**, 41 (1977).
[17] K. Yano, E. T. Davies, *Ann. Mate. Pura Appl.* **37**, 1 (1954); *Rend. Circ. Mate. Palermo* **12**, 211 (1963).
[18] K. Kondo, S. Amari, *RAAG Memoirs* **4**, 225 (1968).
[19] S. Ikeda, *Prog. Theor. Phys.* **64**, 2265 (1980).
[20] G. Neugebauer, *Int. J. Theor. Phys.* **16**, 241 (1977).